PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

DECOMPOSITION THEOREMS FOR 3-CONVEX SUBSETS OF THE PLANE

MARILYN BREEN

Let S be a 3-convex subset of the plane. If $(\text{cl}\, S \sim S) \subseteq \text{int}\,(\text{cl}\, S)$ or if $(\text{cl}\, S \sim S) \subseteq \text{bdry}\,(\text{cl}\, S)$, then S is expressible as a union of four or fewer convex sets. Otherwise, S is a union of six or fewer convex sets. In each case, the bound is best possible.

1. Introduction. Let S be a subset of R^d. Then S is said to be 3-*convex* iff for every three distinct points in S, at least one of the segments determined by these points lies in S. Valentine [2] has proved that for S a closed, 3-convex subset of the plane, S is expressible as a union of three or fewer closed convex sets. We are interested in obtaining a similar decomposition without requiring the set S to be closed. The following definitions and results obtained by Valentine will be useful.

For $S \subseteq R^d$, a point x in S is a *point of local convexity of* S iff there is some neighborhood U of x such that, if $y, z \in S \cap U$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Let S be a closed, connected, 3-convex subset of the plane, and let Q denote the closure of the set of isolated lnc points of S. Valentine has proved that for S not convex, then $\text{card}\, Q \geqq 1$, Q lies in the convex kernel of S, and $Q \subseteq \text{bdry}\,(\text{conv}\, Q)$. An *edge* of $\text{bdry}\,(\text{conv}\, Q)$ is a closed segment (or ray) in $\text{bdry}\,(\text{conv}\, Q)$ whose endpoints are in Q. We define a *leaf* of S in the following manner: In case $\text{card}\, Q \geqq 3$, let L be the line determined by an edge of $\text{bdry}\,(\text{conv}\, Q)$, L_1, L_2 the corresponding open halfspaces. Then L supports $\text{conv}\, Q$, and we may assume $\text{conv}\, Q \subseteq \text{cl}\,(L_1)$. We define $W = \text{cl}\,(L_2 \cap S)$ to be a *leaf* of S. For $2 \geqq \text{card}\, Q \geqq 1$, constructions used by Valentine may be employed to decompose S into two closed convex sets, and we define each of these convex sets to be a *leaf* of S.

By Valentine's results, every point of S is either in $\text{conv}\, Q$ or in some leaf W of S (or both), and every leaf W is convex. Moreover, Valentine obtains his decomposition of S by showing that for any collection $\{s_i\}$ of disjoint edges of $\text{bdry}\,(\text{conv}\, Q)$, with $\{W_i\}$ the corresponding collection of leaves, $\text{conv}\, Q \cup (\bigcup W_i)$ is closed and convex.

Finally, we will use the following familiar definitions: For x, y in S, we say *x see y via S* iff the corresponding segment $[x, y]$ lies in S. A subset T of S is *visually independent via S* iff for every

x, y in T, x does not see y via S.

Throughout the paper, S will denote a 3-convex subset of the plane, Q the closure of the set of isolated lnc points of $\mathrm{cl}\, S$.

2. Preliminary lemmas. The following sequence of lemmas will be useful in obtaining the desired representation theorems. We begin with an easy result.

LEMMA 1. *Cl S is 3-convex.*

Proof. Let x, y, z be distinct points in $\mathrm{cl}\, S$ and select disjoint sequences (x_i), (y_i), (z_i) in S converging to x, y, z respectively. For each i, one of the corresponding segments is in S, and for one pair, say x and y, infinitely many of the segments $[x_i, y_i]$ lie in S. Since these segments converge to $[x, y]$, $[x, y]$ lies in $\mathrm{cl}\, S$.

The remaining lemmas are technical in nature. Lemmas 2, 3, and 4 reveal various pleasant features of $\mathrm{int}\,(\mathrm{cl}\, S) \sim S$, while 5 and 6 are concerned with lnc points of $\mathrm{cl}\, S$.

LEMMA 2. *If $p \in \mathrm{int}\,(\mathrm{cl}\, S) \sim \ker\,(\mathrm{cl}\, S) \neq \emptyset$, then $p \in S$.*

Proof. Since $p \notin \ker\,(\mathrm{cl}\, S)$, there is some point x in $\mathrm{cl}\, S$ for which $[x, p] \not\subseteq \mathrm{cl}\, S$. Moreover, x may be chosen in S (for if p saw every member of S via $\mathrm{cl}\, S$, then p would see every member of $\mathrm{cl}\, S$ via $\mathrm{cl}\, S$ and p would lie in $\ker\,(\mathrm{cl}\, S)$).

There is a convex neighborhood N of p, no point of which sees x via $\mathrm{cl}\, S$, with $N \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$. For any s, t distinct points in $N \cap S$, necessarily $[s, t] \subseteq S$ by the 3-convexity of S, so $N \cap S$ is convex. Since $N \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$, p is interior to some triangle $\mathrm{conv}\,\{w, y, z\}$ with vertices belonging to $N \cap S$. Then since $N \cap S$ is convex, $\mathrm{conv}\,\{w, y, z\} \subseteq S$, and $p \in S$. In fact, $p \in \mathrm{int}\, S$.

COROLLARY. *If $p \in \mathrm{cl}\, S \sim S$, then either $p \in \mathrm{bdry}\,(\mathrm{cl}\, S)$ or $p \in \ker\,(\mathrm{cl}\, S)$ (or both).*

LEMMA 3. *Let $T \neq \emptyset$ be the set of points p of $\mathrm{cl}\, S \sim S$ for which $p \notin \mathrm{bdry}\,(\mathrm{cl}\, S)$. Then every connected component of T is either an isolated point of $\mathrm{cl}\, S \sim S$ or an interval. Moreover, there can be at most one isolated point, and all components of T lie on a common line.*

Proof. If T is a singleton point, the result is immediate, so assume that T contains at least two distinct points x, y. Let $L(x, y)$ denote the line determined by these points. It is clear that not both x and y can be isolated in $\mathrm{cl}\, S \sim S$, for otherwise, since $x, y \in \mathrm{int}\,(\mathrm{cl}\, S)$, it would be easy to select three points of S on $L(x, y)$ visually independent via S.

