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THE EXTENDED CENTRALIZER OF AN S-SET

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Let S be a semigroup with zero. The extended centralizer $Q(M_S)$ of a right S-set M_S is defined. Necessary and sufficient conditions are given for $Q(M_S)$ to be a regular semigroup. In particular, $Q(S_S)$ is shown to be a regular semigroup when S is regular. We also show that whenever the singular congruence on S is the identity, then $Q(S_S)$ is the injective hull of S_S and is right self injective.

1. Introduction. In [3], R. E. Johnson developed the extended centralizer $Q(M_R)$ of an R-module M and noted that $Q(M_R)$ is always a (Von Neumann) regular ring. In this paper, we analogously define the extended centralizer $Q(M_S)$ of a right S-set M_S. McMorris [4] gave an example which illustrated the fact that $Q(S_S)$ is not always a regular semigroup. We give a necessary and sufficient condition for $Q(M_S)$ to be regular and show that when S is regular, $Q(S_S)$ is also regular.

Johnson showed that the ring R is embedded in $Q(R_R)$ when the singular ideal is zero. Analogously we define the singular congruence on an S-set and show that when the singular congruence is the identity, S is embedded in $Q(S_S)$. In this case we also note that $Q(S_S)$ is the injective hull of S considered as a S-set and that, moreover, $Q(S_S)$ is self injective.

2. Preliminaries. Throughout this paper each semigroup will contain a zero (0) unless otherwise specified. Let S be a semigroup. A (centered right) *S-set* M_S is a set M, with an associative scalar operation on M by elements of S, which contains an element (necessarily unique) θ such that $\theta = \theta s = m0$ for all $m \in M$ and for all $s \in S$. The symbol θ will be called the zero of M. Since the distinction between the zero of M and the zero of S is clear from the context, we shall denote both by the same symbol 0. Note that if R is a right ideal of S then R becomes an S-set R_S under ordinary multiplication. A *sub S-set* N_S of an S-set M_S is a subset N of M such that $NS \subseteqq N$. If $m, n \in M_S$ and if $E \subseteqq S$ we shall say that mE is *pointwise equal* to nE when $ms = ns$ for each $s \in E$. This will be denoted as $mE \doteq nE$.

Let M_S and N_S be S-sets. A function $f\colon M_S \rightarrow N_S$ is an *S-homomorphism* if for each $m \in M$ and $s \in S$, $f(ms) = f(m)s$. The collection of all such S-homomorphisms will be denoted by $\mathrm{Hom}_S(M, N)$. If there exists $f \in \mathrm{Hom}_S(M, N)$ which is $1-1$ and onto, we say M_S

is S-isomorphic to N_S and write $M_S \underset{S}{\approx} N_S$.

If f is an S-homomorphism the domain of f will be denoted by D_f and the range of f by R_f. The zero map from M_S will be denoted by 0_M and the identity map on M by 1_M. If $f: M_S \to N_S$ and if $A_S \subseteq N_S$ then $f^{-1}(A) = \{m \in M: f(m) \in A\}$.

An *S-congruence* τ on M_S is an equivalence relation on M such that whenever $(m, n) \in \tau$, then $(ms, ns) \in \tau$ for all $s \in S$. The identity S-congruence on M_S will be denoted by ι_M.

If S has an identity 1 the S-set M_S is said to be *unital* when $m1 = m$ for each $m \in M$. For each semigroup S we shall define S^1 by $S^1 = S \cup \{1\}$ where 1 is a symbol not in S and where multiplication on S is extended to S^1 by defining $1x = x1 = x$ for each $x \in S^1$. With the operation so defined, S^1 is a semigroup. Note that this definition for S^1 differs from the standard one. However, with the definition given here each S-set M_S becomes a unital S^1-set by defining $m1 = m$ for each $m \in M$.

The following definitions and theorem are due to Berthiaume [1]. A sub S-set N_S of M_S is said to be *large* (*essential*) in M_S if for each $f \in \mathrm{Hom}_S(M, K)$ such that $f|N$ is $1-1$ then f is $1-1$. In this case M_S is called an *essential extension* of N_S. The following lemma characterizes large sub S-sets in terms of S-congruences.

