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A TWO-POINT BOUNDARY PROBLEM FOR NONHOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

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This paper is concerned with second order nonhomogeneous differential equations, together with boundary conditions specified at two points. The existence of eigenvalues is established and the oscillatory behavior of the associated eigenfunctions is studied. The results of this paper are obtained by considering the nonhomogeneous problem without regard for existence of solutions of the associated homogeneous boundary problem.

Consider the linear differential equation

$$(r(x, \lambda)y')' + q(x, \lambda)y = f(x, \lambda)\,, \tag{1}$$

and the associated homogeneous equation

$$(r(x, \lambda)u')' + q(x, \lambda)u = 0\,, \tag{2}$$

where $r(x, \lambda)$, $q(x, \lambda)$, and $f(x, \lambda)$ are real-valued functions on X: $a \leqq x \leqq b$, L: $\lambda_\# - \delta < \lambda < \lambda_\# + \delta$, $0 < \delta \leqq \infty$, $-\infty < a < b < \infty$. We shall consider (1) together with two-point boundary conditions of the form

$$\begin{aligned} &\text{(a)}\quad \alpha(\lambda)y(a, \lambda) - \beta(\lambda)(ry')(a, \lambda) = 0\,, \\ &\text{(b)}\quad \gamma(\lambda)y(b, \lambda) - \delta(\lambda)(ry')(b, \lambda) = 0\,. \end{aligned} \tag{3}$$

It is well known that for those values of λ for which the associated homogeneous boundary problem (2, 3) has no solution, the nonhomogeneous problem (1.3) yields a unique solution. Further, for those values of λ for which (2, 3) has a solution, the problem (1, 3) either has no solution or an infinite number of solutions.

In either case the homogeneous problem must be solved or shown to have only the trivial solution. This paper establishes the existence of characteristic values for (1, 3) independent of the corresponding reduced problem. The methods used will be analogous to those of W. M. Whyburn [6, 7, 8], and G. J. Etgen [2, 3].

The following hypotheses on the coefficients involved in the boundary problem will be assumed throughout:

(H_1) For each $x \in X$, each of $r(x, \lambda)$, $q(x, \lambda)$, and $f(x, \lambda)$ is continuous on L.

(H_2) For each $\lambda \in L$, each of $r(x, \lambda)$, $q(x, \lambda)$, and $f(x, \lambda)$ is measurable on X.

(H_3) There exists a Lebesgue integrable function $M(x)$ on X such that $|1/r(x, \lambda)| \leqq M(x)$, $|q(x, \lambda)| \leqq M(x)$, and $|f(x, \lambda)| \leqq M(x)$ on XL.

(H_4) $r(x, \lambda) > 0$ on XL.

(H_5) Each of the functions $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$, and $\delta(\lambda)$ is continuous on L.

(H_6) $\alpha^2(\lambda) + \beta^2(\lambda) > 0$ and $\gamma^2(\lambda) + \delta^2(\lambda) > 0$ on L. In particular, without loss of generality, we assume $\alpha^2(\lambda) + \beta^2(\lambda) \equiv 1$ on L.

(H_7) $\delta(\lambda) > 0$ on L. Also, without loss of generality, we assume $0 < \arcsin(\delta(\lambda_\sharp)/[\gamma^2(\lambda_\sharp) + \delta^2(\lambda_\sharp)]^{1/2}) < \pi$.

2. Preliminary definitions and results. Hypotheses $H_1 - H_3$ allow the application of fundamental existence and uniqueness theorems [1, Ch. 2] for differential equations to obtain the existence of a pair of solutions $\{u(x, \lambda), v(x, \lambda)\}$ of (2) such that $W(x, \lambda) \equiv 1$ on XL, where $W(x, \lambda) = r(x, \lambda)[v'(x, \lambda)u(x, \lambda) - u'(x, \lambda)v(x, \lambda)]$. Such a pair of solutions will be called a *normalized solution basis* of (2). It is now easily verified that given a normalized solution basis $\{u(x, \lambda), v(x, \lambda)\}$ of (2), every solution of (1) is of the form

$$\begin{aligned} y(x, \lambda) = &\left[c_1(\lambda) - \int_a^x f(t, \lambda)v(t, \lambda)dt\right]u(x, \lambda) \\ &+ \left[c_2(\lambda) + \int_a^x f(t, \lambda)u(t, \lambda)dt\right]v(x, \lambda) . \end{aligned} \tag{4}$$