Again using the 3-convexity of S, $L(x, y) \cap S$ has at most two components, and $L(x, y) \cap T \subseteq \ker(\mathrm{cl}\, S)$ has at most three components. By an earlier argument, at most one component of $L(x, y) \cap T$ is an isolated point, and clearly each component is either an isolated point or an interval.

To complete the proof, it suffices to show that $T \subseteq L(x, y)$. Let $t \in \mathrm{int}\,(\mathrm{cl}\, S) \sim L(x, y)$ to show $t \notin T$. Since $L(x, y) \cap T$ contains at most one isolated point, $L(x, y) \cap T$ contains at least one interval $(r, s) \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$, and we may choose some point u in S for which (u, t) cuts (r, s). Then select a convex neighborhood N of t, $N \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$, so that for every q in N, (u, q) cuts (r, s). By techniques similar to those used in the proof of Lemma 2, $N \cap S$ is convex and $t \in S$. Hence $t \notin T$ and $T \subseteq L(x, y)$.

LEMMA 4. *If* $\mathrm{cl}\, S \sim S$ *contains an interval* (r, s) *disjoint from* $\mathrm{bdry}\,(\mathrm{cl}\, S)$, *then every lnc point of* $\mathrm{cl}\, S$ *lies on* $L(r, s)$.

Proof. Assume that for some lnc point t of $\mathrm{cl}\, S$, $t \notin L(r, s)$. As in the proof of Lemma 3, choose a point u and a neighborhood N of t so that u sees no point of $N \cap S$ via S. Since t is an lnc point of $\mathrm{cl}\, S$, N contains points v, w in S which are visually independent via S. Hence u, v, w are visually independent via S, a contradiction, and t must lie on $L(r, s)$.

LEMMA 5. *If p is in* $\ker(\mathrm{cl}\, S)$ *and q, r are in Q, then* $q \notin (p, r)$ (where p, q, r are distinct points).

Proof. Assume, on the contrary, that the points are collinear, with $p < q < r$. Let L denote the line containing p, q, r, L_1, L_2 the corresponding open halfspaces. Since $p \in \ker(\mathrm{cl}\, S)$ and $\mathrm{cl}\, S$ is not convex, there must be some point x of $\mathrm{cl}\, S$ not on L, say in L_1. Our hypothesis implies that $\mathrm{cl}\, S$ is connected, so by [2], Corollary 1, $r \in \ker(\mathrm{cl}\, S)$, and the triangle $\mathrm{conv}\,\{p, x, r\}$ has its boundary in $\mathrm{cl}\, S$. It is easy to see that the closed, 3-convex set $\mathrm{cl}\, S$ is simply connected, so $\mathrm{conv}\,\{p, x, r\} \subseteq \mathrm{cl}\, S$. Thus since q is an lnc point for $\mathrm{cl}\, S$, there must be some point y of $\mathrm{cl}\, S$ in L_2, $\mathrm{conv}\,\{p, y, r\} \subseteq \mathrm{cl}\, S$, and q cannot be an lnc point for $\mathrm{cl}\, S$, clearly impossible. Our assumption is false, and $q \notin (p, r)$.

COROLLARY. *No three members of Q are collinear.*

LEMMA 6. *If* $p \in \operatorname{conv} Q$, $q \in Q$, $q \neq p$, *and* W_1, W_2 *are leaves of* $\operatorname{cl} S$ *containing* q, *then* W_1, W_2 *are in opposite closed halfspaces determined by* $L(p, q)$.

Proof. Clearly the hypothesis implies that $\operatorname{cl} S$ is connected and that $\operatorname{card} Q \geqq 2$. If $\operatorname{card} Q = 2$, the result is an immediate consequence of an argument used by Valentine (Case 2, Theorem 3 of [2]), so we may assume that $\operatorname{card} Q \geqq 3$. Let r lie on the edge of $\operatorname{bdry}(\operatorname{conv} Q)$ which defines W_1, $r \neq q$. If $r \in L(p, q) \equiv L$, then by the definition of W_1, it is obvious that W_1 is in one of the closed halfspaces determined by L, say $\operatorname{cl} L_1$. Otherwise, without loss of generality, assume that r is in the open halfspace L_1. Clearly p and W_1 are separated by $L(r, q)$. Now if any point x of W_1 lay in L_2, then q would lie interior to the triangle $\operatorname{conv}\{p, x, r\} \subseteqq \operatorname{cl} S$, and q could not be an lnc point for $\operatorname{cl} S$, a contradiction. Hence W_1 lies in $\operatorname{cl} L_1$ in either case.

Since $W_1 \cup \operatorname{conv} Q$ is convex (by Valentine's results) and q is an lnc point for $\operatorname{cl} S$, W_2 necessarily contains points in L_2, and $W_2 \subseteqq \operatorname{cl} L_2$, finishing the proof.

3. Decomposition theorems.

With the preliminary lemmas behind us, we begin to investigate conditions under which S may be represented as a union of four or fewer convex sets, dealing primarily with the case for $(\operatorname{cl} S \sim S) \subseteqq \operatorname{int}(\operatorname{cl} S)$.

The first theorem, allowing us to restrict attention to the case for $\operatorname{cl} S = \operatorname{cl}(\operatorname{int} S)$, will be helpful later.

THEOREM 1. *If* $\operatorname{cl} S \neq \operatorname{cl}(\operatorname{int} S)$, *then* S *is a union of two or fewer convex sets.*

Proof. Without loss of generality, assume S is connected, for otherwise the result is trivial. Let $x \in S \sim \operatorname{cl}(\operatorname{int} S) \neq \emptyset$, and let N be a convex neighborhood of x disjoint from $\operatorname{int} S$. Since S is connected, x is not an isolated point of S, and it is clear that $N \cap S$ contains at least one segment.