LEMMA 2.1. *N_S is large in M_S iff for every S-congruence ρ on M_S such that $\rho \neq \iota_M$ we have $\rho|N \neq \iota_N$.*

An S-set M_S is *injective* if for each $A_S \subseteq B_S$ and for each $f \in \mathrm{Hom}_S(A, M)$ there exists $f' \in \mathrm{Hom}_S(B, M)$ such that $f'|A = f$. If $M_S \subseteq N_S$ and if N_S is injective then N_S is called an *injective extension* of M_S. The following theorem due to Berthiaume [1] guarantees the existence of a minimal injective extension which is unique up to S-isomorphism.

THEOREM 2.2. *The S-set M_S is a maximal essential extension of N_S iff M_S is a minimal injective extension of N_S. Every S-set N_S has such an extension which is unique up to S-isomorphism over N_S.*

The minimal injective extension of N_S given in the above theorem is called the *injective hull* of N_S. Note that M_S is the injective hull of N_S iff N_S is essential in M_S and M_S is injective.

A semigroup S will be called *self injective* if S_S is injective.

The S-set M_S is *weakly injective* if for each right ideal R of S and for each $f \in \mathrm{Hom}_S(R, M)$ there exists $m \in M$ such that $f(s) = ms$ for each $s \in R$. In ring theory it is well-known that the corresponding concepts of "injective" and "weakly injective" are equivalent.

However, for semigroups Berthiaume proved the following lemma and gave a counterexample for the converse.

LEMMA 2.3. *If the S-set M_S is injective then M_S is weakly injective.*

3. The singular congruence on an S-set. The following definition is a generalization of a corresponding concept in ring theory. A sub S-set N_S of M_S is *intersection large* in M_S if for each $0 \neq m \in M$ there exists $s \in S^1$ such that $0 \neq ms \in N$. Note that N_S is intersection large in M_S if and only if the intersection of N with any nonzero sub S-set of M_S is always nonzero. Properties of intersection large S-sets are given by the following lemmas which are immediate from the definition.

LEMMA 3.1. *If $X_S \subseteqq Y_S \subseteqq Z_S$ are S-sets then X_S is intersection large in Z_S if and only if X_S is intersection large in Y_S and Y_S is intersection large in Z_S.*

LEMMA 3.2. *Let M_S and N_S be S-sets and let $\phi \in \mathrm{Hom}_S(M, N)$. If A_S is intersection large in N_S then $\phi^{-1}(A)$ is intersection large in M_S.*

Note that if N_S is intersection large in M_S then $m^{-1}N = \{s \in S: ms \in N\}$ is intersection large in S_S for all $m \in M$. In order to show this, define $\phi_m: S \rightarrow M$ by $\phi_m(s) = ms$. Then $\phi_m \in \mathrm{Hom}_S(S, M)$ and $\phi_m^{-1}(N) = m^{-1}N$ is intersection large in S_S by the lemma.

The class of all intersection large sub S-sets of the S-set M_S will be denoted by $\mathscr{P}(M_S)$. This class is closed under finite intersections since $A \cap B = 1_A^{-1}(B)$ where $A, B \in \mathscr{P}(M_S)$.

Let $\mathscr{P} = \mathscr{P}(S_S)$ and for each S-set define

$$\psi = \psi(M_S) = \{(m_1, m_2) \in M \times M: m_1D \mp m_2D \text{ for some } D \in \mathscr{P}\} .$$

It is easily seen from the properties noted above that ψ is an S-congruence on M_S which is a two-sided congruence if $M = S$. The S-congruence ψ is called the *singular congruence* or *$\mathscr{P}$-torsion congruence* on M_S. When $\psi = \iota_M$ we say that M_S is *$\mathscr{P}$-torsion free.*

Feller and Gantos [2] showed that every large sub S-set of an S-set M_S is intersection large in M_S. The converse is not generally true. For, consider the semilattice $S = \{0, e, 1\}$ which has $0 < e < 1$ under the natural partial ordering. The right ideal eS is clearly intersection large in S. Define $f: S \rightarrow S$ by $f(x) = \begin{Bmatrix} e \text{ if } x \in \{e, 1\} \\ 0 \text{ if } x = 0 \end{Bmatrix}$. Then $f \in \mathrm{Hom}_S(S, S)$ and $f | eS$ is $1-1$. However, f is not $1-1$. Therefore, eS is not large in S_S.