Moreover, there exists a solution $y(x, \lambda)$ of (1) satisfying

$$y(a, \lambda) \equiv \beta(\lambda),\ (ry')(a, \lambda) \equiv \alpha(\lambda) \tag{5}$$

on L. In fact, if $\{u(x, \lambda), v(x, \lambda)\}$ is the normalized solution basis of (2) satisfying the initial conditions

$$\begin{aligned} u(a, \lambda) \equiv 1,\ v(a, \lambda) \equiv 0 , \\ (ru')(a, \lambda) \equiv 0,\ (rv')(a, \lambda) \equiv 1 , \end{aligned} \tag{6}$$

on L, then

$$\begin{aligned} y(x, \lambda) = &\left[\beta(\lambda) - \int_a^x f(t, \lambda)v(t, \lambda)dt\right]u(x, \lambda) \\ &+ \left[\alpha(\lambda) + \int_a^x f(t, \lambda)u(t, \lambda)dt\right]v(x, \lambda) \end{aligned} \tag{7}$$

satisfies (5). Thus the solution $y(x, \lambda)$ defined by (7) satisfies (3a).

We establish the existence of values of λ on L for which there corresponds a solution of (1) satisfying (3a, b). Such values are called *eigenvalues* of the respective boundary problem.

Let $\{u(x, \lambda), v(x, \lambda)\}$ be the normalized solution basis of (2) defined by (6). Applying the polar coordinate transformation, we obtain

$$u(x, \lambda) = \rho_1(x, \lambda) \sin \theta_1(x, \lambda),\ v(x, \lambda) = \rho_2(x, \lambda) \sin \theta_2(x, \lambda)\ ,$$
$$(ru')(x, \lambda) = \rho_1(x, \lambda) \cos \theta_1(x, \lambda),\ (rv')(x, \lambda) = \rho_2(x, \lambda) \cos \theta_2(x, \lambda)\ ,$$

where $\rho_i(x, \lambda)$ and $\theta_i(x, \lambda)$ are solutions of

$$(8)\quad \begin{aligned} \rho_i'(x, \lambda) &= \rho_i(x, \lambda)\left[\frac{1}{r(x, \lambda)} - q(x, \lambda)\right] \sin \theta_i(x, \lambda) \cos \theta_i(x, \lambda) \\ \theta_i'(x, \lambda) &= \frac{1}{r(x, \lambda)} \cos^2 \theta_i(x, \lambda) + q(x, \lambda) \sin^2 \theta_i(x, \lambda)\ , \end{aligned}$$

$i = 1, 2$, satisfying $\rho_1(a, \lambda) \equiv \rho_2(a, \lambda) \equiv 1$, $\theta_1(a, \lambda) \equiv \pi/2$, $\theta_2(a, \lambda) \equiv 0$ on L.

LEMMA 1. *The following inequality holds on XL: $0 < \theta_1(x, \lambda) - \theta_2(x, \lambda) < \pi$.*

Proof. Using the polar form of $u(x, \lambda)$ and $v(x, \lambda)$, it follows that

$$1 \equiv W(x, \lambda) = \rho_1(x, \lambda)\rho_2(x, \lambda) \sin [\theta_1(x, \lambda) - \theta_2(x, \lambda)]\ ,$$

where $W(x, \lambda) = r(x, \lambda)[v'u - u'v]$. Hence $\sin [\theta_1(x, \lambda) - \theta_2(x, \lambda)] = 1/\rho_1(x, \lambda)\rho_2(x, \lambda) > 0$ on XL. Since $\theta_1(a, \lambda) - \theta_2(a, \lambda) \equiv \pi/2$ on L, we have $0 < \theta_1(x, \lambda) - \theta_2(x, \lambda) < \pi$ on XL.