We examine the maximal segments of $N \cap S$ (i.e., the segments which are not proper subsets of segments in $N \cap S$). It is easy to show that $N \cap S$ has at most two maximal segments, for otherwise, the 3-convexity of S together with the simple connectedness of $\operatorname{cl} S$ would yield an open region in $\operatorname{cl} S \cap N$. Since by Lemma 3 the points of $\operatorname{int}(\operatorname{cl} S) \sim S$ are collinear, this would imply that $N \cap S$ has interior points, clearly impossible by our choice of N.

In case $N \cap S$ has exactly two maximal segments, an argument similar to the one above may be used to show that any point of S

lies on one of the corresponding lines, and S is a union of two segments (possibly infinite). If $N \cap S$ has just one segment, let K_1 denote a maximal convex subset of S containing it, and let $K_2 \equiv \text{conv}(S \sim K_1)$. Again using the facts that N contains no interior points of cl S and cl S is simply connected, it is not hard to show that $K_2 \subseteqq S$, and $S = K_1 \cup K_2$, completing the proof.

Theorems 2 and 3 show that a decomposition is possible when $(\text{cl}\, S \sim S) \subseteqq \text{int}(\text{cl}\, S)$. There are two cases to consider, depending on the cardinality of Q.

THEOREM 2. *If* $(\text{cl}\, S \sim S) \cap \text{bdry}(\text{cl}\, S) = \emptyset$, *and* card $Q = n$ *for* n *an odd integer*, $n > 1$, *then* S *is expressible as a union of four or fewer convex sets.*

Proof. Clearly the hypothesis implies that $\text{cl}\, S = \text{cl}(\text{int}\, S)$. By the Corollary to Lemma 2, $\text{cl}\, S \sim S \subseteqq \ker(\text{cl}\, S)$, and by Lemma 3, every component of $\text{cl}\, S \sim S$ is either an isolated point or an interval. Since card $Q \geqq 3$ and (by the corollary to Lemma 5) no three members of Q can be collinear, Lemma 4 implies that $\text{cl}\, S \sim S$ cannot contain an interval. Hence $\text{cl}\, S \sim S$ consists of exactly one isolated point p in $\ker(\text{cl}\, S)$.

Select $q \in Q$ in the following manner: If $p \in \text{conv}\, Q$, choose $q \in Q$ so that the line $L(p, q)$ contains no other member of Q. (Clearly this is possible since card Q is odd and no three members of Q are collinear.) If $p \notin \text{conv}\, Q$, let $\{e_i: 1 \leqq i \leqq n\}$ denote the edges of conv Q, $\{E_i: 1 \leqq i \leqq n\}$ the corresponding lines, with conv Q in the closed halfspace $\text{cl}(E_{i1})$ for each i. Then $p \in E_{i2}$ for exactly one i, for otherwise, if $p \in E_{12} \cap E_{22}$, then int conv $(\{p\} \cup e_1 \cup e_2)$ would contain an lnc point of cl S, clearly impossible since $\{p\} \cup e_1 \cup e_2 \subseteqq \ker(\text{cl}\, S)$ and $\text{conv}(\{p\} \cup e_1 \cup e_2) \subseteqq \text{cl}\, S$. Thus we may choose some $q \in Q$ so that $p \in \text{cl}\, E_{i1}$ for each edge e_i containing q. Then (p, q) contains points of conv Q. Since all points of $L(p, q) \cap \text{conv}\, Q$ are on the open ray at p emanating through q, Lemma 5 implies that $L(p, q)$ contains no other members of Q (and in fact p cannot lie on any line E_i).

To review, in either case we have chosen q in Q so that $L(p, q)$ contains no other member of Q and (p, q) contains points of conv Q. Letting L_1, L_2 denote distinct open halfspaces determined by $L = L(p, q)$, define $A \equiv \text{cl}(S \cap L_1)$, $B \equiv \text{cl}(S \cap L_2)$. If W_1, W_2 are leaves of cl S containing q, then by Lemma 6, W_1 and W_2 are in opposite closed halfspaces determined by L, say $W_1 \subseteqq \text{cl}\, L_1$, $W_2 \subseteqq \text{cl}\, L_2$.

Let R_1, R_2 denote opposite closed rays at p, $R_1 \cup R_2 = L$, labeled so that $q \in R_2$. Each of $R_1 \cap S$, $R_2 \cap S$ is an interval by the 3-convexity of S. Points of $R_1 \cap S$ necessarily lie in $A \cap B$, for otherwise

R_1 would contain an lnc point of cl S, clearly impossible. If there are any points of $R_2 \cap S$ not in $A \cap B$, without loss of generality we may assume such points lie in W_1 and hence in $A \sim B$. Then $R_2 \cap S \subseteq A$.

By Case 4 in Theorems 2 and 3 of [2], cl $(S \sim W_2)$ is a union of two closed convex sets C_1, C_2, selected as in Valentine's proof. Since $A =$ cl [cl $(S \sim W_2) \cap L_1]$, A is the union of the two closed convex sets A_1, A_2, where $A_i =$ cl $(C_i \cap L_1)$, $i = 1, 2$. Moreover, $(R_1 \cap S) \cup (p, q]$ lies in one of these sets, say A_1, and $R_2 \sim (p, q]$ is either in A_1 or in A_2.

Using an identical argument for B and cl $(S \sim W_1)$, we may write B as a union of two closed convex sets B_1, B_2 with $(R_1 \cap S) \cup (p, q]$ in B_1, and $R_2 \sim (p, q]$ disjoint from B.

