

EQUICONVERGENCE OF DERIVATIONS

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This paper is a study of bounded point derivations on the classical Banach algebras of analytic functions of a complex variable. The results are positive in character. The higher-order Gleason metrics d^p of $R(X)$ are introduced and conditions are studied under which convergence takes place with respect to these metrics. In particular, if $R(X)$ admits a p th-order bounded point derivation at a point $x \in \partial X$ and \dot{X} satisfies a cone condition at x , then $d^p(y, x)$ tends to 0 as y tends to x along the midline of the cone. Similar results hold for the other classical function algebras. In the case of the algebra $H^\infty(U)$, for open $U \subset C$, the analogous results hold only for regular derivations (a regular p th-order derivation maps z^p to a nonzero complex number). The points of the maximal ideal space of $H^\infty(U)$ at which regular bounded point derivations exist are characterized in terms of analytic capacity, following Hallstrom.

1. Let x be a point of the plane C and A be a class of functions analytic in a disc D centered at x , each function having modulus bounded by 1. Then, as is clear from Cauchy's integral formula, the family $\{f' \mid f \in A\}$ is equicontinuous at x , and for every sequence $\{x_n\} \rightarrow x$, the sequence $\{f'(x_n)\}$ converges to $f'(x)$, uniformly on A , i.e., $\{f'(x_n)\}$ is equiconvergent to $f'(x)$. More generally, for any integer $p \geq 1$, $\{f^{(p)}(x_n)\}$ is equiconvergent to $f^{(p)}(x)$.

Now, given a C -algebra A of continuous functions on a compact set $X \subset C$ which are analytic on \dot{X} , it is often possible to find points on ∂X at which nonzero point derivations exist on A . A (first order) point derivation at $x \in X$ on A is a linear functional $D: A \rightarrow C$ such that

$$D(fg) = f(x)Dg + g(x)Df,$$

whenever $f, g \in A$. This notion generalizes that of derivative at a point. For points $y \in \dot{X}$ all point derivations are of the form $f \rightarrow \alpha f'(y)$ for some complex constant α (independent of f) provided A contains the polynomials. Suppose A contains the identity map z and D is a normalized point derivation at x on A , i.e., $Dz = 1$. A natural question is:

Q1. When is there a sequence of points $x_n \in \dot{X}$, converging to x , such that the sequence $\{f'(x_n)\}$ converges to Df for all $f \in A$?

A bounded point derivation is a point derivation that is continuous

with respect to the uniform norm on X . If A admits a bounded point derivation D at a point x we may ask:

Q2. Can we find $x_n \rightarrow x$, $x_n \in \overset{\circ}{X}$, such that $f'(x_n)$ is equiconvergent to Df on $A_1 = A \cap \{f \mid \|f\|_X \leq 1\}$?

We shall concern ourselves with Q2, which lends itself to treatment by Banach algebra techniques.

2. We treat first the case $A = R(X)$, the uniform closure on X of $R_0(X)$, the class of rational functions with poles off X . $R(X)$ is a function algebra on X [2, p. 2]. The Gleason metric d^0 on X , with respect to $R(X)$, is defined by

$$d^0(x, y) = \sup \{ \|f(x) - f(y)\| \mid f \in R_0(X), \|f\|_X \leq 1 \},$$

for $x, y \in X$. Here $\|f\|_X$ denotes the sup norm of f on X . The properties of X with respect to this metric have been thoroughly investigated. An account may be found in [2], [4]. If x and y belong to the same component of $\overset{\circ}{X}$, then $d^0(x, y) < 2$. If x is a peak point for $R(X)$, then $d^0(x, y) = 2$ whenever $y \neq x$. This prompted the definition of Gleason part. A part P of the algebra $R(X)$ is a subset of X which forms an equivalence class under the relation $x \sim y \iff d^0(x, y) < 2$. The structure of parts can be very complicated. Davie has shown that P may be disconnected, and the Swiss cheese example shows that P may have no interior (cf. [4]). However, a nontrivial part (a part which does not just consist of one peak point) has full area density at each of its points, and in fact Browder [2, p. 177] has shown that every Gleason ball $\{x \in X \mid d^0(x, a) < \varepsilon\}$ ($\varepsilon > 0$) about a nonpeak point a has full area density at a .

