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LIFTING BRAUER CHARACTERS OF p-SOLVABLE GROUPS

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Let φ be an irreducible Brauer character for the prime p of the finite p-solvable group, G. By the Fong-Swan theorem, there exists an ordinary character, χ, which agrees with φ on p-regular elements. This character is not, in general unique. It is proved here that χ can be chosen to be p-rational, i.e. its values lie in a field of the form $Q[\varepsilon]$ with $\varepsilon^n = 1$ and $p \nmid n$. If $p \neq 2$, the character so chosen is unique and every irreducible constituent of its restriction to a normal subgroup is also p-rational and is modularly irreducible.

1. Introdution. Let G be a finite group. We use the notation $\mathrm{Irr}(G)$ to denote the set of ordinary (complex) irreducible characters of G. For a fixed prime p, we write $\mathrm{IBr}(G)$ for the set of irreducible Brauer characters of G, chosen with respect to some fixed pull-back of modular p'-roots of unity to the complex numbers. If χ is an ordinary character, let χ^* denote the restriction of χ to the set of p-regular elements of G so that χ^* is a nonnegative integer linear combination of $\varphi \in \mathrm{IBr}(G)$.

Now suppose that G is p-solvable. A theorem of Fong and Swan (see [2], Theorem 72.1) asserts that if $\varphi \in \mathrm{IBr}(G)$, then there exists $\chi \in \mathrm{Irr}(G)$ with $\chi^* = \varphi$. The character, χ, is not uniquely determined by the equation $\chi^* = \varphi$. Furthermore, if $N \lhd G$ and μ is an irreducible constituent of φ_N, then χ_N does not necessarily have a constituent, ψ, with $\psi^* = \mu$. (An example is given in § 9.) The main result of this paper is that if $p \neq 2$ and $\varphi \in \mathrm{IBr}(G)$, then there exists a unique p-rational character (as defined below) $\chi \in \mathrm{Irr}(G)$, such that $\chi^* = \varphi$. Also, χ behaves well with respect to normal subgroups.

DEFINITION 1.1. Let χ be an ordinary character of G. Then χ is p-rational provided that the values of χ lie in a field of the form $\boldsymbol{Q}[\varepsilon]$ where $\varepsilon^n = 1$, $p \nmid n$.

THEOREM 1.2. *Let G be p-solvable with $p \neq 2$ and let $\varphi \in \mathrm{IBr}(G)$. Then there exists a unique, p-rational $\chi \in \mathrm{Irr}(G)$ with $\chi^* = \varphi$. Furthermore, if $N \lhd G$ and ψ is an irreducible constituent of χ_N, then ψ is p-rational and $\psi^* \in \mathrm{IBr}(N)$.*

In the situation of Theorem 1.2, if μ is any irreducible constituent of φ_N, then there clearly exists a constituent, ψ, of χ_N such that μ is a constituent of ψ^*. Thus $\mu = \psi^*$ and ψ is the unique p-rational lift of μ.

If χ and ψ are as in the theorem and $M \lhd N$, then by application of the theorem to N, we conclude that any irreducible constituent, θ, of ψ_M is p-rational and satisfies $\theta^* \in \mathrm{IBr}(M)$. Repeated application of this argument shows that χ is "subnormally p-rational".

DEFINITION 1.3. Let $\mathscr{M}$ be a set of subgroups of G and let $\chi \in \mathrm{Irr}(G)$. Then χ is $\mathscr{M}$-p-rational if for every $M \in \mathscr{M}$ and irreducible constituent, θ, of χ_M, we have θ is p-rational. If $\mathscr{M}$ is the set of subnormal subgroups of G, we say that χ is *subnormally p-rational*. The set of subnormally p-rational characters of G is denoted $\mathscr{S}(G)$.

THEOREM 1.4. *Let G be p-solvable with $p \neq 2$, and let $\chi \in \mathrm{Irr}(G)$. The following are equivalent*

(a) *χ is p-rational and $\chi^* \in \mathrm{IBr}(G)$,*

(b) *$\chi \in \mathscr{S}(G)$,*

(c) *χ is $\mathscr{M}$-p-rational where $\mathscr{M}$ is a subnormal series in G whose factor groups are p-groups and p'-groups.*

Also, $*$ defines a one-to-one correspondence from $\mathscr{S}(G)$ onto $\mathrm{IBr}(G)$.

Note that in the situation of this theorem, it suffices to check that χ is $\mathscr{M}$-p-rational where $\mathscr{M}$ is the set of characteristic subgroups of G in order to prove that χ is subnormally p-rational. In § 7, we discuss some other conditions sufficient to guarantee $\chi \in \mathscr{S}(G)$. We also raise some questions there. In § 8, the theory of characters of solvable groups is invoked to obtain some partial answers.

This paper also contains a digression in which some of our methods are applied to give a new proof of the Fong-Swan theorem which does work when $p = 2$. This proof constructs p-rational characters but it is not clear that they are uniquely defined.

2. Frobenius reciprocity.

One of the most useful tools for working with ordinary characters is Frobenius reciprocity. We discuss a situation where it works for Brauer characters. If $H \subseteq G$ and φ is a Brauer character of H, we define φ^G by the familiar formula

$$\varphi^G(g) = (1/|H|) \sum_{x \in G} \varphi^\circ(xgx^{-1})$$

for p-regular $g \in G$, where $\varphi^\circ(y) = 0$ if $y \notin H$. Clearly, if ψ is an ordinary character of H, then $(\psi^G)^* = (\psi^*)^G$.

If φ is afforded by an $F[H]$-module, W, for a suitable field, F, of characteristic p, then φ^G is the Brauer character afforded by the $F[G]$-module, W^G. (This fact is somewhat less trivial than the corresponding relationship between induction of ordinary characters and $C[G]$-modules. See § 25 of [1].)

If F is any field and U and V are $F[G]$-modules, define

$$I(U, V) = \dim_F (\hom_G (U, V)) .$$

The following result occurs in [3]. Its proof is routine.

LEMMA 2.1. *Let $H \subseteq G$ and let F be any field. Suppose U is an $F[H]$-module and V is an $F[G]$-module. Then*

$$I(U^G, V) = I(U, V_H) .$$

If μ and ν are Brauer characters of G, we may write

$$\mu = \sum_{\varphi \in \mathrm{IBr}(G)} a_\varphi \varphi$$

and

$$\nu = \sum_{\varphi \in \mathrm{IBr}(G)} b_\varphi \varphi .$$

We shall use the notation

$$I(\mu, \nu) = \sum a_\varphi b_\varphi$$

so that $I(\mu, \mu) = 1$ iff $\mu \in \mathrm{IBr}(G)$ and in that case $I(\mu, \nu) \neq 0$ iff μ is a constituent of ν. Note that if μ, ν are afforded by $F[G]$-modules U and V respectively, then $I(U, V)$ need not equal $I(\mu, \nu)$. However, if F is a splitting field and U and V are completely reducible, then equality does occur.

In general, if $H \subseteq G$, $\mu \in \mathrm{IBr}(H)$ and $\varphi \in \mathrm{IBr}(G)$, we cannot conclude that $I(\mu^G, \varphi) = I(\mu, \varphi_H)$. If $H \lhd G$, then it is not hard to see (using Clifford's theorem and Lemma 2.1) that $I(\mu^G, \varphi) \neq 0$ iff $I(\mu, \varphi_H) \neq 0$. More is true if $p \nmid |G:H|$.

THEOREM 2.2. *Let $H \lhd G$, $\mu \in \mathrm{IBr}(H)$ and $\varphi \in \mathrm{IBr}(G)$. Suppose either $p \nmid |G:H|$ or $\mu^G \in \mathrm{IBr}(G)$. Then $I(\mu^G, \varphi) = I(\mu, \varphi_H)$.*

Proof. Let F be a splitting field for G and H with $\mathrm{char}(F) = p$. Let W be an $F[H]$-module which affords μ and V an $F[G]$-module which affords φ. Now V is irreducible and hence by Clifford's theorem, V_H is completely reducible and $I(\mu, \varphi_H) = I(W, V_H)$. By Lemma 2.1, it suffices to show that $I(\mu^G, \varphi) = I(W^G, V)$ to complete the proof. We do this by proving that W^G is completely reducible.