COROLLARY. *For each $x \in X$, the zeros of $u(x, \lambda)$ and $v(x, \lambda)$ separate each other on L.*

We can write (7) as $y(x, \lambda) = A(x, \lambda)u(x, \lambda) + B(x, \lambda)v(x, \lambda)$, where

$$(9)\quad \begin{aligned} A(x, \lambda) &= \beta(\lambda) - \int_a^x f(t, \lambda)v(t, \lambda)dt\ , \quad \text{and} \\ B(x, \lambda) &= \alpha(\lambda) + \int_a^x f(t, \lambda)u(t, \lambda)dt\ . \end{aligned}$$

It then follows that $y(\bar{x}, \lambda) = y'(\bar{x}, \lambda) = 0$ for some $\bar{x} \in X$ if and only if $A(\bar{x}, \lambda) = B(\bar{x}, \lambda) = 0$, [Lemma 3.3, Theorem 3.12; 5].

If for some $\lambda = \bar{\lambda}$, $y(b, \bar{\lambda}) = y'(b, \bar{\lambda}) = 0$, where the solution $y(x, \lambda)$ is defined by (7), then the boundary condition (3b) is satisfied and $\bar{\lambda}$ is an eigenvalue. We note this possibility could be ruled out if we assume that $\beta(\lambda) + 1 > \exp \int_a^b M(t)dt$ on L, where $M(t)$ is defined in H_3 [Theorem 3.4, 5]. So in the following we assume $y(b, \lambda)$ has no double zeros on L.

In order to establish the existence of eigenvalues for (1, 3a, b), we introduce the functions

$$(10)\quad \begin{aligned} U(x, \lambda) &= \gamma(\lambda)u(x, \lambda) - \delta(\lambda)(ru')(x, \lambda)\ , \\ V(x, \lambda) &= \gamma(\lambda)v(x, \lambda) - \delta(\lambda)(rv')(x, \lambda)\ , \end{aligned}$$

and

$$s(x, \lambda) = A(b, \lambda)U(x, \lambda) + B(b, \lambda)V(x, \lambda) , \quad t(x, \lambda) = A(b, \lambda)V(x, \lambda) - B(b, \lambda)U(x, \lambda) , \tag{11}$$

where $\{u(x, \lambda), v(x, \lambda)\}$ is the solution basis of (2) defined by (6), and where $A(b, \lambda)$ and $B(b, \lambda)$ are defined by (9).

It follows that

$$s^2(x, \lambda) + t^2(x, \lambda) = [A^2(b, \lambda) + B^2(b, \lambda)][U^2(x, \lambda) + V^2(x, \lambda)] .$$

Writing $u(x, \lambda)$ and $v(x, \lambda)$ in polar form, we have

$$s^2(x, \lambda) + t^2(x, \lambda) = (A^2(b, \lambda) + B^2(b, \lambda))(\gamma^2(\lambda) + \delta^2(\lambda)) \cdot (\rho_1^2(x, \lambda)\sin^2[\theta_1(x, \lambda) - \tau(\lambda)] + \rho_2^2(x, \lambda)\sin^2[\theta_2(x, \lambda) - \tau(\lambda)]) ,$$

where

$$\sin\tau(\lambda) = \delta(\lambda)/[\gamma^2(\lambda) + \delta^2(\lambda)]^{1/2} , \quad \text{and} \quad \cos\tau(\lambda) = \gamma(\lambda)/[\gamma^2(\lambda) + \delta^2(\lambda)]^{1/2} . \tag{12}$$

From Lemma 1, we have that $0 < [\theta_1(x, \lambda) - \tau(\lambda)] - [\theta_2(x, \lambda) - \tau(\lambda)] < \pi$ on XL, implying $\sin[\theta_1(x, \lambda) - \tau(\lambda)]$ and $\sin[\theta_2(x, \lambda) - \tau(\lambda)]$ cannot vanish simultaneously for any $x \in X, \lambda \in L$. Using H_6, we conclude that $U^2(x, \lambda) + V^2(x, \lambda) > 0$ on XL. By our assumption that $y(b, \lambda)$ has no double zeros on L, $A^2(b, \lambda) + B^2(b, \lambda) > 0$ on L. Consequently, $s^2(x, \lambda) + t^2(x, \lambda) > 0$ on XL and the complex-valued function $\Delta(x, \lambda)$, defined by

$$\Delta(x, \lambda) = (s(x, \lambda) + it(x, \lambda))/(s(x, \lambda) - it(x, \lambda)) \tag{13}$$

exists on XL.

