PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

GENERATORS FOR EVOLUTION SYSTEMS WITH QUASI CONTINUOUS TRAJECTORIES

JAMES V. HEROD

With G a normed space, this paper provides conditions on a nonlinear function A from $R \times G$ to G in order to insure that if P is in G then there will be a (not necessarily continuous) solution Y for

$$Y(x) = P + \int_0^x d_t A(t, Y(t)) .$$

Early work in the study of the Stieltjes integral equation

$$M(x, z) = 1 + \int_x^z dFM(I, z)$$

was done by H. S. Wall [25] and T. H. Hildebrandt [8]. In Wall's paper, F is a continuous matrix valued function which is of bounded variation on each finite interval. Hildebrandt dropped the requirement of continuity and used a modified Stieltjes integral. J. S. Mac Nerney carefully analysed these ideas in a series of papers which led to the fundamental relationships found in [15], [16], and [17].

The papers [15] and [17] establish two classes OA and OM of functions and a one-to-one pairing of the classes made possible through a continuously continued sum, a continuously continued product, and a Stieltjes integral equation. In [17], if V is in OA, M is in OM, S is a linearly ordered set, and P is contained in a complete, normed, Abelian group, then V and M are related by $M(x, y)P = {}_x\prod^y [1+V]P$, $V(x, y)P = {}_x\sum^y [M-1]P$, and $M(x, y)P = P + \int_x^y VM(I, y)P$.

The results in [15] may be identified with analogous results in ordinary differential equations associated with nonautonomous, continuous, linear systems and [17] may be identified with Lipschitz systems. An indication of the nature of the generality obtained in the Stieltjes integral equation theory is found in [16], or in David L. Lovelady's discussion of interface problems [11, p. 184], or in a recent paper by Robert H. Martin [20] which investigates a linear operator equation and which identifies the linearly ordered set as the positive integers. Additional results related to [15] were found by B. W. Helton and Davis-Chatfield (see [2] or [3]). Also, this author determines a characterization of subsets of the two classes OA and OM which give rise to invertible evolution operators M in [4], for the linear case, and in [7] for the nonlinear (but Lipschitz) case.

In [9] Don Hinton and in [1] Carl Bitzer develop a theory for Stieltjes-Volterra equations. Reneke shows in [21] and [23] that much of the classical Volterra theory is contained in [15] or [17].

Questions concerning bounds for solutions of Stieltjes equations, as well as perturbations of these solutions have been investigated by Schamedeke and Sell [24], Herod [5], Martin [19], Reneke [22], and Lovelady [10], [11], and [12]. Also, Marrah and Proctor [18] have found results concerning periodic solutions.

In [6], this author extends the classes OA and OM by using some of the ideas of analytic semi-group theory. In that investigation, similar to Mac Nerney's, two classes OA and OM are paired by a continuously continued sum, a continuously continued product, and a Riemann-Stieltjes equation. (In this setting, also, Lovelady [14] has generalized earlier results of his involving perturbations of the systems.) The Lipschitz condition of [17] was dropped in [6] at the expense of requiring that $M(\cdot, y)P$, in addition to being of bounded variation on each finite interval, be continuous and that S should be the real line. The results which follow relax these requirements.

We suppose that S is a nondegenerate set with a linear ordering and that $\{S, \geqq\}$ has the least upper bound property. Also, $\{G, +, |\cdot|\}$ denotes a complete, normed Abelian group with zero element 0. Further, suppose that D is a closed subset of G and that V is a function such that if each of x and y is in S and $x \geqq y$ then $V(x, y)$ is a function from D into G having the following properties:

(i) If $x \geqq y \geqq z$ and P is in D then $V(x, y)P + V(y, z)P = V(x, z)P$,

(ii) If $a > b$ then there is a nondecreasing, numerical valued function β defined on S such that if $\varepsilon > 0$ and P is in D then there is a positive number δ having the property that if Q is in D such that $|Q - P| < \delta$ and $a \geqq x \geqq y \geqq b$ then $|V(x, y)P - V(x, y)Q| \leqq [\beta(x) - \beta(y)]\varepsilon$,