At last, define sets A_1', A_2', B_1', B_2' in the following manner: If $(R_2 \cap S) \sim (p, q] \subseteq A_2$, let

$$A_1' \equiv A_1 \sim R_2 , \quad A_2' \equiv A_2 \sim R_1 ,$$
$$B_1' \equiv B_1 \sim R_1 , \quad B_2' \equiv B_2 \sim R_2 .$$

And if $(R_2 \cap S) \sim (p, q] \subseteq A_1$, let

$$A_1' \equiv A_1 \sim R_1 , \quad A_2' \equiv A_2 \sim R_2 ,$$
$$B_1' \equiv B_1 \sim R_2 , \quad B_2' \equiv B_2 \sim R_1 .$$

We assert that these are convex subsets of S whose union is S: Clearly each is a convex subset of S, and $S \sim L$ is contained in their union. For $(R_2 \cap S) \sim (p, q] \subseteq A_2$, $R_2 \cap S \subseteq A_2' \cup B_1'$, $R_1 \cap S \subseteq A_1'$. For $(R_2 \cap S) \sim (p, q] \subseteq A_1$, $R_2 \cap S \subseteq A_1'$, $R_1 \cap S \subseteq B_1'$. Hence in either case $S \cap L$ is contained in the union of these sets, and $S = A_1' \cup A_2' \cup B_1' \cup B_2'$, completing the proof of the theorem.

THEOREM 3. *If* (cl $S \sim S) \cap$ bdry (cl $S) = \emptyset$ *and* card $Q = n \geqq 0$, *where* n (possibly infinite) *is not an odd integer greater than one, then* S *is expressible as a union of four or fewer convex sets.*

Proof. If S is not connected, the result is trivial. Otherwise, by Theorem 3 of Valentine [2], cl S may be expressed as a union of two or fewer closed convex sets A, B. Using Lemma 3, let L be a line containing cl $S \sim S$, L_1, L_2 the corresponding open halfspaces. Since S is 3-convex and A is convex, $S \cap A$ is 3-convex, and hence $(S \cap A) \cap L$ has at most two components, say C_1, C_2. Let R_1, R_2 denote opposite rays on L with $C_1 \subseteq R_1$, $C_2 \subseteq R_2$.

Define

$$A_1 \equiv (A \cap S \cap \text{cl } L_1) \sim R_1 ,$$
$$A_2 \equiv (A \cap S \cap \text{cl } L_2) \sim R_2 .$$

Then A_1, A_2 are convex subsets of S whose union is $A \cap S$.

Similarly define convex sets B_1, B_2 whose union is $B \cap S$. Clearly $S = A_1 \cup A_2 \cup B_1 \cup B_2$, the desired result.

COROLLARY. *If* $(\mathrm{cl}\, S \sim S) \cap \mathrm{bdry}\,(\mathrm{cl}\, S) = \emptyset$, *then* S *is expressible as a union of four or fewer convex sets. The number four is best possible.*

That the number four in the corollary is best possible is evident from Example 1.

EXAMPLE 1. Let S be the set in Figure 1, with $p \in S$. Then S is not expressible as a union of fewer than four convex sets.

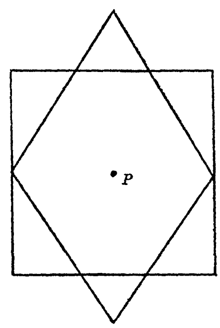

FIGURE 1

The preceding theorems allow us to obtain the following decomposition for open sets.

THEOREM 4. *If* S *is open, then* S *is expressible as a union of four or fewer convex sets. The result is best possible.*

Proof. Let $T \equiv S \cup \mathrm{bdry}\,(\mathrm{cl}\, S)$. Applying arguments identical to those used in the proofs of Theorems 2 and 3, T is expressible as a union of four or fewer convex sets A_i, $1 \leqq i \leqq 4$. Define $B_i \equiv A_i \cap S$, $1 \leqq i \leqq 4$. We assert that each B_i is convex. The proof follows:

By Valentine's results, $\mathrm{cl}\, S$ is expressible as a union of three or fewer closed convex sets C_j, $1 \leqq j \leqq 3$, each consisting of an appropriate selection of leaves of $\mathrm{cl}\, S$, together with $\mathrm{conv}\, Q$. Examining the proofs of Theorems 2 and 3, it is clear that each A_i may be considered as a subset of some C_j set. Thus we may assume $B_1 \subseteqq A_1 \subseteqq C_1$ for an appropriate C_1.

Let $x, y \in B_1$, and let $p \in (x, y)$ to show $p \in B_1$. If x (or y) is interior to some leaf W, then $W \subseteqq C_1$, y sees a neighborhood of x via

C_1, and p is interior to $\operatorname{cl} S$. Since $p \in A_1$ and $p \notin \operatorname{bdry}(\operatorname{cl} S)$, p is in $A_1 \cap S = B_1$. A similar argument holds if x (or y) is interior to $\operatorname{conv} Q$. Since neither x nor y is in $\operatorname{bdry}(\operatorname{cl} S)$, the only other possibility to consider is the case in which $x, y \in \operatorname{bdry}(\operatorname{conv} Q) \sim Q \subseteq \ker(\operatorname{cl} S)$. Then $x \in \operatorname{int}(\operatorname{cl} S)$, $y \in \ker(\operatorname{cl} S)$, y sees some neighborhood of x via $\operatorname{cl} S$, and $p \in \operatorname{int}(\operatorname{cl} S)$. Again $p \in A_1 \cap S = B_1$ and B_1 is indeed convex. Thus S is the union of the convex sets B_i, $1 \leqq i \leqq 4$, and the theorem is proved.

To see that the number four is best possible, let S denote the set in Example 1 with its boundary deleted. Then S is an open 3-convex set not expressible as a union of fewer than four convex sets.

4. The general case.

It remains to investigate the case for S an arbitrary 3-convex subset of the plane. A decomposition of S into six convex sets may be obtained from our previous results, together with Theorems 5 and 6, which deal with the case for $(\operatorname{cl} S \sim S) \subseteq \operatorname{bdry}(\operatorname{cl} S)$.

The following result by Lawrence, Hare, and Kenelly [1, Theorem 2] will be useful:

LAWRENCE, HARE, KENELLY THEOREM. Let T be a subset of a linear space such that each finite subset $F \subseteq T$ has a k-partition, $\{F_1, \cdots, F_k\}$, where $\operatorname{conv} F_i \subseteq T$, $1 \leqq i \leqq k$. Then T is a union of k convex sets.

