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STRONGLY UNIQUE BEST APPROXIMATES TO A FUNCTION ON A SET, AND A FINITE SUBSET THEREOF

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Let X be a compact Hausdorff space and let $C(X)$ denote the space of continuous real valued functions defined on X, normed by the supremum norm $\|f\| = \max_{x \in X} |f(x)|$. Let M be a finite dimensional subspace of $C(X)$. This note examines the problem of whether every best (unique best, strongly unique best) approximate to f on X is also a best (respectively: unique best, strongly unique best) approximate to f on some finite subset of X. Appropriate converse results are also considered.

The Kolmogorov criterion for best approximates shows that $\pi \in M$ is a best approximate to f on X if and only if it is a best approximate to f on a finite subset of

$$E_\pi = \{x \in X : |f(x) - \pi(x)| = \|f - \pi\|\} .$$

Example 1 shows that the corresponding result does not hold for unique best approximates. It can easily be shown that when π is a strongly unique best approximate to f in $C[a, b]$ from a Haar subspace then there is a finite subset A of $[a, b]$ such that π is a strongly unique best approximate to f on A. In Theorem 2 the latter result is extended to an arbitrary finite dimensional subspace M of $C(X)$ and in Theorem 3 a converse is proven in this general setting.

The second algorithm of Remez [11] is an important method for the computation of the best approximate to a function f in $C[a, b]$ from a finite dimensional Haar subspace. This algorithm depends on the fact that a best approximate to f on $[a, b]$ is a best approximate to f on some finite subset of $[a, b]$. (One can think of the algorithm as a search for this subset.) In fact, the proof of the convergence of the algorithm given by E. W. Cheney [3] indicates that the algorithm depends more precisely on the facts that the best approximate π to f on $[a, b]$ is strongly unique and that π is also a strongly unique best approximate to f on some finite subset of $[a, b]$.

It would also be natural to consider in $L^p[a, b]$ for $1 \leqq p < \infty$ the relationship between strongly unique best approximates on $[a, b]$ and on finite subsets of $[a, b]$. However, D. E. Wulbert ([15], [16]) has shown that strong unicity does not occur (nontrivially) in any smooth space and $L^p[a, b]$ for $1 \leqq p < \infty$ is smooth. In the last section a different proof of Wulbert's result is given because the method of the proof enables one to study strong unicity in L^1. It should be observed (see Example 3) that even though there are no finite dimensional subspaces of $L^1[a, b]$ containing a unique best approximate to every f in $L^1[a, b]$, a given f in $L^1[a, b]$ may have a strongly unique best approximate.

The result mentioned above on the relationship between the best approximates to f on X and the best approximates to f on a finite subset of X can be found in [8], [13], and [18].

The results of this note hold with obvious modifications for the complex case.

2. DEFINITIONS. An element π in M is a best approximate to f in $C(X)$ if $\|f - m\| \geqq \|f - \pi\|$ for all m in M; π is a unique best approximate if the inequality is strict for all m in M, $m \neq \pi$; and π is a strongly unique best approximate to f if there exists a real number $r > 0$ such that $\|f - m\| \geqq \|f - \pi\| + r\|\pi - m\|$ for all m in M.

Let M have dimension n. The subspace M is called a Haar (Chebyshev) subspace if no nonzero function in M has more than $n - 1$ zeros in X. If X is the finite interval $[a, b]$, then M is called a weak Chebyshev subspace if no nonzero function in M has more than $n - 1$ sign changes on $[a, b]$. (For properties of Haar and weak Chebyshev systems, see e.g. [4], [5], [6], and [17].) In particular it is known that if M is a Haar subspace of $C[a, b]$ then π is a best approximate to f on a closed set X in $[a, b]$ (where X contains at least $n + 1$ points) if and only if there exists an equioscillation set for $f - \pi$, i.e., a subset A of X containing $n + 1$ points $x_1 < x_2 < \cdots < x_{n+1}$ such that $f(x_{i+1}) - \pi(x_{i+1}) = -[f(x_i) - \pi(x_i)]$, $i = 1, 2, \cdots, n$ and $|f(x_i) - \pi(x_i)| = \|f - \pi\|$, $i = 1, 2, \cdots, n + 1$.

One of the principal tools of the investigation is the following strong Kolmogorov criterion [2] characterizing strongly unique best approximates.