(iii) If $a > b$ then D is contained in the range of $[1 - V(a, b)]$ and if P and Q are in D then $|[1 - V(a, b)]P - [1 - V(a, b)]Q| \geqq |P - Q|$, and

(iv) If $a > b$ and P is in D then there is a nondecreasing, numerical function α such that if $\{s_p\}_0^n$ is a nonincreasing sequence with values in $[b, a]$ and $a \geqq x \geqq y \geqq b$ then $|V(x, y) \prod_{p=1}^n [1 - V(s_{p-1}, s_p)]^{-1}P| \leqq \alpha(x) - \alpha(y)$.

If f is a function from S with values in G and y is in S then $f(y^-)$ is a member g of G having the property that if $\varepsilon > 0$ then there is a member x of S such that $x < y$ and if $x \leqq t < y$ then $|g - f(t)| < \varepsilon$. In a similar manner, $f(y^+)$ may be defined.

The following theorems are established:

THEOREM I. *If* $a > b$, β *is as in* (ii), P *is in* D, *and* $\varepsilon > 0$ *then there is a subdivision* s *of* $\{a, b\}$ *such that if* t *is a refinement of* s *then*

$$|\prod_s [1 - V]^{-1}P - \prod_t [1 - V]^{-1}P| < \{4 + 2[\beta(a) - \beta(b)]\}\varepsilon .$$

Let M be a function defined as follows: If $x \geqq y$ and P is in D then $M(x, y)P = {}_x\prod^y [1 - V]^{-1}P$.

THEOREM II. *If* $a > b$ *then* $M(a, b)$ *is a function from* D *to* D *and*

(1) *If each of* P *and* Q *is in* D *then* $|M(a, b)P - M(a, b)Q| \leqq |P - Q|$,

(2) *If* $x \geqq y \geqq z$ *and* P *is in* D *then* $M(x, y)M(y, z)P = M(x, z)P$,

(3) *If* P *is in* D, *and* $a \geqq x \geqq y \geqq b$ *then* $|M(x, b)P - M(y, b)P| \leqq \alpha(x) - \alpha(y)$,

(4) *If* $a \geqq b$, $\varepsilon > 0$, *and* P *is in* D *then there is a positive number* δ *having the property that if* Q *is in* D *such that* $|Q - P| < \delta$ *and* $a \geqq x \geqq y \geqq b$ *then* $|[M(x, y) - 1]P - [M(x, y) - 1]Q| \leqq [\beta(x) - \beta(y)]\varepsilon$.

THEOREM III. *If* P *is in* D *and* b *is a member of* S *then the only function* g *which is of bounded variation on each finite interval of* S *and which satisfies the integral equation* $g(x) = P + (L) {}_x\int^b V[g]$ *for each* $x \geqq b$ *is given by* $g(x) = M(x, b)P$ *for* $x \geqq b$.

Proof of Theorem I.

LEMMA 1. *If* $a > b$, P *is in* D, *and* α *is as in* (iv), *then*

(1) $\lim_{x \downarrow b}([1 - V(x, b)]^{-1}P)$ *exists and is* $[1 - V(b^+, b)]^{-1}P$ *and*

(2) *If* t *is a subdivision of* $\{a, b\}$ *then* $|\prod_t [1 - V]^{-1}P - [1 - V(b^+, b]^{-1}P| \leqq \alpha(a) - \alpha(b^+)$,

(3) $\lim_{x \uparrow a}([1 - V(a, x)]^{-1}P)$ *exists and is* $[1 - V(a, a^-)]^{-1}P$ *and*

(4) *If* t *is a subdivision of* $\{a, b\}$ *then* $|\prod_t [1 - V]^{-1}P - [1 - V(a, a^-)]^{-1}P| \leqq \alpha(a^-) - \alpha(b)$.