The following proposition gives a sufficient condition for the converse to be true.

PROPOSITION 3.3. *Let M_S be a right S-set such that M_S is $\mathscr{P}$-torsion free. Then $\mathscr{P}(M_S)$ is the set of large sub S-sets of M_S.*

Proof. Let $A_S \in \mathscr{P}(M_S)$ and let $f \in \mathrm{Hom}_S(M, B)$ such that $f|A$ is $1-1$ where B_S is an S-set. Suppose $f(x_1) = f(x_2)$. Let $D = x_1^{-1}A \cap x_2^{-1}A = \{s \in S: x_1 s \in A \text{ and } x_2 s \in A\}$. Then $D \in \mathscr{P}(S)$ and since $f(x_1) = f(x_2)$, we have $f(x_1 s) = f(x_2 s)$ for all $s \in D$. However, $x_1 s$, $x_2 s \in A$ for all $s \in D$ and $f|A$ is $1-1$. Thus $x_1 D \doteqdot x_2 D$ and it follows that $x_1 = x_2$ since M_S is $\mathscr{P}$-torsion free.

It was noted in §2 that an injective S-set M_S is always weakly injective but that a weakly injective S-set is not necessarily injective. In the following proposition we show that the two concepts are equivalent whenever M_S is $\mathscr{P}$-torsion free.

PROPOSITION 3.4. *Let M_S be a weakly injective S-set such that M_S is $\mathscr{P}$-torsion free. Then M_S is injective.*

Proof. Let $A_S \subseteqq B_S$ and let $f \in \mathrm{Hom}_S(A, M)$. Let M^* be the injective hull of M_S. Then M_S is large in M_S^* and hence is intersection large in M_S^*. Also, by Lemma 2.1 we see that M_S^* is $\mathscr{P}_I$-torsion free since $\psi(M_S^*)|M_S = \psi(M_S) = \iota_M$. Thus, since M_S^* is injective, there exists $f' \in \mathrm{Hom}_S(B, M^*)$ such that $f'|A = f$. We claim that $f' \in \mathrm{Hom}_S(B, M)$. Let $b \in B$ and let $f'(b) = n$. By the note following Lemma 3.2 we have $D = n^{-1}M \in \mathscr{P}(S)$. Define $\phi: n^{-1}M \to M$ by $\phi(s) = ns$. Thus we have $\phi \in \mathrm{Hom}_S(n^{-1}M, M)$ and since M_S is weakly injective there exists $m \in M$ such that $\phi(s) = ms$ for each $s \in n^{-1}M$. Therefore, $mD \doteqdot nD$ and since $\psi_I(M_S^*) = \iota_{M^*}$ it follows that $n = m \in M$.

4. The extended centralizer of an *S*-set. The construction of the extended centralizer Q of an S-set M_S is similar to that given by Johnson [3] for rings over modules and is outlined as follows:

Let $\mathscr{P} = \mathscr{P}(M_S)$ be the class of intersection large sub S-sets of the S-set M_S. Let $F = \bigcup_{D \in \mathscr{P}} \mathrm{Hom}_S(D, M)$ and define multiplication on S by $fg = h$ where $h: D_g \cap g^{-1}(D_f) \to M$ by $h(x) = f(g(x))$. Then under this multiplication F is a semigroup. Define a binary relation ω on the semigroup F by $(f, g) \in \omega$ if there exists $D \in \mathscr{P}$ such that $f|D = g|D$. Then ω is a two-sided congruence on F. The semigroup $Q = Q(M_S) = F/\omega$ is called the *extended centralizer of M_S*. The elements of Q will be denoted by $\bar{f}$ where $f \in F$.