THEOREM. *Let M be finite dimensional. There exists a real number $r > 0$ such that*

$$\|f - m\| \geqq \|f - \pi\| + r\|\pi - m\| \forall m \in M$$

if and only if

$$\max_{x \in E_\pi} [f(x) - \pi(x)]m(x) > 0 \quad \forall m \in M, \quad m \not\equiv 0.$$

In proofs we assume without loss of generality that the best approximate to f is 0.

3. Results. The relationship between a strongly unique best approximate to a given f on $[a, b]$ and on a finite subset A of $[a, b]$ is especially simple when M is a Haar subspace. Recall that when M is a Haar subspace of $C[a, b]$ every f in $C(X)$, where X is a compact subset of $[a, b]$, has a strongly unique best approximate from M [9]. Hence by the strong Kolmogorov criterion we have the following result:

THEOREM 1. *Let π be a best approximate from the Haar subspace M of $C[a, b]$ to a given f in $C[a, b]$. Then for every equioscillation set $A \subseteqq E_\pi$,*

$$\max_{x \in A} [f(x) - \pi(x)]m(x) > 0 \quad \forall m \in M,\ m \not\equiv 0 .$$

If we only assume that π is a strongly unique best approximate from a weak Chebyshev subspace, then the conclusion of the previous theorem does not hold. For example, in $C[0, 4\pi]$ let $f(x) = \sin x$ and let M be the linear span of

$$g(x) = \begin{cases} 3\pi/2 - x & 0 \leqq x \leqq 3\pi/2 \\ 0 & 3\pi/2 \leqq x \leqq 5\pi/2 \\ 5\pi/2 - x & 5\pi/2 \leqq x \leqq 4\pi . \end{cases}$$

Then 0 is strongly unique to f since $\max_{x \in E_0} f(x)m(x) > 0$, $\forall m \in M$, $m \neq 0$, but $\max_{x \in A} f(x)(-g(x)) = 0$ where $A = \{5\pi/2, 7\pi/2\}$ is an equioscillation set for $f - 0$.

However, we now show that when π is a strongly unique best approximate from an arbitrary subspace M in $C(X)$, it follows that there does exist some finite subset A of E_π such that π is a strongly unique best approximate to f on A.

THEOREM 2. *Let π be a strongly unique best approximate from a subspace M of $C(X)$ to an element f in $C(X)$. Then there exists a finite subset A of E_π with $\leqq 2n$ points such that*

$$\max_{x \in A} [f(x) - \pi(x)]m(x) > 0 \quad \forall m \in M,\ m/A \not\equiv 0 .$$

Proof. Let M be the span of $\{g_1, \cdots, g_n\}$. Let $\hat{E}_0 = \{(f(x)g_1(x), \cdots, f(x)g_n(x)) \colon x \in E_0\}$. Then it follows ([2], Theorem 6) that 0 is in the interior of the convex hull of $\hat{E}_0$. Hence (see e.g. Theorem 3.13 in [14]) 0 is in the interior of the convex hull of $\hat{A}$, where $\hat{A}$ is a finite subset of $\hat{E}_0$ consisting of $\leqq 2n$ points. It follows ([2], Theorem 6) that 0 is a strongly unique best approximate to f on A. By the strong Kolmogorov criterion $\max_{x \in A} f(x)m(x) > 0$ for all m in M with $m/A \not\equiv 0$.

It is not known in general whether it is possible to find a finite set A satisfying the conditions of the previous theorem such that if m is in M and $m/A = 0$, then $m \equiv 0$. However, if E_π is finite then by setting $A = E_\pi$ one can add to the conclusion of Theorem 2 that $m/E_\pi = 0$ implies $m \equiv 0$. This follows from the strong Kolmogorov criterion. Also if E_π is not finite but it is known that any nonzero function in M has at most $N - 1$ zeros for some integer N (for example $N = n$ when M is a Haar set), then one can just add to the set A of the previous theorem enough points of E_π so that A has N or more points.

It would be of interest to determine whether the $2n$ of the theorem is in general best possible.

If π is a unique best approximate to f on X, then it does not follow that π is a unique best approximate to f on E_π. This can be seen in the next example which will also be used later.

