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PROBABILITIES OF WIENER PATHS CROSSING DIFFERENTIABLE CURVES

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Let $\{W(t); t \geqq 0\}$ be the standard Wiener process. The probabilities $P[\sup_{0 \leqq t \leqq T} W(t) \geqq b]$ and $P[\sup_{0 \leqq t \leqq T} W(t) - at \geqq b]$ are well known. This paper gives the probabilities of the type $P[\sup_{0 \leqq t \leqq T} W(t) - f(t) \geqq b]$ for a large class of differentiable functions $f(t)$ by the use of integral equation techniques.

1. Introduction. Let $\{W(t), t \geqq 0\}$ be the standard Wiener process such that (i) $P[W(0) = 0] = 1$, (ii) $EW(t) = 0$ for all $t \geqq 0$, and (iii) $\mathrm{Cov}\,[W(s), W(t)] = \min(s, t)$. It is well known that for $b \geqq 0$

$$P[\sup_{0 \leqq t \leqq T} W(t) \geqq b] = 2P[W(T) \geqq b] = 2\Psi(bT^{-1/2}) \tag{1.1}$$

where

$$\Psi(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-u^2/2)du ,$$

and that

$$\begin{aligned} &P[\sup_{0 \leqq t \leqq T} W(t) - at \geqq b] \\ &= \Psi[(aT + b)T^{-1/2}] + \exp(-2ab)\Phi[(aT - b)T^{-1/2}] , \end{aligned} \tag{1.2}$$

where $\Phi(x) = 1 - \Psi(x)$.

The identity (1.1) can be found in [2:392], [5:286], and [11:256] while the identity (1.2) can be found in [6], [7:348-349], and [9:80-82]. Doob [3:397-399] gives a very interesting proof of (1.2) for $T = \infty$ case only. Shepp's proof for (1.2) is based on his transformation theorem in [7]. Cameron-Martin translation theorem in [1] also gives the same result using Shepp's argument.

The main purpose of this paper is to find the probability $P[\sup_{0 \leqq t \leqq T} W(t) - f(t) \geqq b]$ for a large class of functions $f(t)$ differentiable in $(0, T]$, which is a generalization of the results (1.1) and (1.2). Durbin [4] gave an integral equation whose solution would be the required probability. However, it turned out to be that his integral equation could not be solved analytically, and hence he presented a numerical approximation method. After that Smith [8] introduced some new techniques to obtain an approximation for the probability. The present authors' integral equation gives explicit expression for the solution, while Durbin's and Smith's do not.

2. Statement of the result and proof.

THEOREM. *For each $T > 0$ let $f(t)$ be continuous on* $[0, T]$, *differentiable in* $(0, T)$, *and satisfy* $|f'(t)| \leqq C/t^p$ $(p < 1/2)$ *for some constant* C. *Then the probability* $P[\sup_{0 \leqq t \leqq T} W(t) - f(t) \geqq b] \equiv F(T)$ *is one if* $f(0) + b \leqq 0$, *and otherwise it is given as the unique continuous solution of the integral equation*

$$F(T) = 2\Psi[(f(T) + b)T^{-1/2}] - 2\int_0^T F(t)M(T, t)dt \,, \tag{2.1}$$

where

$$\Psi(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-u^2/2)du$$

and

$$M(z, t) = \begin{cases} (2\pi)^{-1/2}\dfrac{\partial}{\partial t}\displaystyle\int_{-\infty}^{[f(z)-f(t)](z-t)^{-1/2}} \exp(-u^2/2)du, & (0 \leqq t < z \leqq T) \\ 0\,, \quad (0 \leqq z \leqq t \leqq T)\,. \end{cases} \tag{2.2}$$