Since W^G affords μ^G, we are done if $\mu^G \in \mathrm{IBr}(G)$ and so we assume $p \nmid |G:H|$. Let U be a submodule of W^G. In order to show that U is a direct summand of W^G, it suffices (by Theorem 2(a) of [4]) to show that U_H is a direct summand of $(W^G)_H$. Since $H \lhd G$, $(W^G)_H$ is a direct sum of G-conjugates of W and thus is completely reducible. The result now follows.

3. **p'-factors.** It is well known that if $p \nmid |G|$, then $\mathrm{IBr}(G) = \mathrm{Irr}(G)$. In this section we prove the following generalization.

THEOREM 3.1. *Let $N \lhd G$ with $p \nmid |G:N|$. Let $\psi \in \mathrm{Irr}(N)$ and assume*

(a) $\psi^* \in \mathrm{IBr}(N)$ *and*

(b) $\psi^g = \psi$ *for those $g \in G$ with $(\psi^*)^g = \psi^*$.*

Then $*$ defines a one-to-one correspondence from $\mathscr{X} = \{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \psi] \neq 0$ onto $\mathscr{Y} = \{\varphi \in \mathrm{IBr}(G) \mid I(\varphi_N, \psi^*) \neq 0\}$.

Proof. Write

$$\psi^G = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$$

and

$$(\psi^*)^G = \sum_{\varphi \in \mathrm{IBr}(G)} b_\varphi \varphi \ .$$

Let $\psi = \psi_1, \psi_2, \cdots, \psi_t$ be the distinct G-conjugates of ψ and let $\mu_i = \psi_i^* \in \mathrm{IBr}(N)$. By Frobenius reciprocity, $\chi \in \mathscr{X}$ iff $a_\chi \neq 0$ and

$$\chi_N = a_\chi \sum \psi_i \quad \text{for} \quad \chi \in \mathscr{X} \ .$$

By hypothesis (b), the μ_i are the distinct G-conjugates of ψ^*. From Theorem 2.2, we may conclude that $\varphi \in \mathscr{Y}$ iff $b_\varphi \neq 0$ and

$$\varphi_N = b_\varphi \sum \mu_i \quad \text{for} \quad \varphi \in \mathscr{Y} \ .$$

Let $d_{\chi\varphi}$ be the decomposition numbers so that

$$\chi^* = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi} \varphi$$

for $\chi \in \mathrm{Irr}(G)$. We have

$$\begin{aligned} \sum_\varphi b_\varphi \varphi = (\psi^*)^G = (\psi^G)^* &= \Big(\sum_\chi a_\chi \chi\Big)^* \\ &= \sum_\varphi \Big(\sum_\chi a_\chi d_{\chi\varphi}\Big)\varphi \ . \end{aligned}$$

By the linear independence of $\mathrm{IBr}(G)$, we conclude that

$$(1) \qquad b_\varphi = \sum_\chi a_\chi d_{\chi\varphi} \quad \text{for} \quad \varphi \in \mathrm{IBr}(G) .$$

Also, for $\chi \in \mathscr{X}$ we have

$$(2) \qquad a_\chi \sum \mu_i = (a_\chi \sum \psi_i)^* = (\chi_N)^* = (\chi^*)_N = \textstyle\sum_\varphi d_{\chi\varphi}\varphi_N .$$

If $d_{\chi\varphi} \neq 0$ with $\chi \in \mathscr{X}$, then it follows from (1) that $b_\varphi \neq 0$ and $\varphi \in \mathscr{Y}$. Thus (2) yields

$$a_\chi \sum \mu_i = \sum_{\varphi \in \mathscr{Y}} d_{\chi\varphi}\varphi_N = \Big(\sum_{\varphi \in \mathscr{Y}} b_\varphi d_{\chi\varphi}\Big) \sum \mu_i$$

and thus

$$(3) \qquad a_\chi = \sum_{\varphi \in \mathscr{Y}} b_\varphi d_{\chi\varphi} \quad \text{for} \quad \chi \in \mathscr{X} .$$

Observe that (3) remains valid if the sum is taken over all $\varphi \in \mathrm{IBr}(G)$.

Substitute (1) into (3) to obtain

$$a_\chi = \sum_{\varphi \in \mathrm{IBr}(G)} \Big(\sum_{\xi \in \mathrm{Irr}(G)} a_\xi d_{\xi\varphi}\Big) d_{\chi\varphi} .$$

Since $\sum_\varphi d_{\chi\varphi} d_{\xi\varphi} = I(\chi^*, \xi^*)$, we conclude that

$$(4) \qquad a_\chi = \sum_\xi I(\chi^*, \xi^*) a_\xi \quad \text{for} \quad \chi \in \mathscr{X} .$$

Now $a_\xi \geqq 0$ and $I(\chi^*, \xi^*) \geqq 0$ for all χ, ξ. Furthermore, $I(\chi^*, \chi^*) \geqq 1$ and $a_\chi > 0$ for $\chi \in \mathscr{X}$. We may now conclude from (4) that $I(\chi^*, \chi^*) = 1$ for $\chi \in \mathscr{X}$ and $I(\chi^*, \xi^*) = 0$ if $\chi, \xi \in \mathscr{X}$ with $\chi \neq \xi$. It follows that $*$ defines a one-to-one map from $\mathscr{X}$ into $\mathscr{Y}$.

Now let $\varphi \in \mathscr{Y}$. Then $b_\varphi \neq 0$ and by (1) we conclude that $a_\chi \neq 0 \neq d_{\chi\varphi}$ for some χ. Thus $\chi \in \mathscr{X}$ and φ is a constituent of χ^*. Since χ^* is now known to be irreducible, we have $\chi^* = \varphi$ and the proof is complete.

4. p-factors. Since we are interested in p-solvable groups, every chief factor will be a p-group or a p'-group. We obtain results for p-factors which are analogous to Theorem 3.1.

LEMMA 4.1. *Let $H \lhd G$ with G/H a p-group. Let $\psi \in \mathrm{Irr}(H)$ and $T = \{g \in G \mid \psi^g = \psi\}$. Suppose ψ is extendible to $\eta \in \mathrm{Irr}(T)$. Assume*

(a) *$\psi^* \in \mathrm{IBr}(H)$ and*

(b) *if $(\psi^*)^g = \psi^*$, then $g \in T$.*

Then $(\eta^G)^*$ is the unique $\varphi \in \mathrm{IBr}(G)$ with $I(\varphi_H, \psi^*) \neq 0$.

Proof. Choose $\varphi \in \mathrm{IBr}(G)$ with $I(\varphi_H, \psi^*) \neq 0$. (Clearly, some irreducible constituent of $(\eta^G)^*$ will work.) Now $(\eta^G)_H$ is the sum of the distinct G-conjugates of ψ and so by (b) we conclude that

$((\eta^G)^*)_H$ is the sum of the distinct conjugates of ψ^*. By Clifford's theorem, $\varphi_H = e((\eta^G)^*)_H$ for some integer e.

Since all of the p-regular elements of G lie in H, it follows that $\varphi = e(\eta^G)^*$. We conclude that $e = 1$ since φ is irreducible. The result follows.

If $H \triangleleft G$, G/H is a p-group and $\mu \in \mathrm{IBr}(H)$, then even without assuming the existence of ψ in 4.1, it is true that there is a unique $\varphi \in \mathrm{IBr}(G)$ with $I(\varphi_H, \mu) \neq 0$. Also φ_H is the sum of the distinct conjugates of μ. We will not need these facts, however.