EXAMPLE 1. Let M be the subspace of $C[0, 3\pi]$ spanned by $g_1(x) = 1$ and

$$g_2(x) = \begin{cases} \pi - x & 0 \leqq x \leqq \pi \\ 0 & \pi \leqq x \leqq 5\pi/2 \\ 5\pi/2 - x & 5\pi/2 \leqq x \leqq 3\pi \,. \end{cases}$$

Let $f(x) = \sin x$. Then M is a weak Chebyshev system, but it is not a Haar set on $[0, 3\pi]$. Because $f(x)$ has a horizontal tangent at $x = 5\pi/2$, the function $-g_2(x)$ is not as good an approximate to $f(x)$ as 0 is. Clearly then, 0 is a unique best approximate to f on $[0, 3\pi]$. Now $E_0 = \{\pi/2, 3\pi/2, 5\pi/2\}$. Since M has dimension 2, E_0 is an equioscillation set for $f - 0$ on $[0, 3\pi]$. Now 0 is not a unique best approximate on $E_0 = A$ since $g_2(x)$ is also a best approximate. Also observe that 0 is not a strongly unique best approximate to f on $[0, 3\pi]$ since $\max_{x \in E_0} f(x)[-g_2(x)] = 0$.

In fact even more holds. Let

$$g_3(x) = \begin{cases} x - \pi/2 & 0 \leqq x \leqq \pi/2 \\ 0 & \pi/2 \leqq x \leqq \pi \\ x - \pi & \pi \leqq x \leqq 3\pi/2 \\ 2(7\pi/4 - x) & 3\pi/2 \leqq x \leqq 7\pi/4 \\ x - 7\pi/4 & 7\pi/4 \leqq x \leqq 3\pi \,. \end{cases}$$

Then let M be the subspace of $C[0, 3\pi]$ spanned by $g_2(x)$ and $g_3(x)$, and let $f(x) = \sin x$. Then by consideration of the values of any $m \in M$ at points $\pi/2$, $3\pi/2$, and $5\pi/2$, it is easy to verify that zero is a unique best approximate to f on $[0, 3\pi]$ and $E_0 = \{\pi/2, 3\pi/2, 5\pi/2\}$.

Moreover on each subset A of E_0, there is a function $g \in M$ such that $g/A \not\equiv 0$ and g is a best approximate to f on A. Thus zero is not a unique best approximate to f on any finite subset A of E_0.

The next proposition summarizes the results for an arbitrary subspace M of $C(X)$. For the result on best approximates see [8], [13], and [18].

PROPOSITION. *If π is a best* (strongly unique best) *approximate to f on X, then there exists a finite subset A of X with less than or equal to $n + 1$* (resp. $2n$) *points such that π is a best* (strongly unique best) *approximate on A.*

REMARK. The Kolmogorov and strong Kolmogorov criteria and Example 1 also yield the relationship between the best approximate to f on X and on all of E_π. As expected, π is a best (strongly unique best) approximate to f on X if and only if it has the same property on E_π. This does not hold for a unique best approximate.

4. Converse results. The Kolmogorov criterion shows part (i) of the next theorem.

THEOREM 3. (i) *If π is a best approximate to f on a finite subset of E_π, then π is a best approximate to f on X.*

(ii) *If π is a unique* (strongly unique) *best approximate to f on a finite subset A of E_π, then π is a unique* (strongly unique) *best approximate to f on X, except possibly for those m in M with $m/A \equiv 0$.*

In fact more than this holds. The following result says that if π is a unique best approximate to f on a finite subset A of X, then π is also a strongly unique best approximate to f on A.

THEOREM 4. *Let π be a unique best approximate to f on a finite subset A of X. Assume $f(x) - \pi(x) \not\equiv 0$ on A. Then*

$$\max_{x \in A} [f(x) - \pi(x)]m(x) > 0 \qquad \forall m \not\equiv 0 \quad \textit{on} \quad A.$$

Proof. (We show that if $\max_{x \in A} f(x)q(x) \leqq 0$ for some $q \in M$, then there exists a real number $\lambda > 0$ such that $-\lambda q$ is a best approximate to f on A.) Let $A' = \{x \in A: f(x)q(x) < 0\}$. Let $\lambda > 0$ be such that both the following hold:

(1) $\lambda \max_{x \in A} |q(x)| < \|f\|$,

(2) $\lambda q^2(x) + 2f(x)q(x) < 0$ for all x in A'.