Indication of proof. Suppose that $x \geqq y > b$. Then

$$\begin{aligned} &|[1 - V(x, b)]^{-1}P - [1 - V(y, b)]^{-1}P| \\ &\leqq |V(x, b)[1 - V(y, b)]^{-1}P - V(y, b)[1 - V(y, b)]^{-1}P| \\ &\leqq \alpha(x) - \alpha(y) . \end{aligned}$$

The existence of $\lim_{x \downarrow b} \alpha(x)$, together with the fact that D is closed, implies the existence of $\lim_{x \downarrow b}([1 - V(x, b)]^{-1}P)$ in D. Let Q be this limit. Then $|[1 - V(x, b)]Q - P| \leqq |Q - [1 - V(x, b)]^{-1}P| + |V(x, b)Q - V(x, b)[1 - V(x, b)]^{-1}P|$. Consequently, $P = \lim_{x \downarrow b}[1 - V(x, b)]Q = [1 - V(b^+, b)]Q$. That is, $Q = [1 - V(b^+, b)]^{-1}P$ so that (1) is established. In order to establish (2), suppose that $\{t_p\}_0^n$ is a subdivision of $\{a, b\}$. With Q as above,

$$\left|\prod_{p=1}^{n}[1 - V(t_{p-1}, t_p)]^{-1}P - Q\right|$$
$$\leqq \left|\prod_{p=1}^{n}[1 - V(t_{p-1}, t_p)]^{-1}P - [1 - V(t_{n-1}, t_n)]^{-1}P\right| + \alpha(t_{n-1}) - \alpha(b^+)$$
$$\leqq \sum_{p=1}^{n-1}|V(t_{p-1}, t_p)[1 - V(t_{n-1}, t_n)]^{-1}P| + \alpha(t_{n-1}) - \alpha(b^+)$$
$$\leqq \alpha(a) - \alpha(b^+)\,.$$

In a similar manner, one can establish (3) and (4).

LEMMA 2. *Suppose that $a > b$, β is as in* (ii), *ε is a positive number, and P is in D. There is a subdivision $\{s_p\}_0^m$ of $\{a, b\}$ such that if $\{t_p\}_0^n$ is a refinement of s and k is a sequence such that $t(k_p) = s_p$, $p = 0, 1, \cdots, m$, then*

$$\sum_{p=1}^{m}\sum_{q=1+k_{p-1}}^{k_p}\left|V(t_{q-1}, t_q)\prod_{i=q}^{k_p}[1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=p+1}^{m}[1 - V(s_{j-1}, s_j)]^{-1}P\right.$$
$$\left. - V(t_{q-1}, t_q)\prod_{i=1+k_{p-1}}^{k_p}[1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=p+1}^{m}[1 - V(s_{j-1}, s_j)]^{-1}P\right|$$
$$< [4 + 2(\beta(a) - \beta(b))]\varepsilon\,.$$

Proof. With the supposition of the lemma, let α be as in (iv). Define functions Δ, δ, and d as follows:

If R is in D then $\Delta(R)$ is the largest number e not exceeding 1 and having the property that if Q is in D, $|Q - R| < e$, and $a \geqq x \geqq y \geqq b$ then $|V(x, y)Q - V(x, y)R| \leqq [\beta(x) - \beta(y)]\varepsilon$,

If $b \leqq z < a$, R is in D, and $Q = \lim_{x \downarrow z}[1 - V(x, z)]^{-1}R$ then $\delta(z, R)$ is defined as follows: If there is no point y such that $z < y < a$ then $\delta(z, R) = a$ and, otherwise, $\delta(z, R)$ is the least upper bound of all x such that $z < x \leqq a$ and such that if $z \leqq y < x$ and t is a subdivision of $\{y, z\}$ then $|\prod_t[1 - V]^{-1}R - Q| < \Delta(Q)$, and

If $b \leqq z < y \leqq a$ and c is a positive number then let x be the greatest lower-bound of all w such that $z \leqq w$ and such that if $w \leqq u < y$ then $\alpha(y^-) - \alpha(u) < c$. If there is no point of S between x and y let $d(y, z, c)$ be x. If there is, let $d(y, z, c)$ be such a point. Note that if u is in S and $d(y, z, c) \leqq u < y$ then $\alpha(y^-) - \alpha(u) < c$.