THEOREM 1. *The complex-valued function $\Delta(x, \lambda)$ has the following properties on X for each $\lambda \in L$:*

(i) $|\Delta(x, \lambda)| = 1$.

(ii) *$\Delta(x, \lambda)$ satisfies the first order equation $d\Delta(x, \lambda)/dx = 2i\Delta(x, \lambda)h(x, \lambda)$, where*

$$h(x, \lambda) = (s(x, \lambda)t'(x, \lambda) - s'(x, \lambda)t(x, \lambda))/(s^2(x, \lambda) + t^2(x, \lambda)) . \tag{14}$$

(iii) *$\Delta(x, \lambda) = 1$ if and only if $t(x, \lambda) = 0$,*
$\Delta(x, \lambda) = -1$ if and only if $s(x, \lambda) = 0$.

(iv) *Let $\sigma(x, \lambda) = \arg \Delta(x, \lambda)$, where it is assumed that $0 \leqq \sigma(a, \lambda_\sharp) < 2\pi$ and that $\sigma(x, \lambda)$ is continued as a continuous function on XL. Then, for each fixed λ,*

$$2\int_a^x h(w, \lambda)dw = \sigma(x, \lambda) - \sigma(a, \lambda) . \tag{15}$$

(v) *If* $\gamma^2(\lambda)/r(x, \lambda) + \delta^2(\lambda)q(x, \lambda) > 0$ *on* XL, *then* $\Delta(x, \lambda)$ *moves monotonically and positively on the unit circle.*

Proof. Properties (i)-(iii) are easily verified. Equation (15) is a result of solving the first order equation in $\Delta(x, \lambda)$ and applying the definition of $\sigma(x, \lambda)$. To prove (v), we note that $h(x, \lambda) = (VU' - UV')/(U^2 + V^2) = (\gamma^2(\lambda)/r(x, \lambda) + \delta^2(\lambda)q(x, \lambda))(urv' - vru')/(U^2 + V^2)$. Since $urv' - vru' \equiv 1$, $h(x, \lambda) > 0$ on XL, and $\sigma(x, \lambda)$ is monotone increasing if $\gamma^2(\lambda)/r(x, \lambda) + \delta^2(\lambda)q(x, \lambda)$ is positive on XL.

Considering equation (13), we note that $\Delta(a, \lambda) = (B(b, \lambda) - iA(b, \lambda))/(B(b, \lambda) + iA(b, \lambda)) \neq 1$ on L. Thus $0 < \sigma(a, \lambda) < 2\pi$ on L, and

$$2\int_a^b h(w, \lambda)dw < \sigma(b, \lambda) < 2\int_a^b h(w, \lambda)dw + 2\pi \tag{16}$$

on L.

3. Existence of eigenvalues.

Using the results of the preceding section, we can now state an existence theorem for eigenvalues of (1, 3a, b).

THEOREM 2. *Let* $y(x, \lambda)$ *be the solution of* (1, 3a), *where* $y(x, \lambda)$ *is defined by* (7). *Define* $Q(\lambda)$ *by*

$$Q(\lambda) = 2\int_a^b h(w, \lambda)dw$$

($h(w, \lambda)$ *defined by* (14)). *Suppose* $\gamma^2(\lambda)/r(x, \lambda) + \delta^2(\lambda)q(x, \lambda) > 0$ *on* XL. *Then* $Q(\lambda) > 0$ *on* L. *Let* $m \geqq 0$ *be the least integer such that* inf $Q(\lambda) < (2m + 1)\pi$ *on* L, *and let* n *be an integer such that* sup $Q(\lambda) > (2n + 1)\pi$ *on* L. *If* $n \geqq m + 1$, *then there exist* p, $p = n - m$, *eigenvalues* $\lambda_0, \lambda_1, \cdots, \lambda_{p-1}$ *of* (1, 3a, b).

Proof. Let $y(x, \lambda)$ be the solution of (1) defined by (7). Let $U(x, \lambda)$, $V(x, \lambda)$, $s(x, \lambda)$, $t(x, \lambda)$, and $\Delta(x, \lambda)$ be defined as above.

If $\gamma^2(\lambda)/r(x, \lambda) + \delta^2(\lambda)q(x, \lambda) > 0$ on XL, then from Theorem 1, we know $\sigma(b, \lambda) - \sigma(a, \lambda) > 0$ on L, and $Q(\lambda) > 0$ on L.