Notice that $H(\lambda) = \max_{x \in A'} \lambda q^2(x) + f(x)q(x)$ is a continuous function of λ with $H(0) < 0$. Since $A' \subseteqq A$ is finite such a λ can be chosen.

Now if $x \in A'$, then letting $\|f\|_A = \max_{x \in A} |f(x)|$ we have

$$(f(x) + \lambda q(x))^2 = (f(x))^2 + \lambda(\lambda q^2(x) + 2f(x)q(x)) < (f(x))^2 \leqq \|f\|_A^2 .$$

If $x \in A - A'$ and $q(x) = 0$, then $|f(x) + \lambda q(x)| = |f(x)| \leqq \|f\|_A$; whereas, if $q(x) \neq 0$, then $f(x) = 0$ and

$$|f(x) + \lambda q(x)| = \lambda |q(x)| < \|f\|_A .$$

Thus $|f(x) + \lambda q(x)| \leqq \|f\|_A$ for any x in A.

COROLLARY. *If π is a unique best approximate to f on a finite subset A of E and $m/A = 0$ implies $m \equiv 0$, then π is a strongly unique best approximate to f on X.*

It follows that if $\|f - m\|_A \geqq \|f - \pi\|_A + r\|\pi - m\|_A$ and $m/A = 0$ implies $m \equiv 0$, then $\|f - m\|_X \geqq \|f - \pi\|_X + r'\|\pi - m\|_X$. It would be of interest to determine the relationship between r and r' here and also in the situation under discussion in Theorem 2.

REMARK. When M is a weak Chebyshev set in $C[a, b]$ one expects to obtain better results than for a general subspace M, but this does not occur here. Indeed, if π is a unique best approximate to f on $[a, b]$, A is a set of equioscillation points and $m/A = 0$ implies $m \equiv 0$, then it need not follow that π is a strongly unique best approximate to f on $[a, b]$ as seen in Example 1. Of course if one also assumes that π is a unique best approximate to f on A, then the above theorem guarantees that π is a strongly unique best approximate to f on $[a, b]$. It should be observed that the proof given in [4] of the de La Vallée Poussin theorem when M is a Haar set also proves the result when M is only a weak Chebyshev set.

5. **Strong unicity in L^p, $1 \leqq p < \infty$.** Let W be a normed linear space with dual space W^*. Let M denote a subspace (not necessarily finite dimensional) of W. As shown in [2], the existence of a subspace M of W which gives strongly unique best approximates to elements of W depends on the character of W^*. To be more specific, let $\langle M, f\rangle$ denote the subspace of W spanned by M and f and let $\langle M, f\rangle^*$ be the dual space of $\langle M, f\rangle$. Also let

$$\mathscr{L}_\pi = \{L \in \langle M, f\rangle^* : L(f - \pi) = \|f - \pi\| \quad \text{and} \quad \|L\| = 1\} ,$$

and

$$K_\pi = \{z \in \langle M, f\rangle : Lz \leqq \|f - \pi\| \ \forall L \in \mathscr{L}_\pi\} .$$

Then ([2]) π is a strongly unique best approximate to f if and only if $K_\pi \cap M$ is bounded. If π is a best approximate to f, then ([2]) Haar's result ([4]) in an abstract setting implies that there is at least one element $L_\pi \in \mathscr{L}_\pi$ defined by $L_\pi(m + af) = a\|f - \pi\|$. Any element m in M is trivially its own strongly unique best approximate.

THEOREM 5. (Wulbert). *Let W be a smooth normed linear space. If M is a proper subspace of W and $f \in W - M$, then the best approximate to f from M is not strongly unique.*

Proof. Since W is smooth, $\mathscr{L}_0$ contains a unique linear functional which is L_0. Thus, $M \subseteqq K_0$ and $M \cap K_0$ is not bounded. Hence 0 is not a strongly unique best approximate.

Let μ be a σ-finite positive measure on a σ-algebra Σ of subsets of a set T. As usual let $L^p(T, \Sigma, \mu)$, $1 \leqq p < \infty$, (briefly L^p) denote the space of functions f on T such that $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p} < \infty$. Let $1/p + 1/q = 1$. Then L^p is smooth for $1 < p < \infty$. Of course, any finite dimensional subspace of L^p, $1 < p < \infty$ does contain a unique best approximate to every element in L^p. It follows that if M is a subspace of L^p, $1 < p < \infty$, then there is no $f \in L^p - M$ with a strongly unique best approximate.