Define the sequence u as follows: $u_0 = b$, $u_2 = \delta(u_0, P)$, $u_1 = d(u_2, u_0, \varepsilon)$, and, if n is a positive integer,

$$u_{2n+2} = \delta\Big(u_{2n}, \prod_{q=1}^{2n} [1 - V(u_{2n-q+1}, u_{2n-q})]^{-1}P\Big)$$

and $u_{2n+1} = d(u_{2n+2}, u_{2n}, \varepsilon/2^n)$. Assume that u is an infinite sequence. Since u is nondecreasing and bounded, let u_∞ be $\lim u_p$ and, for each positive integer j, let $R_j = \prod_{q=1}^{j} [1 - V(u_{j-q+1}, u_{j-q})]^{-1}P$. If $m > n$ then, as in [6, p. 250] $|R_m - R_n| \leqq \alpha(u_m) - \alpha(u_n)$. Because $\lim_{x \uparrow u_\infty} \alpha(x)$ exists, $\{R_p\}_{p=1}^{\infty}$ converges. For each integer n, let $Q_n = \lim_{x \downarrow u_n} [1 - V(x, u_n)]^{-1}R_n$. The sequence $\{Q_p\}_{p=1}^{\infty}$ converges for suppose that γ is a positive number. Let $R_\infty = \lim R_p$ and let v be a member of S such that if $u_\infty > x \geqq v$ then $\alpha(x^+) - \alpha(x) < \gamma/2$. Let N be a positive integer such that if $n > N$ then $|R_\infty - R_n| < \gamma/2$ and $u_\infty > u_n \geqq v$. Then $\lim Q_p = R_\infty$ for $|R_\infty - Q_n| < \alpha(u_n^+) - \alpha(u_n) + \gamma/2$. By [6, Lemma 2.1] there is a positive number ξ such that if n is a positive integer then $\Delta(Q_n) > \xi$. Again, using the fact that $\lim_{x \uparrow u_\infty} \alpha(x)$ exists, there is an integer N such that if $m > n > N$ then $\alpha(u_m) - \alpha(u_n) < \xi$ and, in this case, if t is a subdivision of $\{u_m, u_n\}$ then $|\prod_t [1 - V]^{-1}R_n - Q_n| < \alpha(u_m) - \alpha(u_n^+) < \xi \leqq \Delta(Q_n)$. Hence, $\delta(u_n, R_n) \geqq u_m$. Because this holds for each integer $m > n$, $\delta(u_n, R_n) \geqq u_\infty$. This is a contradiction to the assumption that u is an infinite sequence.

Let m be the least integer such that $u_{2m} = a$, and define s_p to be u_{2m-p} for $p = 1, 2, \cdots, 2m$. Let $\{t_q\}_{q=0}^{n}$ be a refinement of s and k be an increasing sequence such that $k_0 = 0$, $k_{2m} = n$, and $t(k_p) = s_p$ for $p = 0, 1, \cdots, 2m$. If p is an integer in $[1, m]$ and q is an integer in $[1 + k_{2p-1}, k_{2p}]$ then $u_{2(m-p)+2} = \delta(u_{2(m-p)}, R_{2(m-p)})$. Hence

$$\Big|\prod_{i=q}^{k2p} [1 - V(t_{i-1}, t_i)]^{-1}R_{2(m-p)} - Q_{2(m-p)}\Big| < \Delta(Q_{2(m-p)})$$

and

$$\begin{aligned}\Big|V(t_{q-1}, t_q)\prod_{i=q}^{k_{2p}} [1 - V(t_{i-1}, t_i)]^{-1}R_{2(m-p)} - V(t_{q-1}, t_q)Q_{2(m-p)}\Big| \\ \leqq [\beta(t_{q-1}) - \beta(t_q)]\varepsilon\,.\end{aligned}$$