In ring theory the extended centralizer is always (von Neumann) regular. An example given by McMorris in [4] shows that this is not the case for semigroups. We can however give a necessary and sufficient condition for Q to be regular in terms of splitting S-homomorphisms, which were studied by Feller and Gantos in [2]. Recall that an S-homomorphism f which maps an S-set M_S onto an S-set N_S is said to *split* if there exists $g \in \mathrm{Hom}_S(N, M)$ such that $fg = 1_N$.

THEOREM 4.1. *The semigroup $Q(M_S)$ is regular if and only if each equivalence class $\bar{f}$ of $Q(M_S) = F/\omega$ contains an element which splits.*

Proof. Assume first that $Q(M_S) = Q$ is regular and let $\bar{f} \in Q$. Then there exists $\bar{g} \in Q$ such that $\bar{f}\bar{g}\bar{f} = \bar{f}$. Hence if $E = \{x \in D_{fgf}: fgf(x) = f(x)\}$ then $E \in \mathscr{P}$. Let $f' = f \mid E$ and $g' = g \mid R_{f'}$. Let $y \in D_{g'}$ and let $x' = g(y)$. Since y is also an element of $R_{f'}$, there exists $x \in E$ such that $f(x) = y$ and we see that $y = f(x) = fgf(x) = fg(y) = f(x')$. Furthermore, $f(x') = f(x) = fgf(x) = fgf(x')$ and it follows that $x' \in E$. Therefore, $y = fg(y) = f(x') = f'(x') = f'g'(y)$ and we see that f' splits.

Conversely, for $\bar{f} \in Q$ there exists $f' \in \bar{f}$ such that f' splits. Hence, there exists $g': R_{f'} \to D_{f'}$ such that $f'g' = \iota_{R_{f'}}$. By Zorn's lemma there is a maximal sub S-set N_S of M_S such that $D_{g'} \cap N = 0$. It easily follows that $D = D_{g'} \cup N \in \mathscr{P}$ in this case. The S-homomorphism g' can be extended to an S-homomorphism $g \in \mathrm{Hom}_S(D, M)$ by defining $g(x) = 0$ if $x \in N$ and $g(x) = g'(x)$ if $x \in D_{g'}$. Hence we have $g \in F$. Let $x \in D_{f'}$. Then $f'gf'(x) = f'g'f'(x) = 1_{R_{f'}}f'(x) = f'(x)$. Therefore, $\bar{f}\bar{g}\bar{f} = \bar{f}'\bar{g}\bar{f}' = \bar{f}' = \bar{f}$ and it follows that Q is regular.

In the case where $M = S$ we have the following theorem.

THEOREM 4.2. *If S is a regular semigroup then $Q(S)$ is regular.*

Proof. By the previous theorem it is sufficient to show that each equivalence class $\bar{f}$ of $Q(S_S)$ contains an element which splits. Let $\bar{f} \in Q$ and let $\mathscr{F} = \{(D_\alpha, g_\alpha): D_\alpha$ is a right ideal of S in D_f such that $f_\alpha = f \mid D_\alpha$ splits on D_α and $g_\alpha: R_\alpha = f_\alpha(D_\alpha) \to D_\alpha$ such that $f_\alpha g_\alpha = 1_{R_\alpha}$. The set $\mathscr{F}$ is nonempty since $(\{0\}, 0) \in \mathscr{F}$ where the zero in the second coordinate is the zero map. Define a partial order $\leqq$ on $\mathscr{F}$ by $(D_\alpha, g_\alpha) \leqq (D_\beta, g_\beta)$ iff $D_\alpha \subseteqq D_\beta$ and $g_\beta \mid R_\alpha = g_\alpha$. By an application of Zorn's lemma, $\mathscr{F}$ contains a maximal element (D_M, g_M). To complete the proof it is sufficient to show that $D_M \in \mathscr{P}(S_S)$. Suppose this is not true. Then D_M is not intersection large in D_f. Hence there exists $e \in D_f$ such that $eS^1 \neq 0$ and $eS^1 \cap D_M = 0$. Since S is regular, we may assume that $e^2 = e$. Let $x = f(e)$ then $xe = f(e)e = f(e^2) = f(e) = x$. We now consider two cases.