In order to apply Lemma 4.1, we need conditions sufficient to guarantee that ψ is extendible to its inertia group, T. There are at least two such sets of extendibility conditions:

(E1) $p \nmid \psi(1)$ and $p \nmid o(\psi)$

(E2) $p \neq 2$ and ψ is p-rational.

Here, $o(\psi)$ is the *determinantal order* of ψ, defined as the order (in the group of linear characters) of $\det(\psi)$, the determinant of a representation which affords ψ.

In order to obtain a proof of the Fong-Swan theorem which works for all p, we shall use (E1). The condition $p \nmid \psi(1)$ causes complications which can be avoided when $p \neq 2$ by using (E2). This will lead to our stronger results in that case.

THEOREM 4.2 (Gallagher). *Let $H \triangleleft G$ with G/H a p-group. Suppose ψ is invariant in G, $p \nmid \psi(1)$ and $p \nmid o(\psi)$. Then ψ has a unique extension, χ, to G with $p \nmid o(\chi)$.*

Proof. First suppose $|G:H| = p$. We may thus extend ψ to $\eta \in \mathrm{Irr}(G)$. Let $\lambda = \det(\eta)$ so that $\lambda_H = \det(\psi)$ has order m with $p \nmid m$. Let $\mu = \lambda^m$ so that $H \subseteq \ker \mu$ and $\mu^b\eta$ is an extension of ψ for any integer, b. Choose b so that $\psi(1)mb \equiv -1 \bmod p$. Let $\chi = \mu^b\eta$. Then

$$\det(\chi) = \mu^{\psi(1)b}\lambda = \lambda^{\psi(1)mb+1} .$$

Since $\mu^p = 1_G$, we have $(\lambda^p)^m = 1_G$. Since $p \mid (\psi(1)mb + 1)$, we conclude $o(\chi) \mid m$ and thus $p \nmid o(\chi)$.

If $\chi_0 \neq \chi$ is an extension of ψ, then $\chi_0 = \alpha\chi$ for some linear character, α, of G/H. Then $\det(\chi_0) = \alpha^{\psi(1)}\det(\chi)$. Since $p = o(\alpha)$ and $p \nmid \psi(1)$, we have $p \mid o(\chi_0)$.

If $|G:H| > p$, choose K, $H < K \triangleleft G$ with $|K:H| = p$. Let ξ be the unique extension of ψ to K with $p \nmid o(\xi)$. Since ξ is unique, it is invariant in G. By induction, ξ has a unique extension, χ to G with $p \nmid o(\chi)$. The uniqueness of χ as an extension of ψ follows since if χ_0 extends ψ with $p \nmid o(\chi_0)$, then $p \nmid o((\chi_0)_K)$ and thus $(\chi_0)_K = \xi$. Therefore $\chi_0 = \chi$.

We need some facts about p-rational characters to prove that (E2) works.

Let $\boldsymbol{Q}_n$ denote the field $\boldsymbol{Q}[e^{2\pi i/n}]$. Then a character, χ, of G is p-rational if for some n with $p \nmid n$ we have $\chi(g) \in \boldsymbol{Q}_n$ for all $g \in G$. Let $|G| = m = p^a r$ with $p \nmid r$. Then $\chi(g) \in \boldsymbol{Q}_m$ for all $g \in G$ and it is not hard to see that χ is p-rational iff its values lie in $\boldsymbol{Q}_r$.

Therefore, the p-rational characters of G are exactly those fixed by the Galois group $\mathfrak{G}(\boldsymbol{Q}_m/\boldsymbol{Q}_r)$ which we shall denote $\mathfrak{G}(G)$. If θ is a character of $H \subseteqq G$, then θ has values in $\boldsymbol{Q}_m$ and θ^σ is defined for $\sigma \in \mathfrak{G}(G)$. It follows that θ is p-rational iff $\theta^\sigma = \theta$ for all such σ.

By Galois theory, we know that the restriction map defines an isomorphism of $\mathfrak{G}(G)$ onto $\mathfrak{G}(\boldsymbol{Q}_{p^a}/\boldsymbol{Q})$. It follows for $p \neq 2$, that $\mathfrak{G}(G)$ is cyclic and also, if $p \mid |G|$, then $\mathfrak{G}(G)$ does not fix a primitive pth root of 1.

THEOREM 4.3. *Let $H \lhd G$ with G/H a p-group, $p \neq 2$. Let $\psi \in \mathrm{Irr}(H)$ be p-rational and invariant in G. Then ψ has a p-rational extension, $\chi \in \mathrm{Irr}(G)$. Furthermore, χ is the unique p-rational irreducible constituent of ψ^G.*

Proof. First suppose $|G\colon H| = p$ and let η be any extension of ψ to G. Let σ generate $\mathfrak{G}(G)$. Then $\psi^\sigma = \psi$ and so η^σ extends ψ and we have $\eta^\sigma = \lambda\eta$ for some $\lambda \in \mathrm{Irr}(G/H)$. If η is not p-rational, then $\lambda \neq 1_G$ and thus $\lambda^\sigma \neq \lambda$ (since $p \neq 2$). We have $\lambda^\sigma = \lambda^m$ for some integer, $m \not\equiv 1 \bmod p$. Now $\lambda^b\eta$ is an extension of ψ for integer b and we may choose b so that $(m-1)b \equiv -1 \bmod p$. Then $mb + 1 \equiv b \bmod p$ and

$$(\lambda^b\eta)^\sigma = \lambda^{mb+1}\eta = \lambda^b\eta$$

and $\lambda^b\eta$ is p-fixed.

Let χ be a p-rational extension of ψ and let χ_0 be another extension. Then $\chi_0 = \alpha\chi$ for some unique $\alpha \in \mathrm{Irr}(G/H)$. If χ_0 is p-rational, it follows that $\chi_0 = \chi_0^\sigma = \alpha^\sigma\chi^\sigma = \alpha^\sigma\chi$ and thus $\alpha^\sigma = \alpha$ and we conclude $\alpha = 1_G$ and $\chi_0 = \chi$.

If $|G\colon H| > p$, choose K, $H < K \lhd G$ with $|K\colon H| = p$. Let ξ be the p-rational extension of ψ to K. By uniqueness, ξ is invariant in G and working by induction, we conclude that ξ has a p-rational extension, $\chi \in \mathrm{Irr}(G)$.

Since $K/H \subseteqq \boldsymbol{Z}(G/H)$, every extension of ψ to K is invariant in G (since each is of the form $\alpha\xi$). If χ_0 is any irreducible constituent of ψ^G, then $(\chi_0)_K = e\eta$ for some extension, η, of ψ. If χ_0 is p-rational, it follows that η is p-rational and thus $\eta = \xi$ and χ_0 is a constituent of ξ^G. By the inductive hypothesis, $\chi_0 = \chi$.

We shall need the following well known fact. It appears, for instance, as Lemma 10.4 of [7].

LEMMA 4.4. *Let* $N \lhd G$, $\theta \in \mathrm{Irr}\,(N)$ *and* $T = \{g \in G \mid \theta^g = \theta\}$. *Then* $\psi \to \psi^G$ *defines a one-to-one correspondence* $\{\psi \in \mathrm{Irr}\,(T) \mid [\psi_N, \theta] \neq 0\}$ *onto* $\{\chi \in \mathrm{Irr}\,(G) \mid [\chi_N, \theta] \neq 0\}$.

In the situation of Lemma 4.4, if ψ is p-rational, then clearly ψ^G is p-rational. Conversely, if χ and θ are both p-rational, then ψ is p-rational since $\psi^G = (\psi^\sigma)^G$ and $[\psi_N, \theta] = [(\psi^\sigma)_N, \theta]$ for $\sigma \in \mathfrak{G}(G)$.