If p is an integer in $[1, m]$ and q is an integer in $[1 + k_{2p-2}, k_{2p-1}]$ then

$$\begin{aligned}\Big| V(t_{q-1}, t_q)\prod_{i=q}^{k_{2p-1}} [1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=2p}^{2m} [1 - V(s_{j-1}, s_j)]^{-1}P \\ - V(t_{q-1}, t_q)\prod_{i=1+k_{2p-2}}^{k_{2p-1}} [1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=2p}^{2m} [1 - V(s_{j-1}, s_j)]^{-1}P\Big|\end{aligned}$$

is zero if $q = 1 + k_{2p-2}$ and does not exceed $2[\alpha(t_{q-1}) - \alpha(t_q)]$ if $1 + k_{2p-2} < q \leqq k_{2p-1}$. Furthermore, $\alpha(t_{k_{2p-2}}-) - \alpha(t_{k_{2p-1}}) = \alpha(s_{2p-2}-) - \alpha(s_{2p-1}) = \alpha(u_{2(m-p)+2}^-) - \alpha(u_{2(m-p+1)}) < \varepsilon/2^{m-p}$. It follows that

$$\sum_{p=1}^{2m}\left\{\sum_{q=1+k_{p-1}}^{k_p}\left|V(t_{q-1}, t_q)\prod_{i=q}^{k_p}[1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=p+1}^{2m}[1 - V(s_{j-1}, s_j)]^{-1}P\right.\right.$$
$$\left.\left. - V(t_{q-1}, t_q)\prod_{i=1+k_{p-1}}^{k_p}[1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=p+1}^{2m}[1 - V(s_{j-1}, s_j)]^{-1}P\right|\right\}$$
$$= \sum_{p=1}^{m}\left\{\sum_{q=1+k_{2p-2}}^{k_{2p-1}}\left|V(t_{q-1}, t_q)\prod_{i=q}^{k_{2p-1}}[1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=2p}^{2m}[1 - V(s_{j-1}, s_j)]^{-1}P\right.\right.$$
$$\left. - V(t_{q-1}, t_q)\prod_{i=1+k_{2p-2}}^{k_{2p-1}}[1 - V(t_{i-1}, t_i)]^{-1}\prod_{j=2p}^{2m}[1 - V(s_{j-1}, s_j)]^{-1}P\right|$$
$$+ \sum_{q=2+k_{2p-1}}^{k_{2p}}\left|V(t_{q-1}, t_q)\prod_{i=q}^{k_{2p}}[1 - V(t_{i-1}, t_i)]^{-1}R_{2(m-p)}\right.$$
$$\left.\left. - V(t_{q-1}, t_q)\prod_{i=1+k_{2p-1}}^{k_{2p}}[1 - V(t_{i-1}, t_i)]^{-1}R_{2(m-p)}\right|\right\}$$
$$\leqq \sum_{p=1}^{m}\varepsilon/2^{m-p} + \sum_{p=1}^{m}2[\beta(s_{2p-1}-) - \beta(s_{2p})]\varepsilon < \{4 + 2[\beta(a) - \beta(b)]\}\varepsilon\,.$$

Indication of proof for Theorem I. The inequalities in the proof of Theorem 2.1 on pages 251 and 252 of [6] carry over almost without change by using the above Lemma 2.

The techniques above also provide the following

COROLLARY. *If* $a > b$, β *is as in* (ii), P *is in* D, *and* $\varepsilon > 0$ *then there is a subdivision* s *of* $\{a, b\}$ *such that if* $\{t_p\}_0^n$ *is a refinement of* s *and* p *is an integer in* $[0, n]$ *then* $|M(t_p, b)P - \prod_{i=p+1}^{n}[1 - V(t_{i-1}, t_i)]^{-1}P| < \varepsilon$.