The concept of an interpolating subspace was introduced in [1], where it was shown that if M is an interpolating subspace then M always contains a strongly unique best approximate to every element $f \in W$. Theorem 6 shows that [1] if W is a smooth normed linear space, then W contains no interpolating subspace. However, there are subspaces which are not interpolating, but from which every element has a strongly unique best approximate.

EXAMPLE 2. In l' let M be the subspace spanned by $(1, 0, 0, \cdots)$ and $(0, 1, 0, \cdots)$. Then [1] M is not an interpolating subspace. Given $f \in l'$, let π in M be given by $(f(1), f(2), 0, \cdots)$. Then for $m \in M$,

$$\begin{aligned}\|f - m\| &= |f(1) - m(1)| + |f(2) - m(2)| + \sum_{i>2} |f(i) - m(i)| \\ &\geqq \sum_{i>2} |f(i)| + r\{|\pi(1) - m(1)| + |\pi(2) - m(2)|\}\end{aligned}$$

where one can choose $r = 1$ to be the strong unicity constant.

The space L^1 contains a finite dimensional subspace M which contains a strongly unique best approximate to every element $f \in L^1 - M$ if and only if (T, Σ, μ) contains an atom ([1], [10]). To obtain further information about strong unicity in L^1, let $f \in L^1$, $\|f\| = 1$ and $f \notin M$. Assume without loss of generality that 0 is a best approximate to f and let $\mathscr{L}_0 = \{L \in \langle M, f\rangle^*: Lf = 1 = \|L\|\}$. For a given $L \in \mathscr{L}_0$, there exists by the Riesz Representation Theorem a function $h \in L^\infty$ such that

$$Lg = \int_T hgd\mu \ \forall g \in L^1 \quad \text{and} \quad \|L\| = \|h\|_\infty .$$

Thus for a given $L \in \mathscr{L}_0$ we have

$$(1) \qquad 1 = \int hfd\mu \leqq \int |h|\,|f|\,d\mu \leqq \|h\|_\infty \|f\|_1 = 1 .$$

The condition for equality in Hölders inequality implies that $|h|\,|f| = \|h\|_\infty |f| = |f|$ a.e. Also (1) shows that $hf = |h|\,|f|$ a.e. Thus $\mathscr{L}_0$ can be identified with

$$\{h \in L^\infty : |f|(|h| - 1) = 0 \text{ a.e.} \quad \text{and} \quad (hf)(1\text{-sgn}\,h\,\text{sgn}\,f) = 0 \text{ a.e.}\} .$$

This characterization of $\mathscr{L}_0$ can be used to study strong unicity in L^1. For example if $\mu\{x: f(x) = 0\} = 0$, then $|h| = 1$ a.e., sgn h sgn $f = 1$ a.e. and therefore h is uniquely determined a.e. Since $\mathscr{L}_0$ contains a unique element it follows as before that 0 is not a strongly unique best approximate to f. We have shown the following:

THEOREM 6. *Let f in $L^1(T, \Sigma, \mu)$ have a strongly unique best approximate π from a subspace M. Then* $\mu\{x: f(x) - \pi(x) = 0\} > 0$.

It should be pointed out that it is possible for an element $f \in L^1$ to have a strongly unique best approximate from a subspace M even when (T, Σ, μ) does not have an atom. It is not known whether a result like Theorem 2 exists for $L^1[a, b]$.

EXAMPLE 3. Let M be the constant functions, a subspace of $L^1[-2, 2]$. Let

$$f(x) = \begin{cases} x + 1 & -2 \leqq x \leqq -1 \\ 0 & -1 \leqq x \leqq 1 \\ x - 1 & 1 \leqq x \leqq 2 . \end{cases}$$

Then one can verify that

$$\|f - c\|_1 = \begin{cases} (|c| + 1)^2 & 1 \geqq |c| \geqq 0 . \\ 4|c| & |c| > 1 . \end{cases}$$

Thus 0 is a best approximate to f and also

$$\|f - c\|_1 \geqq \|f\|_1 + 1/2\|c\|_1 .$$

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