THEOREM 5. *If* $\operatorname{cl} S$ *is convex and* $(\operatorname{cl} S \sim S) \subseteq \operatorname{bdry}(\operatorname{cl} S)$, *then* S *is a union of three or fewer convex sets. The bound of three is best possible.*

Proof. Consider the collection of all intervals in $\operatorname{bdry}(\operatorname{cl} S)$ having endpoints in S and some relatively interior point not in S. Each interval determines a line L, and by the 3-convexity of S, $L \cap S$ has exactly two components. Let $\mathscr{L}$ denote the collection of all such lines. By the Lawrence, Hare, Kenelly Theorem, without loss of generality we may assume that $\mathscr{L}$ is finite. Hence the set $\bigcup \{L \cap S : L \text{ in } \mathscr{L}\}$ has finitely many components, and we may order these components in a clockwise direction along $\operatorname{bdry}(\operatorname{cl} S)$. If c_i denotes the ith component in our ordering, let

$$A' \equiv \{c_i : i \text{ odd}, \ i < n\},$$
$$B' \equiv \{c_i : i \text{ even}, \ i < n\},$$
$$C' \equiv \{c_n\}.$$

Define

$$A \equiv S \sim (B' \cup C') ,$$
$$B \equiv S \sim (A' \cup C') ,$$
$$C \equiv S \sim (A' \cup B') .$$

We assert that A, B, C are convex sets whose union is S. The proof follows:

For x, y in A, if $[x, y]$ contains any point of $\text{int}(\text{cl}\, S)$, then $(x, y) \subseteqq \text{int}(\text{cl}\, S) \subseteqq A$, and $[x, y] \subseteqq A$. Otherwise, $[x, y]$ lies in the boundary of the convex set $\text{cl}\, S$. If the corresponding line $L(x, y)$ is not in $\mathscr{L}$, the result is clear, so suppose $L(x, y) \in \mathscr{L}$. Then x, y must lie in the same c_i set for some i odd, $i < n$, again giving the desired result. Hence A is convex. Similarly, B, C are convex. It is easy to see that $A \cup B \cup C = S$ and the proof is complete.

The surprising fact that three is best possible is illustrated by Example 2.

EXAMPLE 2. Let S denote the set in Figure 2, where dotted lines represent segments not in S. Then S is not expressible as a union of fewer than three convex sets.

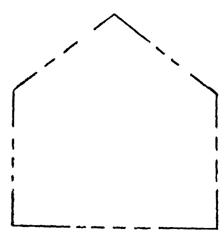

FIGURE 2

THEOREM 6. *If* $(\text{cl}\, S \sim S) \subseteqq \text{bdry}(\text{cl}\, S)$, *then S is a union of four or fewer convex sets. The number four is best possible.*

Proof. We assume that S is connected and $\text{cl}\, S = \text{cl}(\text{int}\, S)$, for otherwise S is a union of two convex sets. Furthermore, by the Lawrence, Hare, Kenelly Theorem, we may assume that $\text{cl}\, S$ has finitely many leaves, and hence $\text{card}\, Q = n$ is finite. Notice also that since $\text{cl}\, S$ is simply connnected and $(\text{cl}\, S \sim S) \subseteqq \text{bdry}(\text{cl}\, S)$, S is simply connected.

For the moment, suppose $3 \leqq n$. Order the points of Q in a clockwise direction along $\text{bdry}(\text{conv}\, Q)$, letting W_i denote the leaf of $\text{cl}\, S$ determined by lnc points q_i, q_{i+1} (where $n + 1 \equiv 1$). By Valentine's results in [2], for any pair of disjoint leaves W_i, W_j of $\text{cl}\, S$, the set $R \equiv \text{conv}\, Q \cup W_i \cup W_j$ is a closed convex set. (In case there are no disjoint leaves, $n = 3$, $W_j = \varnothing$, and $R \equiv \text{conv}\, Q \cup W_i$ is closed and convex.) Consider the collection of intervals in $\text{bdry}\, R$ having endpoints x, y in S and some relatively interior point p not in S. Either such an interval is contained in one leaf, or $x \in W_i \cup \operatorname{conv} Q$, $y \in W_j \cup \operatorname{conv} Q$. We examine the latter case. It is clear that for an appropriate labeling, $j = i + 2$, so to simplify notation, say $i = 1$, $j = 3$, and $L(x, y)$ supports W_2. Clearly not both x, y can lie in conv Q, for then $p \in \operatorname{int} S \subseteqq S$. However, we assert that either x or y must lie in conv Q and that $W_2 \cap S$ is convex. The proof follows:

Assume that x is not an lnc point and that $x < p \leqq q_2 < q_3$, where q_2, q_3 are the lnc points in $W_1 \cap W_2$, $W_2 \cap W_3$ respectively. Then $q_2 \leqq y$. For w in $W_2 \cap S$, w cannot see x via S, so necessarily w sees y via S, by the 3-convexity of S. This implies that $y \leqq q_3$ (for otherwise q_3 could not be an lnc point for cl S). Moreover, since no two points of $W_2 \cap S$ see x via S, the 3-convexity of S together with the convexity of W_2 imply that $W_2 \cap S$ is convex.

Here we digress briefly for future reference. The set $L(x, y) \cap S$ has two components, and by the above argument, one must lie in the interval $[q_2, q_3]$, the other in $W_1 \sim Q$ (by our labeling). For general W_{i-1}, W_{i+1} (disjoint if and only if $n > 3$), we let T_i denote the connected set of all the somewhat troublesome points y in $[q_i, q_{i+1}] \cap S$ having the above property. That is, there exist points x in exactly one of $(W_{i-1} \cap S) \sim Q$, $(W_{i+1} \cap S) \sim Q$ for which $[x, y] \not\subseteq S$ $(n + 1 \equiv 1)$.