Proof of Theorem II. Parts (1) and (2) follow from the corresponding inequalities for the approximations to M; further details are indicated in Theorem 2.2 of [6]. To establish part 3 of Theorem II, suppose that $a \geqq x \geqq y \geqq b$ and P is in D. Let α be as in (iv), and t and s be a subdivision of $\{x, y\}$ and $\{y, b\}$ respectively. Then

$$|M(x, b)P - M(y, b)P| \leqq |M(x, b)P - \textstyle\prod_t[1 - V]^{-1}\prod_s[1 - V]^{-1}P|$$
$$+ |\{\textstyle\prod_t[1 - V]^{-1} - 1\}\prod_s[1 - V]^{-1}P|$$
$$+ |\textstyle\prod_s[1 - V]^{-1}P - M(y, b)P|\,.$$

Also,

$$|\{\textstyle\prod_t[1 - V]^{-1} - 1\}\prod_s[1 - V]^{-1}P|$$
$$= |\textstyle\sum_{p=1}^{n}V(t_{p-1}, t_p)\prod_{i=p}^{n}[1 - V(t_{i-1}, t_i)]^{-1}\prod_s[1 - V]^{-1}P|$$
$$\leqq \alpha(t_0) - \alpha(t_n)\,.$$

For part (4) of Theorem II, suppose that $a > b$, β is as in (iv), $\varepsilon > 0$, and P is in D. Since $M(\cdot, b)P$ is quasi continuous, $M([b, a], b)P$ is compact. Hence, there is a positive number δ such that if Q is in $M([b, a], b)P$, R is in D such that $|Q - R| < \delta$, and $a \geqq x \geqq y \geqq b$

then $|V(x, y)Q - V(x, y)R| \leqq [\beta(x) - \beta(y)]\cdot\varepsilon/3$. Suppose that Q is in D such that $|Q - P| < \delta$, $\{t_p\}_0^n$ is a subdivision of $\{x, y\}$ such that if R is P or Q and p is an integer in $[1, n]$ then

$$\left|\prod_{i=p}^{n} [1 - V(t_{i-1}, t_i)]^{-1}R - M(t_{p-1}, b)R\right| < \delta .$$

Then

$$\begin{aligned}
&|\{\Pi_t [1 - V]^{-1} - 1\}P - \{\Pi_t[1 - V]^{-1} - 1\}Q| \\
&\quad\leqq \sum_{p=1}^{n}\left| V(t_{p-1}, t_p)\prod_{i=p}^{n} [1 - V(t_{i-1}, t_i)]^{-1}P - V(t_{p-1}, t_p)\prod_{i=p}^{n} [1 - V(t_{i-1}, t_i)]^{-1}Q\right| \\
&\quad\leqq [\beta(x) - \beta(y)]\varepsilon .
\end{aligned}$$

Proof of Theorem III. This theorem established that the evolution operator M which was found in Theorem II provides a solution to the initial value problem indicated in Theorem III. Note that the integral used is the Cauchy-left integral: If f is a function from $[b, a]$ with values in D then $(L)\int_a^b V[f]$ is approximated by $\sum_{p=1}^{n} V(t_{p-1}, t_p)f(t_{p-1})$ where t is a subdivision of $\{a, b\}$.