In particular, a is not isolated in the part metric d^0 , and there is a sequence of points $x_n \in P \setminus \{a\}$ which converges to a simultaneously in the Euclidean and Gleason metrics. In plain language, as $n \rightarrow +\infty$, $|x_n - a| \rightarrow 0$, and $\{f(x_n)\}$ is equiconvergent to $f(a)$ for $f \in R_0(X) \cap \{f \mid \|f\|_X \leq 1\} = R_0(X, 1)$.

For $p \geq 1$ we define the p th order Gleason metric on X by

$$d^p(x, y) = \sup \{ \|f^{(p)}(x) - f^{(p)}(y)\| \mid f \in R_0(X, 1) \},$$

for $x, y \in X$.

The first thing to note is that $d^p(x, y)$ may be $+\infty$, so we are using the word "metric" a little loosely. An ordinary metric may be obtained from d^p by composing it with the arctangent function, but we would rather not do this. We extend d^p to $C \times C$ by writing

$d^p(x, y) = d^p(y, x) = +\infty$ whenever one of the elements x, y fails to be in X .

For $p \geq 0$ we say that a (normalized) p th order bounded point derivation on $R(X)$ exists at a point $x \in X$ if and only if the functional $f \rightarrow f^{(p)}(x)$ on $R_0(X)$ extends to a continuous linear functional D_x^p on $R(X)$, i.e., if and only if

$$s^p(x) = \sup \{ \| f^{(p)}(x) \| \mid f \in R_0(X, 1) \} = \| D_x^p \|$$

is finite. Suppose this happens, and x_n is a sequence of points of \dot{X} tending to x (in Euclidean norm). Then to say that $f^{(p)}(x_n) \rightarrow D_x^p f$ equiconvergently on $R(X, 1)$ is the same thing as saying that $d^p(x_n, x) \rightarrow 0$.

Notice that the two definitions so far available for a normalized first order bounded point derivation on $R(X)$ agree.

For purposes of computation it is usually easier to work with the function d_0^p , defined by

$$d_0^p(x, y) = \sup \{ \| f^{(p)}(y) \| \mid f \in R_0(X, 1) \text{ and } f(x) = f'(x) = \dots = f^{(p)}(x) = 0 \}.$$

3. The elementary properties of the functions d^p, s^p, d_0^p are summarized in the following theorem. Here, as usual, p is a non-negative integer.

THEOREM 1. *Let $x, y \in C$. Then*

- (1) $|s^p(x) - s^p(y)| \leq d^p(x, y) \leq s^p(x) + s^p(y)$;
- (2) $d^p(x, y) \geq (p + 1)! |x - y| / (\text{diam } X)^{p+1}$;
- (3) for $x \in \dot{X}$,

$$s^{p+1}(x) = \lim_{y \rightarrow x} \frac{d^p(x, y)}{|x - y|};$$

(4) for each compact subset K of a component of \dot{X} there is a constant $L > 0$ such that

$$d^p(x, y) \leq L |x - y|,$$

for $x, y \in K$, so d^p is continuous on \dot{X} ;

- (5) s^p is continuous on \dot{X} ;
- (6) $d_0^p(x, y) \leq d^p(x, y) \leq \{1 + \exp(\text{diam } X)\} \{\sup_{0 \leq \nu \leq p} s^\nu(x)\} d_0^p(x, y)$;
- (7) if X_n is a decreasing sequence of compact sets, each containing X in its interior, whose intersection is X , then $s_n^p \uparrow s^p$ and $d_n^p \uparrow d^p$, where s_n^p and d_n^p are respectively, the s^p -function and the d^p -function associated with X_n ;

- (8) s^p and d^p are lower semi-continuous;
 (9) if $|x_n - x| \rightarrow 0$ and $\{s^p(x_n)\}$ is a bounded sequence, then $s^p(x)$ is finite;
 (10) if $s^p(w) < +\infty$ for some $w \neq x$, then $s_p(x) = +\infty$ if and only if $d^p(x, y) = +\infty$ for every $y \neq x$;
 (11) x is an interior point of X if and only if

$$\sup_{n \geq 1} \left[\frac{1}{n} \log \frac{s_n(x)}{n!} \right] < +\infty .$$

Proof.