Continuing the argument, delete W_2 and consider the 3-convex set $(S \sim W_2) \cup (S \cap L(x, y))$. Renumber the lnc points and leaves for this set so that the old W_1 and W_3 are contained in the new leaf U_1. Since we are assuming card Q is finite, repeating the procedure finitely many times yields a 3-convex set S_0 having the following property: For V_i, V_j disjoint leaves of cl S_0, x in $V_i \cap S_0$, y in $V_j \cap S_0$, then $[x, y] \subseteqq S_0$. In addition, without loss of generality we may assume that for each leaf V_i of cl S_0, $V_i \cap S_0$ is not convex, for otherwise, V_i may be deleted by the above procedure.

To avoid confusion, let Q_0 denote the set of lnc points of cl S_0, $Q_0 \subseteqq Q$, card $Q_0 = m \leqq n$. For $3 \leqq m$, let V_i denote the leaf determined by lnc points p_i, p_{i+1} in Q_0 (where $p_{m+1} = p_1$). For $m = 2$, let V_1, V_2 denote the leaves of cl S_0 as defined in the introduction to this paper. If $0 \leqq m \leqq 1$, let $V_1 = V_2 = \operatorname{cl} S_0$.

For each i, consider the collection of intervals in bdry V_i having endpoints in $V_i \cap S_0$ and some relatively interior point not in S_0. Each interval determines a line L, and for $m \neq 1$, $L \cap V_i \cap S_0$ has exactly two components, each in bdry V_i. In case $m = 1$, an obvious adjustment may be made (by deleting any ray of L which contains interior points of cl S_0) to yield the same result. For each i, let $\mathscr{L}_i$ denote the collection of all such lines. Again using the Lawrence, Hare, Kenelly Theorem, we may assume that each $\mathscr{L}_i$ is finite. The set $\bigcup \{L \cap V_i \cap S_0 \colon L \text{ in } \mathscr{L}_i\}$ has finitely many components, and we may order them in a clockwise direction along bdry V_i. Let c_{ij} denote the jth such component for V_i, and let $\mathscr{C}_i$ denote the collection of all the c_{ij} sets corresponding to V_i. Clearly each c_{ij} is either a point, an interval, or the union of two noncollinear intervals. Moreover, for $m \geqq 2$, no components for V_i, V_{i+1} may have common points. (Such a point would necessarily be p_{i+1}, and if $s_i \in V_i \cap S_0$, $s_{i+1} \in V_{i+1} \cap S_0$ with some interior point of each of $[s_i, p_{i+1}]$, $[p_{i+1}, s_{i+1}]$ not in S_0, then s_i, p_{i+1}, s_{i+1} would be visually independent via S_0, clearly impossible.)

For each V_i, select every c_{i2j}. That is, select the members of $\mathscr{C}_i$ having second subscript even. No two components selected correspond to the same line, and for $m \neq 0$, we have chosen one component corresponding to each line in $\mathscr{L}_i$. If $m = 0$, without loss of generality we may assume $\mathscr{C}_1$ is ordered in a clockwise direction from some point in $Q \cap \operatorname{cl} S_0 \neq \emptyset$. In case no component has been chosen for some line L in $\mathscr{L}_1$, then L must contain points of both the first and last members of $\mathscr{C}_1$, and by a previous argument, one of these components must lie in conv Q.

For $m \neq 1$, since V_i is convex, it is easy to show that conv $\{c_{i2j}: 1 \leqq j\}$ is a subset of S_0 (and this is certainly true even if cl S_0 is convex). We will prove that $B_0 \equiv$ conv $\{c_{i2j}: 1 \leqq i \leqq m, 1 \leqq j\}$ is in S_0 and hence in S. If cl S_0 is convex (or empty) the result is immediate, so assume cl S_0 has at least one lnc point. For convenience, in case cl S_0 has only one lnc point, call it p_2, and let $V_1 = V_2$ follow p_2 in our clockwise ordering.

Recall that $V_i \cap S_0$ is not convex for any i, so no $\mathscr{C}_i$ is empty. Let c_0 denote the last member of $\mathscr{C}_1$ selected, x the last point of cl c_0 (relative to our ordering). If $x \neq p_2$, let $L = L(x, p_2)$. Otherwise, by the 3-convexity of S_0, $c_0 = \{p_2\}$, and in this case let L denote the corresponding member of $\mathscr{L}_1$. Let L_1, L_2 be the open halfspaces determined by L, with $Q_0 \subseteqq \operatorname{cl} L_1$. Since p_2 is an lnc point of S_0 and S_0 is 3-convex, it is clear that at most one member of $\mathscr{C}_2$, namely c_{21}, may contain points in L_2. We assert that c_0 sees c_{22} via S_0. The proof follows:

In case $L \in \mathscr{L}_1$, $L \cap V_1 \cap S_0$ has two components, each in bdry V_1, and one of these must be $\{p_2\}$. Then by the 3-convexity of S_0, $c_{22} \subseteqq L_1$ and c_0 sees c_{22} via S_0. Otherwise, $c_0 \sim \{x\} \subseteqq L_1$. If $x \notin S_0$, then since $c_{22} \subseteqq \operatorname{cl} L_1$, it is clear that c_0 sees c_{22} via S_0. If $x \in S_0$ and $p_2 \in S_0$, then again the result is clear. If $x \in S_0$ and $p_2 \notin S_0$, then $c_{22} \subseteqq L_1$ and c_0 sees c_{22} via S_0, finishing the argument.

In case V_1, V_2 are the only leaves for cl S_0, $V_1 \neq V_2$, then repeating the argument for the last member of $\mathscr{C}_2$ and c_{12} and using the fact that S_0 is simply connected, we have $B_0 \subseteqq S_0 \subseteqq S$. (If $V_1 = V_2$, the result is immediate.) Otherwise, $3 \leqq m$ and an inductive argument may be used to show that B_0 is in S.