THEOREM 4.5. *Let* $H \lhd G$ *with* G/H *a* p*-group,* $p \neq 2$. *Suppose* $\theta \in \mathrm{Irr}\,(H)$ *is* p*-rational. Then* θ^G *has a unique* p*-rational irreducible constituent,* χ. *Furthermore, suppose* $\theta^* = \mu \in \mathrm{IBr}\,(H)$ *and that* $\theta^g = \theta$ *for those* $g \in G$ *with* $\mu^g = \mu$. *Then* $\chi^* \in \mathrm{IBr}\,(G)$. *Also* χ^* *is the unique* $\varphi \in \mathrm{IBr}\,(G)$ *with* $I(\varphi_H, \mu) \neq 0$.

Proof. Let $T = \{g \in G \mid \theta^g = \theta\}$. By Theorem 4.3, let η be the p-rational extension of θ to T so that η is the unique p-rational element of $\{\psi \in \mathrm{Irr}\,(T) \mid [\psi_H, \theta] \neq 0\}$. By Lemma 4.4 and the remarks following it, we conclude that $\chi = \eta^G$ is the unique p-rational irreducible constituent of θ^G. The final statements follow from Lemma 4.1.

Connections between Theorems 4.2 and 4.3 are given by the following.

COROLLARY 4.6. *Let* $\chi \in \mathrm{Irr}\,(G)$ *be* p*-rational with* $p \neq 2$. *Then* $p \nmid o(\chi)$.

Proof. We have $(\det \chi)^\sigma = \det(\chi^\sigma) = \det \chi$ for $\sigma \in \mathfrak{G}(G)$. If $p \mid o(\chi)$, then $\det(\chi)$ takes on the value $e^{2\pi i/p}$, a contradiction.

COROLLARY 4.7. *In the situation of Theorem 4.2, if* ψ *is* p*-rational, then so is* χ.

Proof. Clearly $o(\chi^\sigma) = o(\chi)$ for $\sigma \in \mathfrak{G}(G)$. Since χ^σ is an extension of ψ, uniqueness forces $\chi^\sigma = \chi$.

5. **The Fong-Swan theorem.** In this section we prove a slight strengthening of the Fong-Swan theorem using Theorem 3.1, Lemma 4.1 and Theorem 4.2. The notion of p-rational characters is only incidental here. Our result also includes a theorem of Huppert (Satz 7 of [5]). We begin with a lemma which is the analog of part of Lemma 4.4 for Brauer characters. In its module version, at least, it is well known and we omit the proof.

LEMMA 5.1. *Let* $N \lhd G$, $\mu \in \mathrm{IBr}\,(N)$ *and* $T = \{g \in G \mid \mu^g = \mu\}$. *Let*

$\varphi \in \mathrm{IBr}(G)$ with $I(\varphi_N, \mu) \neq 0$. Then there exists a unique $\tau \in \mathrm{IBr}(T)$ such that $\tau^G = \varphi$ and $I(\tau_N, \mu) \neq 0$.

We need the following corollary of Theorem 3.1.

COROLLARY 5.2. *Let $N \lhd G$ with $p \nmid |G: N|$ and let $\psi \in \mathrm{Irr}(N)$ be p-rational and satisfy*

(a) $\psi^* \in \mathrm{IBr}(N)$ and

(b) $\psi^g = \psi$ for those $g \in G$ with $(\psi^*)^g = \psi^*$.

Then every irreducible constituent of ψ^G is p-rational.

Proof. Let χ be an irreducible constituent of ψ^G and let $\sigma \in \mathfrak{G}(G)$. Then $(\chi^\sigma)^* = \chi^*$ and $[\chi^\sigma, \psi^G] = [\chi^\sigma, (\psi^\sigma)^G] \neq 0$. By Theorem 3.1, $*$ is one-to-one on irreducible constituents of ψ^G. Thus $\chi = \chi^\sigma$ and the result follows.

DEFINITION 5.3. Suppose $\chi \in \mathrm{Irr}(G)$ satisfies

(a) $\chi^* \in \mathrm{IBr}(G)$ and

(b) $\chi^\alpha = \chi$ for those $\alpha \in \mathrm{Aut}(G)$ with $(\chi^*)^\alpha = \chi^*$.

Then χ is *automorphic*.

Note that if a subset $\mathscr{X} \subseteq \mathrm{Irr}(G)$ can be found such that $\mathscr{X}^\alpha = \mathscr{X}$ for all $\alpha \in \mathrm{Aut}(G)$ and $*$ defines a one-to-one correspondence from $\mathscr{X}$ onto $\mathrm{IBr}(G)$, then every $\chi \in \mathscr{X}$ is automorphic. In § 6 we shall prove that $\mathscr{X} = \mathscr{S}(G)$ has these properties if G is p-solvable with $p \neq 2$.

THEOREM 5.4. *Let G be p-solvable and let $\varphi \in \mathrm{IBr}(G)$. Then there exists p-rational automorphic $\chi \in \mathrm{Irr}(G)$ with $\chi^* = \varphi$. Furthermore, if $p \mid \varphi(1)$, then there exists N char G such that the number of distinct irreducible constituents of φ_N is divisible by p.*

Proof. Use induction on $|G|$.

Case 1. There exists N char G such that the number of irreducible constituents of φ_N is divisible by p.

Let μ be one of these constituents so that $p \mid |G: T|$ where $T = \{g \in G \mid \mu^g = \mu\}$. By Lemma 5.1, find $\tau \in \mathrm{IBr}(T)$ with $\tau^G = \varphi$ and $I(\tau_N, \mu) \neq 0$. Since $T < G$, choose a p-rational, automorphic $\psi \in \mathrm{Irr}(T)$ with $\psi^* = \tau$. Let $\chi = \psi^G$. Then $\chi^* = (\psi^G)^* = \tau^G = \varphi$ and hence $\chi \in \mathrm{Irr}(G)$. Certainly, χ is p-rational. If $\alpha \in \mathrm{Aut}(G)$ with $\varphi^\alpha = \varphi$, then $N^\alpha = N$ and μ^α is a constituent of φ_N so that $\mu^\alpha = \mu^g$ for some $g \in G$. Define $\beta \in \mathrm{Aut}(G)$ by $x^\beta = (x^\alpha)^{g^{-1}}$. Then $\varphi^\beta = \varphi$, $\mu^\beta = \mu$ and $T^\beta = T$. From the uniqueness of τ we conclude that $\tau^\beta = \tau$ and thus $\psi^\beta = \psi$ and $\chi^\beta = \chi$. Since $\chi^\beta = \chi^\alpha$, we have shown that χ is automorphic.

Case 2. No N as in Case 1 exists.

Suppose $M < G$ is characteristic and let μ be an irreducible constituent of φ_M. If N char M, then N char G. The number of distinct irreducible constituents of μ_N divides the number for φ_N and hence is prime to p. By the inductive hypothesis, $p \nmid \mu(1)$. Also, we may choose p-rational, automorphic $\theta \in \mathrm{Irr}(M)$ with $\theta^* = \mu$. Thus $\theta^g = \theta$ whenever $\mu^g = \mu$. Also, $p \nmid \theta(1)$ and the number of distinct conjugates of θ in G is prime to p.

If $O^p(G) < G$, take $M = O^p(G)$. By the last sentence of the preceding paragraph, θ is invariant in G. Since $O^p(M) = M$ and $o(\theta) = |M\colon \ker(\det(\theta))|$, we have $p \nmid o(\theta)$. By Theorem 4.2, θ has a unique extension, $\chi \in \mathrm{Irr}(G)$ with $p \nmid \chi(1)$. By Lemma 4.1, $\chi^* = \varphi$ and by Corollary 4.7, χ is p-fixed. Also $\varphi(1) = \chi(1) = \theta(1)$ and so $p \nmid \varphi(1)$.