(1) is clear.

(2): Take $f(z) = (z - y)^{p+1}/(\text{diam } X)^{p+1}$. Then $f \in R_0(X, 1)$, so

$$d^p(x, y) \geq |f^{(p)}(x) - f^{(p)}(y)| .$$

(3) requires a lengthy but straightforward argument, using the Cauchy integral formula.

(4) follows from (3), using compactness.

(5) follows from (1) and (4).

(6): For the second inequality, let $f \in R_0(X, 1)$, and form

$$g(z) = f(z) - \sum_{\nu=0}^p \frac{f^{(\nu)}(x)}{\nu!} (z - x)^\nu .$$

Then $g(x) = g'(x) = \dots = g^{(p)}(x) = 0$, and

$$\begin{aligned} \|g\|_x &\leq 1 + \sum_{\nu=0}^p \frac{s^\nu(x)}{\nu!} (\text{diam } X)^\nu \\ &\leq \left\{ 1 + \sum_{\nu=0}^p \frac{(\text{diam } X)^\nu}{\nu!} \right\} \left\{ \sup_{0 \leq \nu \leq p} s^\nu(x) \right\} \\ &\leq \{1 + \exp(\text{diam } X)\} \left\{ \sup_{0 \leq \nu \leq p} s^\nu(x) \right\} . \end{aligned}$$

(7) follows from the fact that each $f \in R_0(X, 1)$ belongs to every $R(X_n)$ from some point on.

(8): By (4), (5), and (7), s^p and d^p are increasing limits of continuous functions.

(9): Take $X_m \downarrow X$ as in (7). For each m , $x \in \dot{X}_m$, so by (5),

$$\begin{aligned} s_m^p(x) &\leq \sup_{n \geq 1} s_m^p(x_n) \\ &\leq \sup_{n \geq 1} s^p(x_n) . \end{aligned}$$

Thus, by (7),

$$s^p(x) = \lim_{m \rightarrow \infty} s_m^p(x) \leq \sup_{n \geq 1} s^p(x_n) < +\infty .$$

(10): We may assume $p > 0$. If $d^p(x, y) = +\infty$ for every $y \neq x$, then by (1),

$$s^p(x) \geq d^p(x, w) - s^p(w) = +\infty .$$

This proves one direction.

If $s^p(x) = +\infty$ and $d^p(x, y) < +\infty$ for some y , then assume p is minimal. We have $x \in X$ and so we may choose a sequence $f_n \in R_0(X, 1)$ such that

$$|f_n^{(p)}(x)| \longrightarrow +\infty ,$$

while $|f_n^{(p)}(x) - f_n^{(p)}(y)| \leq M$ for all n , for some constant M . Form

$$g_n(z) = (2z - x - y)f_n(z) .$$

Then

$$g_n^{(p)}(z) = 2pf_n^{(p-1)}(z) + (2z - x - y)f_n^{(p)}(z) .$$

Thus

$$\begin{aligned} & |g_n^{(p)}(x) - g_n^{(p)}(y)| \\ &= |2pf_n^{(p-1)}(x) + (x - y)f_n^{(p)}(x) - 2pf_n^{(p-1)}(y) - (y - x)f_n^{(p)}(y)| \\ &\geq |x - y| |f_n^{(p)}(x) + f_n^{(p)}(y)| \\ &\quad - 2p |f_n^{(p-1)}(x) - f_n^{(p-1)}(y)| \longrightarrow +\infty \text{ as } n \longrightarrow +\infty . \end{aligned}$$

(11): The point x is an interior point of X if and only if

$$s_n(x) \leq M^n n !$$

for some constant $M > 0$. ("Only if" is clear, and "if" is true because the inequality implies that every function in $R_0(X, 1)$ is actually analytic in a full disc centered at x . This forces $x \in X$.) (11) is just a way of rewriting this.