Suppose that m and n are integers with the properties described in the hypothesis. Then there exists a value of λ, say λ^*, such that $Q(\lambda^*) < (2m + 1)\pi$ and a value of λ, say $\bar{\lambda}$, such that $Q(\bar{\lambda}) > (2n + 1)\pi$. Clearly, $\lambda^* \neq \bar{\lambda}$, and so we may assume $\lambda^* < \bar{\lambda}$. From (16), we have $Q(\lambda) < \sigma(b, \lambda) < Q(\lambda) + 2\pi$ on L. Therefore, $\sigma(b, \lambda^*) < (2m + 3)\pi$ and $\sigma(b, \bar{\lambda}) > (2n + 1)\pi$. Since $n = m + p$, $p \geqq 1$, there exist p values of λ, $\lambda_0, \lambda_1, \cdots, \lambda_{p-1}$ on $(\lambda^*, \bar{\lambda})$ such that $\sigma(b, \lambda_j) = [2(m + j) + 3]\pi$, for $j = 0, 1, \cdots, p - 1$. We assume that $\lambda_0 < \lambda_1 < \cdots < \lambda_{p-1}$ since $\sigma(b, \lambda)$ is continuous in λ. Now $\sigma(b, \lambda) = \arg \Delta(x, \lambda)$ implies that $\Delta(b, \lambda_j) = -1$ for $j = 0, 1, \cdots, p - 1$, and consequently, $s(b, \lambda_j) = 0$

for $j = 0, 1, \cdots, p-1$.

Considering (3b) we have

$$\gamma(\lambda)y(b, \lambda) - \delta(\lambda)(ry')(b, \lambda) = A(b, \lambda)U(b, \lambda) + B(b, \lambda)V(b, \lambda) = s(b, \lambda) \, .$$

Hence for λ_j, $j = 0, 1, \cdots, p-1$, the condition (3b) is satisfied and the λ_j are the eigenvalues for (1, 3a, b).

COROLLARY. *Under the hypotheses of Theorem 2, if the integer n can be chosen arbitrary large, then there exist infinitely many eigenvalues $\lambda_0, \lambda_1, \lambda_2, \cdots$ for* (1, 3a, b).

The following theorem also gives a criterion for the existence of eigenvalues for this nonhomogeneous boundary problem.

THEOREM 3. *Let $y(x, \lambda)$ be the solution of* (1, 3a) *defined by* (7). *Then $\theta_2(b, \lambda) - \tau(\lambda) > -\pi$ on L, where $\theta_2(x, \lambda)$ and $\tau(\lambda)$ are defined by* (8) *and* (12) *respectively. Suppose $A(b, \lambda) \neq 0$ on L, where $A(x, \lambda)$ is defined by* (9). *Let $m \geqq 0$ be the least integer so that* $\inf [\theta_2(b, \lambda) - \tau(\lambda)] < m\pi$ *on L, and let n be an integer such that* $\sup [\theta_2(b, \lambda) - \tau(\lambda)] > n\pi$ *on L. If $n \geqq m + 2$, then there exist at least $p - 1$, $p = n - m$, nonempty sets of eigenvalues $T_0, T_1, \cdots, T_{p-2}$ for the boundary problem* (1, 3a, b).

Proof. From (7), (9) and the polar representation for the normalized solution basis $\{u(x, \lambda), v(x, \lambda)\}$ of (2), defined by (6), we have $y(x, \lambda) = A(x, \lambda)\rho_1(x, \lambda) \sin \theta_1(x, \lambda) + B(x, \lambda)\rho_2(x, \lambda) \sin \theta_2(x, \lambda)$. Further, we can write the boundary condition (3b) in the form

$$\begin{aligned} P(\lambda) = [\gamma^2(\lambda) + \delta^2(\lambda)]^{1/2}\{&\rho_1(b, \lambda)A(b, \lambda) \sin [\theta_1(b, \lambda) - \tau(\lambda)] \\ &+ \rho_2(b, \lambda)B(b, \lambda) \sin [\theta_2(b, \lambda) - \tau(\lambda)]\} \, , \end{aligned} \tag{17}$$

where $\rho_i(x, \lambda)$, $\theta_i(x, \lambda)$, $i = 1, 2$ are defined by (8), and $\tau(\lambda)$ is defined by (12).