If $\alpha \in \mathrm{Aut}(G)$ and $\varphi^\alpha = \varphi$ then $\mu^\alpha = \mu$ since $\mu = \varphi_M$. Therefore, $\theta^\alpha = \theta$ and by the uniqueness of χ and the fact that $o(\chi) = o(\chi^\alpha)$, we conclude that $\chi = \chi^\alpha$.

We may now suppose $O^p(G) = G$ so that $O^{p'}(G) < G$ and we take $M = O^{p'}(G)$. By Theorem 3.1, there exists a unique irreducible constituent, χ, of θ^G with $\chi^* = \varphi$. If $\alpha \in \mathrm{Aut}(G)$ with $\varphi^\alpha = \varphi$, then $M^\alpha = M$ and $\mu^\alpha = \mu^g$ for some $g \in G$. Define $\beta \in \mathrm{Aut}(G)$ by $x^\beta = (x^\alpha)^{g^{-1}}$ so that $\mu^\beta = \mu$ and $\theta^\beta = \theta$. Now χ^β is a constituent of θ^G and $(\chi^\beta)^* = \varphi^\beta = \varphi$. By the uniqueness of χ, $\chi^\beta = \chi$. Since $\chi^\beta = \chi^\alpha$, we have proved that χ is automorphic. As is well known, $(\chi(1)/\theta(1)) \mid |G\colon M|$. It follows that $\chi(1) = \varphi(1)$ is prime to p. Also, χ is p-rational by Corollary 5.2.

6. The main theorems.

LEMMA 6.1. *Let $N \lhd G$ and $\mu \in \mathrm{IBr}(N)$. Suppose μ is invariant in G and that $\mu^G \in \mathrm{IBr}(G)$. Then $N = G$.*

Proof. Let $\varphi = \mu^G$. By Theorem 2.2,

$$1 = I(\varphi, \varphi) = I(\mu^G, \varphi) = I(\mu, \varphi_N)\,.$$

Since μ is invariant in G, Clifford's theorem yields $\varphi_N = |G\colon N|\,\mu$ and thus

$$1 = I(\mu, \varphi_N) = |G\colon N|\,.$$

Proof of Theorem 1.2. Use induction on $|G|$. Let $\chi \in \mathrm{Irr}(G)$ be

p-rational and suppose $\chi^* \in \mathrm{IBr}(G)$. Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ with $[\chi_N, \theta] \neq 0$. We show by induction on $|N|$ that θ is p-rational and $\theta^* \in \mathrm{IBr}(N)$.

Let $T = \{g \in G \mid \theta^g = \theta\}$ and $\psi \in \mathrm{Irr}(T)$ with $\psi^G = \chi$ and $[\psi_N, \theta] \neq 0$. Let $S = \{g \in G \mid \theta^g = \theta^\sigma$ for some $\sigma \in \mathfrak{G}(G)\}$. Note that S is a subgroup since for $x \in G$, $\sigma \in \mathfrak{G}(G)$ we have $(\theta^\sigma)^x = (\theta^x)^\sigma$. Also, $S \supseteq T$ so that $\eta = \psi^S \in \mathrm{Irr}(S)$. Now $(\eta^*)^G = (\eta^G)^* = \chi^* \in \mathrm{IBr}(G)$, and we conclude that $\eta^* \in \mathrm{IBr}(S)$.

We claim that η is p-rational. Let $\sigma \in \mathfrak{G}(G)$. Then $\chi^\sigma = \chi$ and hence $\theta^\sigma = \theta^g$ for some $g \in G$. Necessarily, $g \in S$ and so θ and θ^σ are conjugate in S and hence $[(\eta^\sigma)_N, \theta] \neq 0$ and we can find $\psi_1 \in \mathrm{Irr}(T)$ with $(\psi_1)^S = \eta^\sigma$ and $[(\psi_1)_N, \theta] \neq 0$. Now $(\psi_1)^G = (\eta^\sigma)^G = \chi^\sigma = \chi$. It follows (Lemma 4.4) that $\psi_1 = \psi$ and hence $\eta = \psi^S = (\psi_1)^S = \eta^\sigma$ and η is p-rational as claimed. If $S < G$, then by the inductive hypothesis on $|G|$, we conclude that θ is p-rational and $\theta^* \in \mathrm{IBr}(N)$.

Suppose then, $S = G$. If $g \in G$, then T^g is the inertia group of $\theta^g = \theta^\sigma$ for some $\sigma \in \mathfrak{G}(G)$. If $\theta^x = \theta$, then $(\theta^\sigma)^x = (\theta^x)^\sigma = \theta^\sigma$ and we conclude that $T^g = T$ and hence $T \lhd G$. Also, $(\psi^g)^G = \chi = (\psi^\sigma)^G$ and $[(\psi^g)_N, \theta^\sigma] \neq 0$. It follows that $\psi^g = \psi^\sigma$ and thus $(\psi^*)^g = (\psi^\sigma)^* = \psi^*$. Since $(\psi^*)^G = \chi^* \in \mathrm{IBr}(G)$, Lemma 6.1 applies and we conclude that $T = G$.

We now have $\chi_N = e\theta$ and thus θ is p-rational. We may assume $N > 1$. Suppose $M < N$, $M \lhd G$. Let ξ be an irreducible constituent of θ_M. By the inductive hypothesis on $|N|$, we conclude that ξ is p-rational and $\xi^* \in \mathrm{IBr}(M)$. If $g \in G$ with $(\xi^*)^g = \xi^*$, then since ξ^g is p-rational, the (uniqueness) inductive hypothesis on $|G|$ yields $\xi = \xi^g$.

If $\mathbf{O}^p(N) < N$, take $M = \mathbf{O}^p(N)$. Then Theorem 4.5 applies to N. Since θ is the (unique) p-rational irreducible constituent of ξ^N, we conclude that $\theta^* \in \mathrm{IBr}(N)$. Otherwise, $\mathbf{O}^{p'}(N) < N$ and we may take $M = \mathbf{O}^{p'}(N)$. In this case, Theorem 3.1 yields $\theta^* \in \mathrm{IBr}(N)$.

Now suppose $\chi_0 \in \mathrm{Irr}(G)$ is p-rational and that $\chi^* = \chi_0^*$. Let M be a maximal normal subgroup of G and let μ be an irreducible constituent of $(\chi^*)_M$. Choose irreducible constituents, ψ and ψ_0 of χ_M and $(\chi_0)_M$ respectively, so that $I(\psi^*, \mu) \neq 0 \neq I((\psi_0)^*, \mu)$. By the first part of the proof we conclude that $\psi^* = \mu = (\psi_0)^*$ and that ψ and ψ_0 are p-rational. By the inductive hypothesis, $\psi = \psi_0$.

If $g \in G$ and $\mu^g = \mu$, then $\psi^g = \psi$, again by the inductive hypothesis. Now conclude that $\chi = \chi_0$ using Theorem 3.1 if $p \nmid |G: M|$ and Theorem 4.5 if $p = |G: M|$.

Given $\varphi \in \mathrm{IBr}(G)$, the existence of p-rational $\chi \in \mathrm{Irr}(G)$ with $\chi^* = \varphi$ follows from Theorem 5.4. Alternatively, in the present situation, it follows immediately by induction applied to a maximal normal subgroup of G, using Theorem 4.5 or Theorem 3.1 together with Corollary 5.2.

COROLLARY 6.2. *Let G be p-solvable with $p \neq 2$ and let $\chi \in \mathrm{Irr}(G)$ be p-rational with $\chi^* \in \mathrm{IBr}(G)$. Then χ is automorphic.*

Proof. Immediate from the uniqueness in Theorem 1.2.