More precisely for $f(0) + b > 0$

$$\begin{aligned} &P[\sup_{0 \leqq t \leqq T} W(t) - f(t) \geqq b] \\ &= h(T) + \sum_{n=1}^\infty 4^n \int_0^T K_n(T, t)h(t)dt \,, \end{aligned} \tag{2.3}$$

where

$$h(T) = 2\Psi[(f(T) + b)T^{-1/2}] - 4\int_0^T M(T, t)\Psi[(f(t) + b)t^{-1/2}]dt \,,$$

$$K_1(T, t) = \int_t^T M(T, z)M(z, t)dz \,,$$

and

$$K_{n+1}(T, t) = \int_t^T K_n(T, z)K_1(z, t)dz \,.$$

Proof. If $f(0) + b \leqq 0$, then since $W(0) = 0$ a.s., it is obvious that the probability is one. Now, let $\tau = \tau(\omega)$ be the first hitting time of the curve $f(t) + b$ by the sample path $W(t, \omega)$, that is to say that $W(\tau, \omega) = f(\tau) + b$, and if $0 \leqq t < \tau$, then $W(t, \omega) < f(t) + b$. If $W(t, \omega)$ never reaches the curve $f(t) + b$, then we simply set $\tau = \infty$. Thus

$$\begin{aligned} F(T) = {} & P[W(T) \geqq f(T) + b] \\ & + P[\sup_{0 \leqq s \leqq T} W(s) - f(s) \geqq b,\ W(T) < f(T) + b] \,. \end{aligned}$$

Using the fact that $P[\tau \leqq t] = P[\sup_{0 \leqq s \leqq t} W(s) - f(s) \geqq b] \equiv F(t)$ and the notation in the theorem, we obtain

$$\begin{aligned} F(\mathrm{T}) &= \Psi[(f(T)+b)T^{-1/2}] \\ &\quad + \int_0^T P[W(T) < f(T)+b \mid \tau = t]dF(t) \\ &= \Psi[(f(T)+b)T^{-1/2}] \\ &\quad + \int_0^T P[W(T) - W(t) < f(T) - f(t) \mid \tau = t]dF(t) . \end{aligned}$$

Since the increment $W(T) - W(t)$ is independent of the condition $\tau = t$, it follows that

$$\begin{aligned} F(T) &= \Psi[(f(T)+b)T^{-1/2}] \\ &\quad + \int_0^T \Phi[(f(T) - f(t))(T-t)^{-1/2}]dF(t) , \end{aligned}$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-u^2/2)du$. As $\lim_{t\uparrow T}[f(T) - f(t)](T - t)^{-1/2} = 0$, integration by parts yields (interpreting the integral in improper sense)

$$F(T) = \Psi[(f(T)+b)T^{-1/2}] + \frac{1}{2}F(T) - \int_0^T F(t)M(T, t)dt ,$$

from which (2.1) follows.

To solve the integral equation (2.1) rewrite $M(z, t)$ by the use of (2.2)

$$M(z, t) = \begin{cases} (2\pi)^{-1/2}(z-t)^{-1/2}\left[-f'(t) + \dfrac{f(z) - f(t)}{2(z-t)}\right]\exp\left\{-\dfrac{[f(z)-f(t)]^2}{2(z-t)}\right\} \\ \qquad\qquad \text{if } \ 0 \leqq t < z \leqq T , \\ 0 \qquad\quad \text{if } \ 0 \leqq z \leqq t \leqq T . \end{cases} \tag{2.4}$$

Apparently $M(z, t)$ is not square integrable on $[0, T]^2$. Hence the integral equation (2.1) can not be solved by usual methods for Volterra integral equations of the second kind (see Tricomi [10, pp. 10–15]). However, using the expression (2.1) for $F(t)$ in the right-hand side of (2.1), we can rewrite (2.1) as:

$$F(T) = G(T) - 2\int_0^T M(T, z)\left[G(z) - 2\int_0^z F(t)M(z, t)dt\right]dz ,$$

where $G(T) = 2\Psi[(f(T)+b)T^{-1/2}]$. Thus the change of order of integration gives

$$\begin{aligned} F(T) &= G(T) - 2\int_0^T M(T, t)G(t)dt \\ &\quad + 4\int_0^T F(t)\left[\int_t^T M(T, z)M(z, t)dz\right]dt . \end{aligned} \tag{2.5}$$

Now, using the conditions on $f(T)$ in the theorem and the Mean

Value Theorem, we obtain from (2.4) with suitable constants C_1 and C_2

$$\begin{aligned}
&\left|\int_t^T M(T, z)M(z, t)dz\right| \\
&\leqq C_1\int_t^T (T-z)^{-1/2}(z-t)^{-1/2}\left[|f'(z)|+\frac{C}{2}z^{-p}\right]\left[|f'(t)|+\frac{C}{2}t^{-p}\right]dz \\
&\leqq C_2 t^{-p}\int_t^T (T-z)^{-1/2}(z-t)^{-1/2}z^{-p}dz\ .
\end{aligned}$$