Since $\theta_2'(x, \lambda) = 1/r(x, \lambda) > 0$ when $v(x, \lambda) = 0$, $\theta_2(x, \lambda)$ is increasing at zeros of $v(x, \lambda)$, for each $\lambda \in L$. Moreover, $\theta_2(a, \lambda) \equiv 0$ implies $\theta_2(b, \lambda) > 0$ on L. Using (12) and H_7, we have $0 < \tau(\lambda) < \pi$ on L, and thus $\theta_2(b, \lambda) - \tau(\lambda) > -\pi$ on L.

Let m and n be integers with the properties described in the hypotheses. Then there exist values of λ, say λ^* and $\bar{\lambda}$, such that $\theta_2(b, \lambda^*) - \tau(\lambda^*) < m\pi$ and $\theta_2(b, \bar{\lambda}) - \tau(\bar{\lambda}) > n\pi$. Clearly, $\lambda^* \neq \bar{\lambda}$, so assume $\lambda^* < \bar{\lambda}$. Since $n = m + p$, $p \geqq 2$, there exist p values of λ, $\lambda_0, \lambda_1, \cdots, \lambda_{p-1}$, on $(\lambda^*, \bar{\lambda})$ such that $\theta_2(b, \lambda_j) - \tau(\lambda_j) = (m + j)\pi$, $j = 0, 1, \cdots, p-1$. From the continuity of $\theta_2(b, \lambda) - \tau(\lambda)$ on L, we may assume $\lambda_0 < \lambda_1 < \cdots < \lambda_{p-1}$.

By Lemma 1, $0 < \theta_1(b, \lambda) - \theta_2(b, \lambda) < \pi$ on L, and therefore $0 < [\theta_1(b, \lambda) - \tau(\lambda)] - [\theta_2(b, \lambda) - \tau(\lambda)] < \pi$ on L, and the zeros of $\sin[\theta_1(b, \lambda) - \tau(\lambda)]$ and $\sin[\theta_2(b, \lambda) - \tau(\lambda)]$ separate each other on L. Thus $\sin[\theta_1(b, \lambda_j) - \tau(\lambda_j)] > 0$ and $\sin[\theta_1(b, \lambda_{j+1}) - \tau(\lambda_{j+1}))] < 0$ for $j = 0, 1, \cdots, p - 2$, or vice versa.

Without loss of generality, assume that $A(b, \lambda) > 0$ on L. Then from (17), $P(\lambda_j) > 0$ and $P(\lambda_{j+1}) < 0$ for each j, or vice versa. In either case, since $P(\lambda)$ is continuous on L, there is a $\bar{\lambda}_j \in (\lambda_j, \lambda_{j+1})$ such that $P(\bar{\lambda}_j) = 0$, and $\bar{\lambda}_j$ is an eigenvalue for (1, 3a, b), $j = 0, 1, 2, p - 2$. Letting T_j be the set of all eigenvalues on $(\lambda_j, \lambda_{j+1})$, $j = 0, 1, \cdots, p - 2$ the theorem is proved.

COROLLARY 1. *Under the hypotheses of Theorem* 3, *if the integer n can be chosen arbitrarily large, then there exist infinitely many sets of eigenvalues $T_0, T_1, \cdots$ for* (1, 3a, b).

COROLLARY 2. *Suppose, in addition to the hypotheses of Theorem* 3, *that $A(x, \lambda) \neq 0$ on X for each $\lambda \in L$. Then there exist $p - 1$ nonempty sets of eigenvalues $J_0, J_1, \cdots, J_{p-2}$ for* (1, 3a, b) *such that if $\rho_j \in J_j$, $j = 0, 1, \cdots, p - 2$, then $\theta_2(b, \rho_j) - \tau(\rho_j) \geqq (m + j)\pi$. Moreover, if $j \geqq 1$, then the corresponding solution $y(x, \rho_j)$ has at least $j - 1$ zeros on X.*

Proof. We know that $\theta_2(b, \lambda) - \tau(\lambda)$ is continuous on L and increases from less than $m\pi$ to more than $n\pi$. Choose λ_j such that $\theta_2(b, \lambda) - \tau(\lambda) \geqq (m + j)\pi$ for $\lambda > \lambda_j$, and let J_j be the set of eigenvalues on $(\lambda_j, \lambda_{j+1})$. From Theorem 3, each J_j is nonempty.