Proof of Theorem 1.4. Repeated application of Theorem 1.2 shows that if $\chi^* \in \mathrm{IBr}(G)$ and χ is p-rational, then $\chi \in \mathscr{S}(G)$. Trivially, if $\chi \in \mathscr{S}(G)$, then χ is $\mathscr{M}$-p-rational for any collection, $\mathscr{M}$, of subgroups of G. Now suppose χ is $\mathscr{M}$-p-rational where $\mathscr{M}$ is some subnormal series for G with factors being p-groups and p'-groups. Let $M \in \mathscr{M}$ with $M \lneqq\!\lhd\, G$ and either $p \nmid |G: M|$ or G/M a p-group. Let θ be an irreducible constituent of χ_M so that θ is $\mathscr{M}_1$-p-rational where $\mathscr{M}_1 = \{H \in \mathscr{M} \mid H \subseteq M\}$ is an appropriate subnormal series of M.

Working by induction on $|G|$, we assume $\theta^* \in \mathrm{IBr}(M)$. By Corollary 6.2, θ is automorphic. Thus by Theorem 3.1 or Theorem 4.5, we conclude that $\chi^* \in \mathrm{IBr}(G)$.

Since (b) implies (a), $*$ defines a map of $\mathscr{S}(G)$ into $\mathrm{IBr}(G)$. This map is one-to-one by Theorem 1.2 and is onto by Theorem 1.2 together with the fact that (a) implies (b). This completes the proof.

7. Corollaries, further results and questions.

The most obvious deficiency in our work is the situation when $p = 2$. We ask

QUESTION 7.1. Let G be solvable and $p = 2$. Let $\varphi \in \mathrm{IBr}(G)$. Does there exist $\chi \in \mathrm{Irr}(G)$ such that for every $N \lhd G$ and every irreducible constituent, ψ, of χ_N, we have $\psi^* \in \mathrm{IBr}(N)$?

For $p \neq 2$, we have much more.

COROLLARY 7.2. *Let G be p-solvable with $p \neq 2$ and let $\varphi \in \mathrm{IBr}(G)$. Then there exists $\chi \in \mathrm{Irr}(G)$ such that for every $M \lhd\lhd G$, $*$ defines a one-to-one map from $\{\psi \in \mathrm{Irr}(M) \mid [\chi_M, \psi] \neq 0\}$ onto*

$$\{\mu \in \mathrm{IBr}(M) \mid I(\varphi_M, \mu) \neq 0\} .$$

Proof. Take $\chi \in \mathscr{S}(G)$ with $\chi^* = \varphi$.

COROLLARY 7.3. *Let G be p-solvable with $p \neq 2$. Let $N \lhd G$ and $\psi \in \mathscr{S}(N)$.*

(a) *If $p \nmid |G: N|$, then every irreducible constituent of ψ^G lies in $\mathscr{S}(G)$.*

(b) *If G/N is a p-group, then exactly one irreducible constituent of ψ^G is p-rational. It lies in $\mathscr{S}(G)$.*

(c) *In general, some irreducible constituent of ψ^G lies in $\mathscr{S}(G)$.*

Proof. (a) We have ψ is automorphic and Theorem 3.1 yields $\chi^* \in \mathrm{IBr}(G)$ for every irreducible constituent, χ, of ψ^G. By Corollary 5.2, χ is p-rational and hence $\chi \in \mathscr{S}(G)$ by Theorem 1.4.

(b) Since ψ is automorphic, Theorem 4.5 and Theorem 1.4 yield the result.

(c) This is immediate by alternate application of (a) and (b).

If ψ is p-rational but $\psi \notin \mathscr{S}(N)$, then for $p \nmid |G: N|$, it is possible that no irreducible constituent of ψ^G is p-rational. An example is in § 9.

If $H \subseteq G$ and $\psi \in \mathscr{S}(H)$, it does not necessarily follow that some irreducible constituent of ψ^G lies in $\mathscr{S}(G)$. An example for $p = 3$ is $G = \mathrm{SL}(2, 3)$, $H \subseteq G$, cyclic of order 6 and $\psi \in \mathrm{Irr}(H)$ with $\psi \neq 1_H = \psi^2$. We ask

QUESTION 7.4. Let G be p-solvable with $p \neq 2$ and $H \subseteq G$. Suppose $\psi \in \mathscr{S}(H)$ and $\psi^G \in \mathrm{Irr}(G)$. Is $\psi^G \in \mathscr{S}(G)$? Suppose $\chi \in \mathscr{S}(G)$ and $\chi_H \in \mathrm{Irr}(H)$. Is $\chi_H \in \mathscr{S}(H)$?

We shall prove some special cases.

THEOREM 7.5. *Let G be p-solvable with $p \neq 2$. Let $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ and $T = \{g \in G \mid \theta^g = \theta\}$. Let $\psi \in \mathrm{Irr}(T)$ with $[\psi_N, \theta] \neq 0$. Let $\chi = \psi^G$. Then $\psi \in \mathscr{S}(T)$ iff $\chi \in \mathscr{S}(G)$.*

Proof. We may assume $N < G$ and choose maximal $M \lhd G$, $M \supseteq N$. Choose an irreducible constituent ξ, of χ_M with $[\xi_N, \theta] \neq 0$. Let $S = M \cap T$ and by Lemma 4.4, choose $\eta \in \mathrm{Irr}(S)$ with $\eta^M = \xi$. Working by induction on $|G|$, we have $\eta \in \mathscr{S}(S)$ iff $\xi \in \mathscr{S}(M)$. Since $[\eta^G, \chi] \neq 0$, it follows from Lemma 4.4 that $[\eta^T, \psi] \neq 0$. Note that $S \lhd T$.

If $p \nmid |G: M|$, then $p \nmid |T: S|$ and it follows from Corollary 7.3(a) that $\xi \in \mathscr{S}(M)$ iff $\chi \in \mathscr{S}(G)$ and $\eta \in \mathscr{S}(S)$ iff $\psi \in \mathscr{S}(T)$. We are done in this case.

Suppose then, $|G: M| = p$. If $\psi \in \mathscr{S}(T)$ then $\eta \in \mathscr{S}(S)$ and $\xi \in \mathscr{S}(M)$. Also $\chi = \psi^G$ is p-rational and thus $\chi \in \mathscr{S}(G)$ by Corollary 7.3(b). Conversely, if $\chi \in \mathscr{S}(G)$, then $\xi \in \mathscr{S}(M)$ and $\eta \in \mathscr{S}(S)$. Also $\theta \in \mathscr{S}(N)$ and hence by the remark following Lemma 4.4, ψ is p-rational Now $\psi \in \mathscr{S}(T)$ by 7.3(b).

THEOREM 7.6. *Let $L \lhd K \lhd G$ and $H \subseteq G$ with $HK = G$, $H \cap K = L$. Let $\theta \in \mathrm{Irr}(K)$ be invariant in G and suppose $\theta_L \in \mathrm{Irr}(L)$. Let $\chi \in \mathrm{Irr}(G)$ with $[\chi_K, \theta] \neq 0$. Then $\chi_H = \psi \in \mathrm{Irr}(H)$. Also, if G is p-solvable with $p \neq 2$ and $\chi \in \mathscr{S}(G)$, then $\psi \in \mathscr{S}(H)$.*

Proof. That $\psi \in \mathrm{Irr}(H)$ is part of Lemma 10.5 of [7]. Suppose G is p-solvable, $p \neq 2$ and $\chi \in \mathscr{S}(G)$. If $K = G$ then $H = L \lhd G$ and $\psi \in \mathscr{S}(H)$. Assume then, that $K < G$ and choose maximal $M \lhd G$ with $M \supseteq K$. Let $U = H \cap M$ so that $KU = M$ and $K \cap U = L$. Let ξ be an irreducible constituent of χ_M with $[\xi_K, \theta] \neq 0$. Then $\xi \in \mathscr{S}(M)$ and working by induction on $|G|$, we may assume $\xi_U = \eta \in \mathscr{S}(U)$.

Now $U \lhd H$ and ψ is a p-rational irreducible constituent of η^H. Since either $p \nmid |H: U|$ or $p = |H: U|$, Corollary 7.3 yields $\psi \in \mathscr{S}(H)$.