Case 1. Suppose $xeS \cap R_M \neq 0$. Then there exists $s \in S$ such that $0 \neq xes \in R_M$. Consider $esS \subseteqq eS$. Let $D' = D_M \cup esS$ and let $f' = f \mid D'$. Then $f'(D') = R_M$. If $y \in R_M$ then $f'g_M(y) = f_M g_M(y) = y$. Hence $(D', g_M) \in \mathscr{F}$ and $(D_M, g_M) < (D', g_M)$ which contradicts the maximality of (D_M, g_M).

Case 2. Suppose $xeS \cap R_M = 0$. Let x' be an inverse of x and define $g': R' = R_M \cup xeS \rightarrow D' = D_M \cup eS$ by $g'(y) = \begin{Bmatrix} g_M(y) \text{ if } y \in R_M \\ ex'y \text{ if } y \in xeS \end{Bmatrix}$. Note that $g' \in \operatorname{Hom}(R', D')$. Now let $f' = f \mid D'$ and let $y \in R'$. If $y \in R_M$ then $f'g'(y) = f_M g_M(y) = 1_{R_M}(y) = y$. On the other hand, if $y \in xeS$, say $y = xes$, then $f'g'(y) = f'g'(xes) = f'(ex'xes) = xx'xes = xes = y$. Hence it follows that $f'g' = 1_{R'}$. Thus, $(D', g') \in \mathscr{F}$ and clearly $(D_M, g_M) < (D', g')$ which again contradicts the maximality of (D_M, g_M).

Therefore, D_M must be intersection large in S and the theorem follows.

An S-set M_S is *intersection uniform* if every nonzero sub S-set of M is intersection large.

THEOREM 4.3. *The semigroup $Q = Q(M_S)$ is a right cancellative semigroup with zero if and only if M_S is intersection uniform.*

Proof. Suppose that Q is a right cancellative semigroup with zero and let N_S be a nonzero sub S-set of M_S. Using Zorn's lemma to find a maximal sub S-set N' of M such that $N \cap N' = 0$, define a function f on $N \cup N'$ by $f(x) = x$ if $x \in N$ and $f(x) = 0$ if $x \in N'$. Then $f^2 = f$ and $f \in F$. If $\bar{f} = \bar{0}$ then there exists $D \in \mathscr{P}(M_S)$ such that $D \subseteqq D_f$ and $f(D) = 0$ which implies that $D \subseteqq N'$. Hence $N' \in \mathscr{P}(M_S)$. But this is impossible since $N \cap N' = 0$. Thus we have $\bar{f} \neq \bar{0}$. Since $\bar{1}_M \bar{f} = \bar{f} = \bar{f}\bar{f}$ and since each nonzero element of Q is right cancellable, it follows that $\bar{1}_M = \bar{f}$. Therefore, there exists $D \in \mathscr{P}$ such that $D \subseteqq N$ and it follows that $N \in \mathscr{P}(M_S)$. The proof of the converse is immediate.

5. The injective hull of a $\mathscr{P}$-torsion free semigroup. Throughout this section we shall consider the semigroup S as an S-set over itself. For $s \in S$ define $\phi_s: S \rightarrow S$ by $\phi_s(t) = st$. Then $\phi_s \in F$ and it easily follows that the map $\phi: S \rightarrow Q$ by $\phi(s) = \bar{\phi}_s$ is a representation of S in $Q = Q(S_S)$. Note also that we can regard Q as a centered right S-set by defining $\bar{f}s = \overline{f\phi_s}$ for each $\bar{f} \in Q$ and for each $s \in S$. The following lemmas are easy consequences of the above remarks.