Using Valentine's results, write $\mathrm{cl}\, S$ as a union of three or fewer convex sets A_j, $j = 1, 2, 3$, where for n odd

$$A_1 \equiv \bigcup \{W_i \colon i \text{ odd}, \ i < n\} \cup \mathrm{conv}\, Q \ ,$$
$$A_2 \equiv \bigcup \{W_i \colon i \text{ even}, \ i < n\} \cup \mathrm{conv}\, Q \ ,$$
$$A_3 \equiv W_n \cup \mathrm{conv}\, Q \ ,$$

and for n even

$$A_1 \equiv \bigcup \{W_i \colon i \text{ odd}, \ i \leqq n\} \cup \mathrm{conv}\, Q \ ,$$
$$A_2 = \bigcup \{W_i \colon i \text{ even}, \ i \leqq n\} \cup \mathrm{conv}\, Q \ ,$$
$$A_3 = \emptyset \ .$$

Define $B_j \equiv S \cap [A_j \sim ((\mathrm{bdry}\, S) \cap B_0)]$, $j = 1, 2, 3$.

Recall the T_i sets defined previously, $T_i \subseteq [q_i, q_{i+1}] \subseteq W_i$, $1 \leqq i \leqq n$. To simplify notation, let $L_i = L(q_i, q_{i+1})$, and define sets F_i, G_i in the following manner: For i even, let $F_i = T_i$ if points from both components of $L_i \cap S$ are in B_1, $F_i = \emptyset$ otherwise. Similarly for i odd, let $F_i = T_i$ if points from both components of $L_i \cap S$ are in B_2, $F_i = \emptyset$ otherwise. For $i = 1$, $i = n - 1$, let $G_i = T_i$ if points from both components of $L_i \cap S$ are in B_3, $G_i = \emptyset$ otherwise. By previous remarks, at least one of G_1, F_1 is empty, and at least one of G_{n-1}, F_{n-1} is empty.

Define

$$D_1 \equiv B_1 \sim \bigcup \{F_i \colon i \text{ even}\} \ ,$$
$$D_2 \equiv B_2 \sim \bigcup \{F_i \colon i \text{ odd}\} \ ,$$
$$D_3 \equiv B_3 \sim \bigcup \{G_i, G_{n-1}\} \ .$$

Finally, letting $P = \{F_i \cap F_j \colon 1 \leqq i < j \leqq n\} \cup \{G_i \cap F_j \colon i = 1, n - 1, 1 \leqq j \leqq n\}$, define $D_0 \equiv \mathrm{conv}\,(B_0 \cup P)$. We assert that the sets D_j, $0 \leqq j \leqq 3$, are convex sets whose union is S. The proof follows:

Suppose that one of the sets D_1, D_2, D_3, say D_1, is not convex to obtain a contradiction. Choose x, y in D_1 for which $[x, y] \not\subseteq D_1$. It is clear that $[x, y] \subseteq \mathrm{bdry}\,(\mathrm{cl}\, D_1) = \mathrm{bdry}\, A_1$. Furthermore, x, y cannot both belong to $W \sim Q$ for any leaf W of $\mathrm{cl}\, S$, for otherwise they would belong to the same leaf of $\mathrm{cl}\, S_0$, and one of x, y would lie in $(\mathrm{bdry}\, S) \cap B_0$ and hence not in D_1, a contradiction. Employing a previous argument, the set $L(x, y) \cap S$ has two components, each having points in B_1, and one of these components is the set $[q_i, q_{i+1}] \cap S = T_i$ for some i even $(n + 1 \equiv 1)$. Let R_i denote the other component of $L(x, y) \cap S$. If $R_i \cap B_0 \neq \emptyset$, then R_i, T_i would lie on the boundary of a leaf of $\mathrm{cl}\, S_0$, $R_i \subseteq B_0$, $T_i \subseteq B_1$, and $[x, y] \subseteq T_i \subseteq D_1$, a contradiction. Thus $R_i \cap B_0 = \emptyset$ and $R_i \subseteq D_1$. However, this implies that one of x, y must lie in F_i and not in D_1, again a contradiction. Our assumption is false and D_1 is convex. Similarly D_2, D_3 are convex, and clearly each is a subset of S.

It remains to show that the convex set D_0 lies in S. Examining the set P, if $F_i \cap F_j \neq \emptyset$ for some $i \neq j$ (or if $G_i \cap F_j \neq \emptyset$), then $F_i = T_i$, $F_j = T_j$, for an appropriate labeling $j = i+1$, and $F_i \cap F_{i+1} = \{q_{i+1}\} \subseteqq S$. We will show that for each z in B_0, $[q_{i+1}, z] \subseteqq S$. The proof follows:

We have seen that $W_i \cap S$, $W_{i+1} \cap S$ are both convex, so for every z in one of these sets, $[q_{i+1}, z] \subseteqq S$. Moreover, we assert that the components of $L(q_i, q_{i+1}) \cap S$, $L(q_{i+1}, q_{i+2}) \cap S$ not in conv Q, call them R_i, R_{i+1}, are disjoint from B_0: If $R_i \cap B_0 \neq \emptyset$, then by an earlier argument, $R_i \subseteqq B_0$, $T_i \cap B_0 = \emptyset$, $T_i \subseteqq D_1 \cap D_2 \cap D_3$, and $F_i = \emptyset$, a contradiction. Hence for z in $B_0 \sim (W_i \cup W_{i+1})$, $(q_{i+1}, z) \subseteqq \operatorname{int} S$, and $[q_{i+1}, z] \subseteqq S$ whenever $z \in B_0$, the desired result.

Certainly for q_i, q_j, q_k in $P \subseteqq S$, conv $\{q_i, q_j, q_k\} \subseteqq S$.

By Carathéodory's theorem in the plane, to prove that $D_0 \equiv \operatorname{conv}(B_0 \cup P)$ is in S, it is sufficient to show that the convex hull of any three points of $B_0 \cup P$ is in S, and from the remarks above, clearly we need only show conv $\{q_i, q_j, z\} \subseteqq S$ for q_i, q_j in P, z in B_0. However, since S is simply connected and bdry (conv $\{q_i, q_j, z\}) \subseteqq S$, conv $\{q_i, q_j, z\} \subseteqq S$ and $D_0 \subseteqq S$, the desired result.