The substitution $z = t + (T-t)u$ in the above yields

$$\begin{aligned}
\left|\int_t^T M(T, z)M(z, t)dz\right| &\leqq C_2 t^{-p}\int_0^1 (1-u)^{-1/2}u^{-1/2}[uT+(1-u)t]^{-p}du \\
&\leqq C_2 t^{-p}T^{-p}\int_0^1 (1-u)^{-1/2}u^{-1/2}u^{-p}du \\
&\leqq (\text{const.})t^{-p}T^{-p}\ .
\end{aligned}$$

Thus the kernel $\int_t^T M(T, z)M(z, t)dz$ in the integral equation (2.5) is indeed square integrable for any $p < 1/2$, and hence the integral equation has a unique continuous solution for $F(T)$, and the solution is given by (2.3) (see Tricomi [10, pp. 5–8]).

REMARK. In some special cases of $f(t)$ the integral equation in the theorem can be solved more directly.

Case 1. If $f(t) \equiv c$ in the theorem, then $M(T, t) \equiv 0$ and hence $F(T) = 2\,\Psi[(c+b)T^{-1/2}]$ which agrees with (1.1).

Case 2. If $f(t) = at$, then

$$\begin{aligned}
M(T, t) &= (2\pi)^{-1/2}\frac{\partial}{\partial t}\int_{-\infty}^{a\sqrt{T-t}} \exp(-u^2/2)du \\
&= \frac{-a}{2(2\pi)^{1/2}}(T-t)^{-1/2}\exp[-a^2(T-t)/2] \equiv N(T-t),\ 0 \leqq t < T\ .
\end{aligned}$$

If we set $G(T) \equiv 2\Psi[(aT+b)T^{-1/2}]$, then the integral equation becomes

$$F(T) = G(T) - 2\int_0^T F(t)N(T-t)dt\ .$$

Taking the Laplace transform $\left(L[F(T)] = \int_0^\infty e^{-sT}F(T)dT\right)$ of both sides, we get

$$L[F(T)] = L[G(T)] - 2L[F(T)]L[N(T)]\ ,$$

or

$$\begin{aligned} L[F(T)] &= L[G(T)]/\{1 + 2L[N(T)]\} \\ &= s^{-1} \exp\left[-ab - b(2s + a^2)^{1/2}\right] . \end{aligned}$$

Therefore,

$$F(T) = 1 - \Phi[(aT + b)T^{-1/2}] + \exp(-2ab)\Phi[(aT - b)T^{-1/2}]$$

which agrees with (1.2).

REFERENCES

1. R. H. Cameron and W. T. Martin, *On transformations of Wiener integrals under a general class of linear transformations*, Ann. Math., **45** (1944), 386-396.
2. J. L. Doob, *Stochastic Processes*, John Wiley & Sons, Inc., 1953.
3. ———, *Heuristic approach to the Kolmogorov-Smirnov theorems*, Ann. Math. Stat., **20** (1949), 393-403.
4. J. Durbin, *Boundary-crossing probabilities for the Brownian motion and poisson processes and techniques fo computing the power of the Kolmogorov-Smirnov test*, J. Appl. Prob., **8** (1971), 431-453.
5. I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, W. B. Saunders Co., 1969.
6. S. Malmquist, *On certain confidence contours for distribution functions*, Ann. Math. Stat., **25** (1954), 523-533.
7. L. A. Shepp, *Radon-Nikodym derivatives of Gaussian measures*, Ann. Math. Stat., **37** (1966), 321-354.
8. C. S. Smith, *A note on boundary-crossing probabilities for the Brownian motion*, J. Appl. Prob., **9** (1972), 857-861.
9. L. Takacs, *Combinatotial Methods in the Theory of Stochastic Processes*, John Wiley & Sons, Inc., 1967.
10. F. G. Tricomi, *Integral Equations*, Interscience Publishers, Inc., 1957.
11. H. G. Tucker, *A Graduate Course in Probability*, Academic Press, 1967.

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