If for fixed λ, $\theta_2(b, \lambda) - \theta_2(a, \lambda) \geqq q\pi$, then $v(x, \lambda) \equiv 0 \pmod{\pi}$ at least q times on X. Further, if $A(x, \lambda) \neq 0$ on X for each $\lambda \in L$, then by a generalization of a theorem by Leighton [Thm. 2.1, 4], we know that the zeros of $y(x, \lambda)$ and $v(x, \lambda)$ separate on X. Suppose $\rho_j \in J_j$, $j \geqq 1$. Then $\theta_2(b, \rho_j) - \tau(\lambda_j) \geqq (m + j)\pi$. Since $\theta_2(a, \lambda) \equiv 0$ on L and $\tau(\lambda) > 0$ on L, this implies that $\theta_2(b, \lambda_j) - \theta_2(a, \lambda_j) \geqq (m + j)\pi + \tau(\rho_j) \geqq (m + j)\pi \geqq j\pi$. We conclude that $v(x, \rho_j)$ has at least j zeros on X, and consequently, $y(x, \rho_j)$ has at least $j - 1$ zeros on X.

COROLLARY 3. *Suppose, in the hypotheses of Theorem* 3, *we assume that $A(b, \lambda)$ does not change sign on L, rather than be nonzero. Then the number of distinct eigenvalues for* (1, 3a, b) *is at least $(p - 1)/2$ if p is odd and at least $p/2$ if p is even.*

Proof. Paraphrasing Theorem 3, choose $\lambda_0, \lambda_1, \cdots, \lambda_{p-1}$ such that $\theta_2(b, \lambda_j) - \tau(\lambda_j) = (m + j)\pi$. Then $\sin[\theta_1(b, \lambda_j) - \tau(\lambda_j)] > 0$ and $\sin[\theta_1(b, \lambda_{i+1}) - \tau(\lambda_{j+1})] < 0$ for $j = 0, 1, \cdots, p - 2$, or vice versa.

Assuming, without loss of generality, that $A(b, \lambda) \geqq 0$ on L, we have $P(\lambda_j) \geqq 0$ and $P(\lambda_{j+1}) \leqq 0$, $j = 0, 1, \cdots, p-2$, or vice versa. In either case, there is a $\bar{\lambda}_j \in [\lambda_j, \lambda_{j+1}]$ such that $P(\bar{\lambda}_j) = 0$, and $\bar{\lambda}_j$ is an eigenvalue for (1, 3a, b). Let T_j be the set of eigenvalues on $[\lambda_j, \lambda_{j+1}]$, $j = 0, 1, \cdots, p-2$. Now it may happen that two sets T_j and T_{j+1} each contain only one eigenvalue, and moreover, that eigenvalue is a common eigenvalue, namely λ_{j+1}. We find, therefore, that the number of distinct eigenvalues for (1, 3a, b) is at least $(p-1)/2$ if p is odd, and at least $p/2$ if p is even.

We remark that the hypotheses of Theorem 3 require that $A(b, \lambda) = \beta(\lambda) - \int_a^b f(t, \lambda)v(t, \lambda)dt \neq 0$ on L. This can be verified if we assume

(i) $\beta(\lambda) + 1 > \exp \int_a^b M(t)dt$ on L,

[Thm. 3.4, 5], or

(ii) (a) $q(x, \lambda)$ is not identically zero on any subinterval of X for each $\lambda \in L$, and is not identically zero on any subinterval of L for each $x \in X$.

(b) $f(x, \lambda)/q(x, \lambda)$ is defined, integrable, nonpositive, and nondecreasing on X for each $\lambda \in L$.

(c) $\rho_1(b, \lambda) \geqq \rho_1(x, \lambda)$ on X for each $\lambda \in L$.

(d) $\beta(\lambda) > -2 \exp\left[\int_a^b M(t)dt\right] f(b, \lambda)/q(b, \lambda)$ on L,

[Thm. 3.5, 5].

Here $M(t)$ is the Lebesgue integrable bound of the functions $1/r(x, \lambda)$, $q(x, \lambda)$, and $f(x, \lambda)$.

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