4. For our purposes all measures will be finite complex Borel regular measures with compact support in C . For $\nu > 0$, the *potential of order ν* of a measure μ is given by

$$\mu^\nu(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|^\nu} ,$$

where $|\mu|$ is the total variation measure of μ . Wherever $\mu^1(z) < +\infty$ we define the *Cauchy transform* of μ by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z} .$$

For every continuous linear functional L on $R(X)$ there is a measure

μ , supported on X , which "represents L on $R(X)$ ", i.e.,

$$\int f d\mu = Lf$$

for every $f \in R(X)$. This fact follows from the Hahn-Banach and Riesz Representation theorems. Also, μ may be chosen to have its support on ∂X , since $R(X)$ and $R(X)|_{\partial X}$ are isomorphic Banach algebras. An annihilating measure for $R(X)$ is a measure μ on X such that

$$\int f d\mu = 0$$

for every $f \in R(X)$. We write $\mu \perp R(X)$. The following easy fact was first noted by Bishop, and plays a central role in our theory (cf. [2, p. 171]).

LEMMA. If $\mu \perp R(X)$, $\mu^1(y) < +\infty$, and $\hat{\mu}(y) \neq 0$, then the measure

$$\frac{1}{\hat{\mu}(y)} \frac{1}{z - y} \mu$$

represents "evaluation at y " on $R(X)$, i.e.,

$$\int f d\mu = f(y)$$

for $f \in R(X)$.

The case $p = 0$ of the following theorem is due to Browder [2, p. 176].

THEOREM 2. Let p be a nonnegative integer. Suppose the measure μ represents a bounded p th order point derivation on $R(X)$ at x . Then for every given $a > 0$ there is a corresponding $b > 0$ such that $d^p(x, y) < a$ whenever

$$(2) \quad \sum_{\nu=1}^{p+1} |x - y|^\nu \mu^\nu(y) < b.$$

Proof. We proceed by induction on p : Suppose p is the least non-negative integer for which the proposition fails. Let μ represent D_x^p and $a > 0$ be given. We may suppose $a < 1$. For $\tau = 0, 1, \dots, p - 1$, $R(X)$ admits a bounded τ th order point derivation at x , represented by

$$\mu_\tau = \frac{\tau!(z-x)^{p-\tau}}{p!} \mu,$$

so there are numbers $b_\tau > 0$ such that

$$(3) \quad \sum_{\nu=0}^{\tau+1} |x-y|^\nu \mu_\tau^\nu(y) < b_\tau$$

forces $d^\tau(x, y) < a/2$. Now

$$\begin{aligned} \mu_\tau^\nu(y) &= \int \frac{\tau! |z-x|^{p-\tau}}{|z-y|^\nu} d|\mu|(z) \\ &\leq \tau! (\text{diam } X)^{p-\tau} \mu^\nu(y), \end{aligned}$$

so, setting $c_\tau = b_\tau \{ \sup_{0 \leq \nu \leq p} \tau! (\text{diam } X)^{p-\tau} \}^{-1}$, and $c = \inf_{0 \leq \tau \leq p-1} c_\tau$, we deduce that $\sum_{\nu=0}^{p+1} |x-y|^\nu \mu^\nu(y) < c$ forces (3) for $\tau = 0, 1, \dots, p-1$.

Let $K = 1 + \exp(\text{diam } X)$,

$$T = 2 \{ \sup_{0 \leq \tau \leq p} s^\tau(x) \} K.$$

Note that $T \geq 2K$, since $s^0(x) = 1$.

Choose $b > 0$ to be smaller than each of the numbers $c, 1/2, p!(\text{diam } X)^{-p-1}$ and $a\{2T(K_p + \|\mu\|)\}^{-1}$, where $K_p > 0$ is a constant, depending only on p , which will be described later.