Finally, by inspection, each $F_i \neq \emptyset$ fails to belong to at most one of the sets D_1, D_2, D_3. Points in intersecting F_i sets are in D_0, so $\bigcup \{D_j \colon 0 \leqq j \leqq 3\} = S$ and the argument for $3 \leqq \operatorname{card} Q$ is complete.

To finish the proof, we must examine the cases for $0 \leqq \operatorname{card} Q \leqq 2$. If card $Q = 2$ or if card $Q = 1$ and $S \sim Q$ is connected, then let W_1, W_2 denote the corresponding leaves of cl S, and use a simplified version of the previous proof to define B_0, B_1, B_2. If one of B_1, B_2, say B_1, is not convex, then letting $T = W_1 \cap W_2 \cap S$, $W_2 \cap S = B_2$ is convex, $T \subseteqq B_2$, and B_0, $B_1 \sim T$, B_2 are the desired convex sets.

In case card $Q = 1$ and $S \sim Q$ is not connected, then for W_1, W_2 the corresponding leaves of cl S, each of $W_1 \cap S$, $W_2 \cap S$ is convex. For card $Q = 0$, the result follows from Theorem 5, and the proof of Theorem 6 is complete.

The number four in Theorem 6 is best possible, as the following example illustrates.

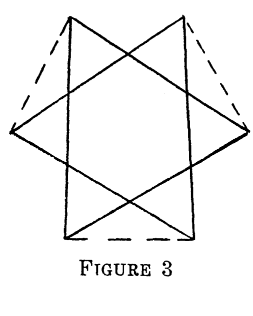

FIGURE 3

EXAMPLE 3. Let S denote the set in Figure 3, where dotted segments are in bdry (cl S) $\sim S$. Then S is a union of no fewer than four convex sets.

At last, using Theorem 6, we have a decomposition theorem for S an arbitrary 3-convex subset of the plane.

THEOREM 7. *The set S is a union of six or fewer convex sets. The result is best possible.*

Proof. By earlier comments, we may assume that S is connected, $\text{cl } S = \text{cl}(\text{int } S)$, and Q is finite. Furthermore, we assume $\text{int}(\text{cl } S) \sim S \neq \emptyset$, for otherwise the result is an immediate consequence of Theorem 6. Let $T \equiv S \cup \text{bdry}(\text{cl } S)$, and let L be the line containing $\text{cl } T \sim T$ described in Theorem 2 or Theorem 3 (whichever is appropriate). Clearly L may be chosen to contain an lnc point q of cl S. If L_1, L_2 are the corresponding open halfspaces, then each of $T_1 \equiv \text{cl}(T \cap L_1) = \text{cl}(S \cap L_1)$, $T_2 \equiv \text{cl}(T \cap L_2) = \text{cl}(S \cap L_2)$ is 3-convex.

Define $S_i \equiv T_i \cap S$, $i = 1, 2,$. We assert that each S_i is 3-convex: For x, y, z in $S_1 = T_1 \cap S$, assume $[x, y]$ lies in the 3-convex set S to show $[x, y] \subseteqq S_1$. If x or y is in L_1, then certainly $(x, y) \subseteqq L_1 \cap S \subseteqq T_1$, and $[x, y] \subseteqq S_1$. If x, y are on L, then since no lnc points of the closed set T_1 are on L, x, y lie in the same leaf of T_1, and $[x, y] \subseteqq T_1 \cap S = S_1$. Thus S_1 is 3-convex. Similarly S_2 is 3-convex. Moreover, $(\text{cl } S_i \sim S_i) \subseteqq \text{bdry}(\text{cl } S_i)$, $i = 1, 2$.

Using Theorem 6, we will show that each S_i is a union of three convex sets: By the proofs of Theorems 2 and 3, $\text{cl } S_1 = T_1$ is a union of two convex sets A_1, A_2, and each A_i may be considered a subset of an appropriate C_j set, $1 \leqq j \leqq 3$, where the C_j sets are those described in Valentine's paper with $\text{cl } S = C_1 \cup C_2 \cup C_3$. In case T_1 has one leaf or an even number of leaves, then clearly the proof of Theorem 6 may be used to write S_1 as a union of three convex sets. If T_1 has n leaves for n odd, $n > 1$, let V be the leaf of T_1 bounded by L, $q \in Q \cap L \subseteqq A_1 \cap A_2$. Order the lnc points of T_1 in a clockwise direction so that V is determined by q_n, q_1, and let U_n, U_{n+1} denote the closed subsets of V bounded by $L(q_n, q)$, $L(q, q_1)$ respectively. Treating $U_1, \cdots, U_n, U_{n+1}$ as leaves of T_1, U_i determined by lnc points q_i, q_{i+1}, $1 \leqq i < n$, the proof of Theorem 6 may be applied to write S_1 as a union of three convex sets. (Of course, in defining B_0, points of V in S_0 belong to the same leaf of S_0.)

By a parallel argument S_2 is a union of three convex sets, and $S = S_1 \cup S_2$ is a union of six or fewer convex sets, finishing the proof of the theorem.

Our final example shows that the bound of six in Theorem 7 is best possible.

EXAMPLE 4. Let S be the set in Figure 4, with dotted segments in $\text{bdry}(\text{cl } S) \sim S$ and $p \in \text{int}(\text{cl } S) \sim S$. Then S cannot be expressed as a union of fewer than six convex sets.

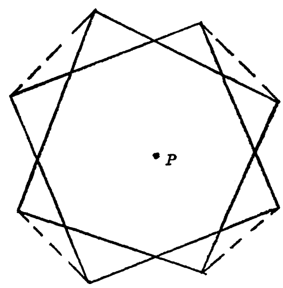

FIGURE 4

REFERENCES

1. J. F. Lawrence, W. R. Hare, and John W. Kenelly, *Finite unions of convex sets*, Proc. Amer. Math. Soc., **34** (1972), 225-228.
2. F. A. Valentine, *A three point convexity property*, Pacific J. Math., **7** (1957), 1227-1235.

Received May 15, 1973.

THE UNIVERSITY OF OKLAHOMA