Sometimes, every p-rational character lies in $\mathscr{S}(G)$. The following gives a sufficient condition for this.

THEOREM 7.7. *Let G be p-solvable with $p \neq 2$, and suppose that the Frobenius group of order $p(p-1)$ is not involved in G. Then every p-rational $\chi \in \mathrm{Irr}(G)$ lies in $\mathscr{S}(G)$.*

Proof. Let $|G| = m$ and $\varepsilon = e^{2\pi i/m}$. Let $\mathfrak{G}(G) = \langle \sigma \rangle$ so that $\varepsilon^\sigma = \varepsilon^k$ for some integer k, with $(m, k) = 1$. It follows from a standard argument, using a counting lemma of Brauer, that the number of p-rational $\chi \in \mathrm{Irr}(G)$ is equal to the number of conjugacy classes, $\mathscr{K}$, such that $x^k \in \mathscr{K}$ whenever $x \in \mathscr{K}$. We claim that these can only be the p-regular classes.

Suppose x is p-singular and $y \in G$ with $x^y = x^k$. Let $u \in \langle x \rangle$ with $o(u) = p$ so that $u^y = u^k$. Since $\langle \sigma \rangle$ permutes the primitive pth roots of unity transitively, we conclude that k is a multiplicative generator of the group of nonzero integers mod p. It follows that all elements of $\langle u \rangle$ are conjugate in $\langle u, y \rangle = H$ and thus $H/C_{\langle y \rangle}(u)$ is a Frobenius group of order $p(p-1)$. This contradiction shows that the number of p-rational $\chi \in \mathrm{Irr}(G)$ is not more than the number of p-regular classes. Now $|\mathscr{S}(G)| = |\mathrm{IBr}(G)|$ is equal to the number of p-regular classes. The result follows.

Combining the information in Theorem 5.4 with our main results, we obtain the following.

COROLLARY 7.8. *Let G be p-solvable with $p \neq 2$ and let $\chi \in \mathscr{S}(G)$. Then there exists $U \subseteq G$ and $\eta \in \mathscr{S}(U)$ such that $\chi = \eta^G$ and $p \nmid \eta(1)$.*

Proof. We may suppose $p \mid \chi(1)$. Then $\chi^* \in \mathrm{IBr}(G)$ and $p \mid \chi^*(1)$ and by Theorem 5.4, there exists $N \lhd G$ such that $(\chi^*)_N$ is not homogeneous. Let θ be an irreducible constituent of χ_N. Since $\theta^* \in \mathrm{IBr}(N)$, it follows that θ is not invariant in G. Let $T = \{g \in G \mid \theta^g = \theta\} < G$ and choose $\psi \in \mathrm{Irr}(T)$ with $\psi^G = \chi$ and $[\psi_N, \theta] \neq 0$.

By Theorem 7.5, $\psi \in \mathscr{S}(T)$. Since $T < G$, induction on $|G|$ yields the result.

We close this section with the observation that if G is p-solvable with $p \neq 2$, then the subnormally p-rational characters can be located in the character table of G. Certainly, the p-rational characters can be found. Also, the p-regular classes can be identified (see Theorem 2.5 in [8]) and thus the functions χ^* can be constructed for $\chi \in \mathrm{Irr}(G)$. We have

COROLLARY 7.9. *Let G be p-solvable with $p \neq 2$. Let $\chi \in \mathrm{Irr}(G)$ be p-rational. Then $\chi \in \mathscr{S}(G)$ iff χ^* is not of the form $\sum a_\psi \psi^*$ for p-rational $\psi \in \mathrm{Irr}(G)$ with $\psi(1) < \chi(1)$ and $a_\psi \geqq 0$, integers.*

8. Solvable groups. In this section we use some of the deeper properties of solvable groups to obtain a partial answer to Question 7.4.

THEOREM 8.1. *Let G be solvable and suppose $p \neq 2$. Let $H \subseteqq G$, $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$. Suppose $2 \nmid \chi(1)$.*
(a) *If $\psi \in \mathscr{S}(H)$ and $\chi = \psi^G$, then $\chi \in \mathscr{S}(G)$.*
(b) *If $\chi \in \mathscr{S}(G)$ and $\psi = \chi_H$, then $\psi \in \mathscr{S}(H)$.*

To prove this theorem we will use the main result (Theorem 9.1 and Corollary 9.2) of [7]. We state part of it here.

THEOREM 8.2. *Let $L \subseteqq K \lhd G$ with $L \lhd G$ and K/L abelian of odd order. Let $\theta \in \mathrm{Irr}(K)$ be invariant in G and suppose $\theta_L = e\varphi$ with $\varphi \in \mathrm{Irr}(L)$ and $e^2 = |K: L|$. Then there exists $U \subseteqq G$ and a character, β, of G such that*
(a) *$UK = G$, $U \cap K = L$;*
(b) *$|\beta(g)|^2 = |C_{K/L}(g)|$ for all $g \in G$;*
(c) *the equation $\chi_U = \beta_U \xi$ defines a one-to-one correspondence between $\{\chi \in \mathrm{Irr}(G) \mid [\chi_K, \theta] \neq 0\}$ and $\{\xi \in \mathrm{Irr}(U) \mid [\xi_L, \varphi] \neq 0\}$ and*
(d) *if $\chi_U = \beta_U \xi$ as in (c), then $\xi^G = \bar{\beta}\chi$.*

THEOREM 8.3. *Let G be solvable and let H be a maximal subgroup. Suppose $2 \nmid |G: H|$. Let $L = \mathrm{core}_G(H)$ and let K/L be a chief factor of G. Suppose $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$ and let φ be an irreducible constituent of ψ_L.*
(a) *If $\chi = \psi^G$ then φ is not invariant G.*
(b) *If $\chi_H = \psi$ and φ is invariant in G, then φ is extendible to K.*

Proof. Since G is solvable, K/L is an abelian q-group. We have $G = KH$ and $L = K \cap H$. Since $2 \nmid |G:H|$, we have $q \neq 2$. Now $C_H(K/L) \lhd G$ and so $C_H(K/L) = L$. We may assume $H \ntriangleleft G$ and thus $H > L$. Let R/L be a chief factor of H. Let $C = [K, R]L$. Since $R \lhd H$, it follows that $C \lhd G$. Since $R \nsubseteq L = C_H(K/L)$, we have $C > L$ and thus $C = K$. It follows that if $\lambda \in \mathrm{Irr}(K/L)$ is fixed by R, then $\lambda = 1_K$. Also R/L is an r-group with $r \neq q$.

Suppose $U \subseteq G$ with $UK = G$ and $U \cap K = L$. Then $(U \cap RK)/L \in \mathrm{Syl}_r(RK/L)$. If we replace H by a conjugate, we may assume that $U \cap RK = R$ and thus $H = N_G(R) = U$.

Now assume φ is invariant in G. By the "going up theorem" (Proposition 3 (2) of [6]) there are two possibilities: (i) $\varphi^K = e\theta$ with $e^2 = |K:L|$ and $\theta \in \mathrm{Irr}(K)$ or (ii) φ is extendible to K.

Suppose (i) occurs. Then $\theta_L = e\varphi$ and Theorem 8.2 applies. We may assume that the subgroup, U, in the conclusion of that theorem is H. We have $\chi_H = \beta_H \xi$ for some $\xi \in \mathrm{Irr}(H)$ with $[\xi_L, \varphi] \neq 0$. Also, $\psi^G = \bar{\beta}\eta$ for some $\eta \in \mathrm{Irr}(G)$ with $[\eta_K, \theta] \neq 0$. If $\psi = \chi_H$, then $\psi = \beta_H \xi$ and $\beta_H \in \mathrm{Irr}(H)$. If $\chi = \psi^G$, then $\chi = \bar{\beta}\eta$ and $\bar{\beta} \in \mathrm{Irr}(G)$. In either case $\beta \in \mathrm{Irr}(G)$ and $[\beta\bar{\beta}, 1_G] = 1$.