Let (2) hold. We will show that $d^p(x, y) < a$. We claim it suffices to show

$$(4) \quad d_0^p(y, x) < a/T.$$

For, assuming (4), we have by Theorem 1(6), (1),

$$\begin{aligned} d^p(x, y) &\leq K \{ \sup_{0 \leq \nu \leq p} s^\nu(y) \} d_0^p(y, x) \\ &\leq K \{ \sup_{0 \leq \nu \leq p} s^\nu(x) + \sup_{0 \leq \nu \leq p} d^\nu(x, y) \} d_0^p(y, x). \end{aligned}$$

Thus, if $d^p(x, y) \geq a$, then $d^p(x, y) = \sup_{0 \leq \nu \leq p} d^\nu(x, y)$, since (3) holds for $\tau = 0, 1, \dots, p-1$, so

$$d^p(x, y) \{ 1 - K d_0^p(y, x) \} \leq \frac{1}{2} T d_0^p(y, x).$$

Since $K d_0^p(y, x) < aK/T < a/2 < 1/2$, we deduce

$$d^p(x, y) \leq T d_0^p(y, x) < a,$$

which is a contradiction.

We proceed to get (4).

The measure $\mu_0 = ((z-x)^p/p!) \mu$ represents evaluation at x on $R(X)$. Thus $\sigma = (z-x)\mu_0$ annihilates $R(X)$. Now

$$\hat{\sigma}(y) = \frac{1}{p!} \int \frac{(z-x)^{p+1}}{z-y} d\mu(z) = 1 + (y-x)\hat{\mu}_0(y),$$

so, since

$$\begin{aligned} |(y-x)\hat{\mu}_0(y)| &\leq |y-x| \mu'_0(y) \\ &\leq \frac{|y-x| (\text{diam } X)^{p+1} \mu^1(y)}{p!} < b < 1, \end{aligned}$$

we have $\hat{\sigma}(y) \neq 0$. Also $\sigma^1(y) < +\infty$, since $\mu^1(y) < +\infty$, by (2). Thus, by the lemma, the measure

$$\frac{\sigma}{\hat{\sigma}(y)(z-y)} = \frac{(z-x)^{p+1}\mu}{p!\hat{\sigma}(y)(z-y)}$$

represents evaluation at y on $R(X)$, so

$$\frac{(z-x)^{p+1}\mu}{\hat{\sigma}(y)(z-y)^{p+1}}$$

annihilates the class

$$B = \{f \in R_0(X, 1) \mid f(y) = f'(y) = \dots = f^{(p)}(y) = 0\},$$

since $\mu^{p+1}(y) < +\infty$, by (2).

Let $e = \hat{\sigma}(y)$. Then $|e| > 1 - b > 1/2$, and also $|1 - e| < b$.

We have

$$\begin{aligned} d_p^0(y, x) &= \sup \{|f^{(p)}(x)| \mid f \in B\} \\ &= \sup \left\{ \left| \int f(z) \left\{ 1 - \frac{(z-x)^{p+1}}{e(z-y)^{p+1}} \right\} d\mu(z) \right| \mid f \in B \right\} \\ &\leq \int \left| \frac{e(z-y)^{p+1} - (z-x)^{p+1}}{e(z-y)^{p+1}} \right| d|\mu|(z) \\ &\leq \frac{1}{1-b} \int \left| \frac{(z-y)^{p+1} - (z-x)^{p+1}}{(z-y)^{p+1}} - (1-e) \right| d|\mu|(z) \\ &\leq 2|x-y| \int \left| \sum_{\nu=0}^p \binom{p}{\nu} \frac{(z-x)^\nu}{(z-y)^{\nu+1}} \right| d|\mu|(z) + 2b\|\mu\|. \end{aligned}$$

Now we observe that $(z-x)^\nu/(z-y)^{\nu+1}$ is a linear combination of terms

$$\frac{1}{z-y}, \frac{x-y}{(z-y)^2}, \dots, \frac{(x-y)^\nu}{(z-y)^{\nu+1}},$$

so that we may continue the inequality:

$$\leq 2K_p |x-y| \sum_{\nu=1}^{p+1} |x-y|^{\nu-1} \mu^\nu(y) + 2b\|\mu\|,$$

where K_p depends only on p , and so, continuing:

$$\begin{aligned} &\leq 2(K_p + \|\mu\|)b \\ &\leq \frac{a}{T}. \end{aligned}$$

This concludes the proof.