LEMMA 5.1. $\psi(S_S) = \phi^{-1} \circ \phi$.

LEMMA 5.2. *For each* $f \in F$ *and for each* $s \in D_f$, $f\phi_s = \phi_{f(s)}$.

When $\psi(S_S) = \iota_S$ *we shall assume that* S *is embedded in* $Q = Q_{\mathscr{P}}(S)$ *under the identification* $s\langle - \rangle \bar{\phi}_s$. *From Lemma* 5.2 *we see that* $\bar{f}s = f(s)$ *for each* $\bar{f} \in Q_{\mathscr{P}}(S)$ *and for each* $s \in D_f$ *under the identification described above. Thus we see that* S_S *is intersection large in* Q_S. *In addition, the next lemma shows that* Q_S *is* $\mathscr{P}$*-torsion free.*

LEMMA 5.3. *If* S *is* $\mathscr{P}$*-torsion free then* Q_S *is* $\mathscr{P}$*-torsion free.*

Proof. Let $(\bar{f}_1, \bar{f}_2) \in \psi(Q_S)$. Then there exists $E \in \mathscr{P}$ such that $\bar{f}_1 E = \bar{f}_2 E$. Let $E' = E \cap D_{f_1} \cap D_{f_2} \in \mathscr{P}$. Then for each $s \in E'$, we have $f_1(s) = \bar{f}_1 s = \bar{f}_2 s = f_2(s)$ and it follows that $\bar{f}_1 = \bar{f}_2$.

The following lemma is immediate from Lemma 2.1 and the remarks preceding the above lemma.

LEMMA 5.4. *If* S *is* $\mathscr{P}$*-torsion free then* S_S *is large in* Q_S.

We now can show that Q_S is the injective hull of S and furthermore Q is injective as a Q-set.

THEOREM 5.5. *If* S *is* $\mathscr{P}$*-torsion free then* $Q_S = Q(S_S)$ *is the injective hull of* S_S.

Proof. Since S_S is large in Q_S by Lemma 5.3, we need only show that Q_S is injective. By Lemma 5.3 and Proposition 3.4 it suffices to verify that Q_S is weakly injective. Let R be a right ideal of S and let $\Phi \in \mathrm{Hom}_S(R, Q)$. Since S_S is intersection large in Q_S, $R' = \Phi^{-1}(S) \in \mathscr{P}$ and $f = \Phi | R' \in F$. We claim that $\Phi(r) = \bar{f}r$ for each $r \in R$. For each $s \in r^{-1}R' = \{s: rs \in R'\}$ we have $\Phi(r)s = \Phi(rs) = f(rs) = (\bar{f}r)s$. Thus, since $r^{-1}R' \in \mathscr{P}$, it follows that $(\Phi(r), \bar{f}r) \in \psi(Q_S)$ which is the identity S-congruence on Q_S. Therefore, $\Phi(r) = \bar{f}r$ for each $r \in R$ and the result follows.

THEOREM 5.6. *If* S *is* $\mathscr{P}$*-torsion free then* $Q = Q(S_S)$ *is self injective.*

Proof. Let $A_Q \subseteqq B_Q$ be Q-sets and let $\Phi' \in \mathrm{Hom}_Q(A, B)$. Then $\Phi' \in \mathrm{Hom}_S(A, Q)$. Since Q_S is the injective hull of S_S, there exists $\Phi \in \mathrm{Hom}_S(B, Q)$ such that $\Phi | A = \Phi'$. We claim that Φ is a Q-homomorphism. Let $b \in B$ and $\bar{f} \in Q$. Then for each $s \in D_f$ we have $\Phi(b\bar{f})s = \Phi(b\bar{f}s) = \Phi(bf(s)) = \Phi(b)f(s) = \Phi(b)\bar{f}s$. Thus $(\Phi(b\bar{f}), \Phi(b)\bar{f}) \in \psi(Q_S)$ which is the identity congruence on Q. Therefore, it follows that $\Phi(b\bar{f}) = \Phi(b)\bar{f}$.

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