LEMMA 3. *Suppose that $a > b$ and f is a function from $[b, a]$ to D which is of bounded variation. It follows that $(L)\int_a^b V[f]$ exists; in fact, if $\varepsilon > 0$ then there is a subdivision s of $\{a, b\}$ such that if $\{t_p\}_{p=0}^{n}$ is a refinement of s then*

$$\sum_{p=1}^{n}\left| V(t_{p-1}, t_p)f(t_{p-1}) - (L)\int_{t_{p-1}}^{t_p} V[f]\right| < \varepsilon .$$

LEMMA 4. *Suppose that b is in S, P is in D, each of f and g is of bounded variation, and, for each $x \geqq b$, $f(x) = P + L\int_x^b V[f]$ and $g(x) = P + (L)\int_x^b V[g]$. It follows that if $x \geqq b$ then $f(x) = g(x)$.*

Proof. With the supposition of the lemma, let x be in S such that $x \geqq b$, ε be a positive number, and $\{t_p\}_{p=0}^{n}$ be a subdivision of $\{x, b\}$ such that

$$\sum_{p=1}^{n}\left\{\left|\int_{t_{p-1}}^{t_p} V[f] - V(t_{p-1}, t_p)f(t_{p-1})\right| + \left|\int_{p-1}^{t_p} V[g] - V(t_{p-1}, t_p)g(t_{p-1})\right|\right\} < \varepsilon .$$

Then

$$|f(x) - g(x)| \leqq |f(x) - g(x)| + \sum_{p=1}^{n} \{|[1 - V(t_{p-1}, t_p)]f(t_{p-1})$$
$$- [1 - V(t_{p-1}, t_p)]g(t_{p-1})| - |f(t_{p-1}) - g(t_{p-1})|\}$$
$$= \sum_{p=1}^{n} \{-|f(t_p) - g(t_p)| + |[1 - V(t_{p-1}, t_p)]f(t_{p-1})$$
$$- [1 - V(t_{p-1}, t_p)]g(t_{p-1})|\} \leqq \sum_{p=1}^{n} \left\{\left|\int_{t_{p-1}}^{t_p} V[f] - V(t_{p-1}, t_p)f(t_{p-1})\right|\right.$$
$$\left. + \left| -\int_{t_{p-1}}^{t_p} V]g] + V(t_{p-1}, t_p)g(t_{p-1})\right|\right\} < \varepsilon .$$

Thus

$$f(x) = g(x) .$$

Indication of proof for Theorem III. Suppose that $a > b$, P is in D, and s is a subdivision of $\{a, b\}$. Then

$$\left|\prod_{p=1}^{n} [1 - V(s_{p-1}, s_p)]^{-1}P - P - \sum_{p=1}^{n} V(s_{p-1}, s_p)M(s_{p-1}, b)P\right|$$
$$= \left|\sum_{p=1}^{n} V(s_{p-1}, s_p) \prod_{i=p}^{n} [1 - V(s_{i-1}, s_i)]^{-1}P - V(s_{p-1}, s_p)M(s_{p-1}, b)P\right| .$$

Using the fact that $M([b, a], b)P$ is compact, together with the above corollary, we get that $M(a, b)P - P - (L)\int_a^b VM(\cdot, b)P = 0$. Lemma 4 shows that this is the only solution to the Stieltjes integral equation.

EXAMPLE. Suppose that g is an increasing, number valued function, A is a function with values in a Banach space G, and that A has the following properties: (Compare [6, p. 258].)

(a) If t is a number then $A(t, \cdot)$ has domain all of G,

(b) If P is in G then $A(\cdot, P)$ is continuous,

(c) If $a > b$, P is in G, and $\varepsilon > 0$ then there is a positive number δ having the property that if $a \geqq u \geqq b$ and Q is in G such that $|Q - P| < \delta$ then $|A(u, Q) - A(u, P)| < \varepsilon$,

(d) If $a > b$ and B is a bounded subset of G then A is bounded on $[b, a] \times B$, and

(e) If t is a number, P and Q are in G, and $c > 0$ then

$$|[P - cA(t, P)] - [Q - cA(t, Q)]| \geqq |P - Q| .$$

Also, as in [6, p. 258] let $V(x, y)P = (L)\int_y^x dgA(\quad, P)$ for $x \geqq y$ and P in G.

Then V is in OA and if c is a number and P is in G then the preceeding provides the only function f such that

$$f(x) = P - (L)\int_x^c dgA(\quad, f) .$$

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Received July 3, 1973.

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