Conclusion (b) of Theorem 8.2 asserts that $\beta\bar{\beta}$ is the permutation character of G acting on all of the elements of K/L. This action cannot be transitive since the identity of K/L is a fixed point. This contradicts $[\beta\bar{\beta}, 1_G] = 1$.

We conclude that (ii) occurs and φ is extendible to K. This completes the proof of (b). We assume that $\chi = \psi^G$ and obtain a contradiction to prove (a).

We have $\chi_K = (\psi^G)_K = (\psi_L)^K$ which is a multiple of φ^K. It follows that the $|K:L|$ distinct extensions of φ to K are transitively permuted by G and hence by H/L. Since $R \lhd H$ and $r \neq q$, it follows that R fixes all of the extensions of φ to K.

Let $\theta_1 \neq \theta_2$ be two extensions of φ. Then $\theta_2 = \lambda\theta_1$ for some unique $\lambda \in \mathrm{Irr}(K/L)$ with $\lambda \neq 1_K$. It follows that R fixes λ and this is the desired contradiction.

Before proceeding with the proof of (8.1), we observe a consequence of Theorem 8.3(a).

COROLLARY 8.4. *Let G be solvable and suppose $\chi \in \mathrm{Irr}(G)$ is quasiprimitive with $2 \nmid \chi(1)$. Then χ is primitive.*

This result was recently proved by T. R. Berger without assuming $2 \nmid \chi(1)$.

Proof of Theorem 8.1. We may assume that H is a maximal subgroup. Let $L = \mathrm{core}_G(H)$ and let φ be an irreducible constituent of ψ_L. In either situation (a) or (b) we have $[\chi_L, \varphi] \neq 0$. Let $T =$

$\{g \in G \mid \varphi^g = \varphi\}$ and $S = H \cap T$. Let $\xi \in \mathrm{Irr}(T)$ with $\xi^G = \chi$ and $[\xi_L, \varphi] \neq 0$ and $\eta \in \mathrm{Irr}(S)$ with $\eta^H = \psi$ and $[\eta_L, \varphi] \neq 0$.

Now assume $\psi^G = \chi$ and $\psi \in \mathscr{S}(H)$. Since $2 \nmid \chi(1)$, we have $2 \nmid |G: H|$ and hence $T < G$ by Theorem 8.3(a). Also $\eta^G = \chi$ and it follows from Lemma 4.4, that $\eta^T = \xi$. Further, $2 \nmid \xi(1)$. By Theorem 7.5, $\eta \in \mathscr{S}(S)$. Working by induction on $|G|$ and using $T < G$, we have $\xi = \eta^T \in \mathscr{S}(T)$. Another application of Theorem 7.5 yields $\chi \in \mathscr{S}(G)$, as desired.

Now suppose $\chi_H = \psi$ and $\chi \in \mathscr{S}(G)$. By Mackey's theorem, $(\xi_S)^H$ is a constituent of $(\xi^G)_H = \psi$. It follows that $(\xi_S)^H = \psi$ and thus by Lemma 4.4, $\xi_S = \eta$. Now $\xi \in \mathscr{S}(T)$ and if $T < G$, the inductive hypothesis yields $\eta \in \mathscr{S}(S)$ since $2 \nmid \xi(1)$. Therefore $\psi \in \mathscr{S}(H)$ as desired.

We now consider the remaining case, where φ is invariant in G. Let K/L be a chief factor of G so that K/L is an abelian q-group, $KH = G$ and $K \cap H = L$. Let θ be an irreducible constituent of χ_K so that $[\theta_L, \varphi] \neq 0$. We claim that $\theta_L = \varphi$. If $q \neq 2$, this follows from Theorem 8.3 (b). If $q = 2$, then $\theta(1)/\varphi(1)$ is a power of 2. Since $\theta(1) \mid \chi(1)$, it follows that $2 \nmid \theta(1)$ and hence $\theta(1) = \varphi(1)$. If θ is invariant in G, the result follows from Theorem 7.6.

Assume then, that $W = \{g \in G \mid \theta^g = \theta\} < G$ and choose $\alpha \in \mathrm{Irr}(W)$ with $\alpha^G = \chi$ and $[\alpha_K, \theta] \neq 0$ so that $\alpha \in \mathscr{S}(W)$. Let $V = W \cap H$ and $\beta = \alpha_V$ so that $\beta^H = (\alpha^G)_H = \psi$. Thus $\beta \in \mathrm{Irr}(V)$ and by the inductive hypothesis, $\beta \in \mathscr{S}(V)$. Now $\psi \in \mathscr{S}(H)$ by part (a) and the proof is complete.

9. Example.

THEOREM 9.1. *Let $p \neq 2$. The following situation can occur*:

(a) *G is p-solvable, $N \lhd G$, $p \nmid |G: N|$*,

(b) $\chi \in \mathrm{Irr}(G)$, $\chi^* \in \mathrm{IBr}(G)$,

(c) $\chi_N = \theta \in \mathrm{Irr}(N)$, $\theta^* \notin \mathrm{IBr}(N)$,

(d) *θ is p-rational and*

(e) *no irreducible constituent of θ^G is p-rational.*

Proof. Let A be abelian of order $p(p-1)^2$ with $A = \langle x, u, v\rangle$, $x^p = u^{p-1} = v^{p-1} = 1$. Let $\alpha \in \mathrm{Aut}(\langle x\rangle)$ of order $p-1$ and define $\sigma \in \mathrm{Aut}(A)$ by $x^\sigma = x^\alpha$, $u^\sigma = uv$ and $v^\sigma = v$ so that $o(\sigma) = p-1$. Let $G = A \rtimes \langle\sigma\rangle$, the semi-direct product.

Define $\lambda \in \mathrm{Irr}(A)$ by $\lambda(x) = \varepsilon = e^{2\pi i/p}$, $\lambda(u) = 1$ and $\lambda(v) = \delta = e^{2\pi i/(p-1)}$. Let $\chi = \lambda^G$. Note that λ has $p-1$ distinct G-conjugates and these take on distinct values at x and at u. It follows that $\chi_H \in \mathrm{Irr}(H)$ where $H = \langle u, v, \sigma\rangle$ and also $\chi_N = \theta \in \mathrm{Irr}(N)$ where $N = \langle x, v, \sigma\rangle$. Since H is a p'-group, we have $\chi^* \in \mathrm{IBr}(G)$. Since

$G' = \langle x, v\rangle \subseteq N$, we have $N \lhd G$.

Now $N' = \langle x\rangle = O_p(N) \subseteq \ker \mu$ for any $\mu \in \mathrm{IBr}(N)$. It follows that $\mu(1) = 1$ and thus $\theta^* \notin \mathrm{IBr}(N)$. Parts (a), (b) and (c) have now been proved.

Note that $\theta = (\lambda_{\langle v,x\rangle})^N$ and thus θ vanishes on $N - \langle v, x\rangle$. We compute $\theta(v^a x^b) = \delta^a \sum_{i=1}^{p-1} \varepsilon^i = -\delta^a \in \boldsymbol{Q}[\delta]$. It follows that θ is p-rational.

Let ψ be an irreducible constituent of θ^G. Then $\psi = \gamma\chi$ for some $\gamma \in \mathrm{Irr}(G/N)$ and thus $\psi_H = \gamma_H \chi_H$ is irreducible and hence $\psi^* \in \mathrm{IBr}(G)$. Since $\theta^* \notin \mathrm{IBr}(N)$, it follows from Theorem 1.2 that ψ is not p-rational. (Or compute $\psi(xu)$ and observe that it does not lie in $\boldsymbol{Q}[\delta]$.) This completes the proof.

In the notation of Theorom 9.1, if χ were p-rational, then (b) and (c) could not both hold. If $\theta \in \mathscr{S}(G)$ then (e) could not hold.

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