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A NOTE ON STARSHAPED SETS, (k)-EXTREME POINTS AND THE HALF RAY PROPERTY

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Let S be a compact subset of R^d, $d \geqq 2$. S is said to have the half-ray property if for each point x of the complement of S there exists a half line with x as vertex having empty intersection with S. It is proven that S is starshaped iff S has the half-ray property and the intersection of the stars of the $(d-2)$-extreme points is not empty.

Let $S \subset R^d$. We say $x \in S$ is a *(k)-extreme* point of S provided for every $k+1$ dimensional simplex $D \subset S$, $x \notin \operatorname{relint} D$ where relint D denotes the interior of D relative to the $k+1$ dimensional space D generates. If $y \in S$ the symbol $S(y)$ is defined as $S(y) = \{z \mid z \in S$ and $[yz] \subset S\}$, where $[yz]$ denotes the closed line segment from y to z. The symbol $E(S)$ denotes the set of all $(d-2)$-extreme points of S. We say S is *starshaped* if $\operatorname{Ker} S \neq \emptyset$, where $\operatorname{Ker} S = \bigcap_{y \in S} S(y)$. In [1] the following is proved:

THEOREM 1. *Let $S \subset R^d$, $d \geqq 2$, be compact and starshaped. Then* $\operatorname{Ker} S = \bigcap_{x \in E(S)} S(x)$.

Theorem 1 certainly yields information about the structure of a starshaped set but at the same time raises several questions. First, has Theorem 1 a converse? Specifically, given that $\bigcap_{x \in E(S)} S(x) \neq \emptyset$, under what hypothesis will S be starshaped? Secondly, can the hypothesis of starshaped be replaced with a seemingly more general hypothesis? We answer the latter question in Theorem 2.

DEFINITION 1. Let $S \subset R^d$ and let $S^\sim$ be the complement of S. We say S has the half-ray property if and only if for every $x \in S^\sim$ there exists a half line l with x as vertex such that $l \cap S = \emptyset$.

THEOREM 2. *Let $S \subset R^d$, $d \geqq 2$, be compact and suppose $\bigcap_{x \in E(S)} S(x) \neq \emptyset$. Then the following are equivalent:*

(1) *S has the half-ray property.*

(2) $\operatorname{Ker} S = \bigcap_{x \in E(S)} S(x)$.

Since for any starshaped set S, S has the half-ray property and $\bigcap_{x \in E(S)} S(x) \neq \emptyset$, the implication $(1) \Rightarrow (2)$ generalizes Theorem 1. Further, the implication $(1) \Rightarrow (2)$ is a type of converse since we assume $\bigcap_{x \in E(S)} S(x) = \emptyset$ and obtain as a conclusion, rather than a hypothesis, that S is starshaped. As a corollary to Theorem 2, *we obtain a new characterization for starshaped sets.*

COROLLARY 1. *Let* $S \subset R^d$, $d \geqq 2$, *be compact. Then the following are equivalent*:

(1) S *is starshaped.*

(2) $\bigcap_{x \in E(S)} S(x) \neq \emptyset$ *and* S *has the half-ray property.*

2. *Proof of Theorem* 2. In the proof the symbol $\| \; \|$ denotes the Euclidean norm and the symbol $[ab_\infty)$ denotes the half line determined by the points a and b with a as vertex.

$(2) \Rightarrow (1)$. This follows immediately since any starshaped set has the half-ray property.

$(1) \Rightarrow (2)$. Let $y \in \bigcap_{x \in E(S)} S(x)$ and we show $y \in \operatorname{Ker} S$. Suppose $y \notin \operatorname{Ker} S$. Then there exists $z \in S$ such that $[yz] \not\subset S$. Let $a \in [yz] \sim S$. Without loss of generality, suppose a is the origin, O_v. By hypothesis there exists a half line $l = [0_v b_\infty)$ with $[0_v b_\infty) \cap S = \emptyset$. Let Q be the two dimensional subspace spanned by y and b. Now rotate l in Q so that the angle between l and $[0_v z_\infty)$ (which is already less than π) decreases. Cease the rotation when S is intersected and let the rotated half line be l^*. Note $l^* \cap S$ is compact and hence $\theta = \sup \{\|x\| \mid x \in l^* \cap S\}$ exists. Let $x \in l^* \cap S$ be such that $\|x\| = \theta$. We claim $x \in E(S)$. Suppose not. Then $x \in \operatorname{relint} D$ where D is a $d-1$ dimensional simplex in S. Since $x \in D \cap Q$, $\dim (D \cap Q) \geqq 1$. For each $z \in D$, $z \neq x$ let $[zx_\infty) \cap D$ be $[ze_z]$ and note $x \in (ze_z)$. Let $w \in D \cap Q$, $w \neq x$. Note $[we_w] \subset Q$. Now, if $[we_w] \subset l^*$, we contradict the definition of x since $x \in (we_w)$ and if $[we_w] \not\subset l^*$, we contradict the definition of l^*. Thus, $x \in E(S)$. Then $[xy] \subset S$ and this contradicts the definition of l^*. Thus, $y \in \operatorname{Ker} S$ and we are done.

In conclusion, we remark that a triangle in E^2 is an example of a nonstarshaped set for which $\bigcap_{x \in E(S)} S(x) \neq \emptyset$ and which does not have the half-ray property. The latter shows that in the implication $(1) \Rightarrow (2)$ of Theorem 2 the hypothesis of S having the half-ray property cannot be deleted.

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REFERENCE

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