5. We now establish a convergence theorem for the d^p metric.

THEOREM 3. *Suppose $p = 0$, and x is not a peak point for $R(X)$, or $p \geq 1$, and $R(X)$ admits a bounded p th order point derivation at x . Suppose there is a positive constant K , and a sequence of points $\{y_n\}$, elements of \dot{X} , which converges to x (in Euclidean norm), such that*

$$(3) \quad \text{dist}[y_n, \partial X] \geq K|y_n - x|$$

for $n = 1, 2, 3, \dots$. Then $\{y_n\}$ converges to x in the d^p metric.

Proof. Select a measure μ , supported on ∂X , with no mass at x , which represents the p th order derivation at x .

By Theorem 2, it suffices to show that $|x - y_n|^\nu \mu^\nu(y_n)$ is small for each ν , $1 \leq \nu \leq p + 1$, provided n is large.

Fix $\varepsilon > 0$, and ν , $1 \leq \nu \leq p + 1$. If $z \in \partial X$, then for each $n \geq 1$,

$$\frac{|z - y_n|}{|x - y_n|} \geq K,$$

by (3). Choose $r_1 > 0$ such that

$$\mu B(x, r_1) < \frac{\varepsilon K}{2}.$$

Choose $r > 0$ such that

$$\frac{r}{r_1} < \min \left\{ \frac{\varepsilon}{2^{\nu+1} \|\mu\|}, \frac{1}{2} \right\}.$$

Then choose N so large that $n \geq N$ ensures $|x - y_n| < r$. Then, for $n \geq N$,

$$\begin{aligned} |x - y_n|^\nu \mu^\nu(y_n) &= |x - y_n|^\nu \int \frac{d|\mu|(z)}{|z - y_n|^\nu} \\ &= |x - y_n|^\nu \left\{ \int_{C \setminus B(x, r_1)} + \int_{B(x, r_1)} \right\} \\ &\leq \frac{r^\nu \|\mu\|}{(r_1/2)^\nu} + \frac{\mu B(x, r_1)}{K^\nu} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof.

COROLLARY 1. *Suppose \dot{X} satisfies a cone condition at a point $x \in \partial X$. Then whenever $p = 0$ and x is not a peak point for $R(X)$, or $p \geq 1$ and $R(X)$ admits a bounded p th order point derivation at x , it follows that $d^p(y, x) \rightarrow 0$ as y approaches x along the midline of the cone.*

This clearly follows from Theorem 3. Using the language of tangent cones [3, p. 233] we can say more.

COROLLARY 2. *Let $x \in \partial X$, E be a compact connected subset of X , $x \in E$, $E \setminus \{x\} \subset \dot{X}$, and suppose that*

$$\text{Tan}(E, x) \cap \text{Tan}(\partial X, x) = (0).$$

Then under the same hypothesis on p , $R(X)$ as before, $d^p(y, x) \rightarrow 0$ as y approaches x in E .

COROLLARY 3. *Suppose \dot{X} satisfies a cone condition at x , and Γ is the midline of the cone. Suppose $R(X)$ admits a bounded p th order point derivations at x ($p \geq 1$). Let D_x^p and D_x^{p-1} denote the normalized point derivations of orders p and $p - 1$ at x . Then*

$$D_x^p f = \lim_{\substack{y \rightarrow x \\ y \in \Gamma}} \left[\frac{f^{(p-1)}(y) - D_x^{p-1} f}{y - x} \right]$$

for every $f \in R(X)$, and the convergence is equiconvergence on $R(X, 1)$.

This follows readily from Corollary 1.

6. For examples to which these results apply, see [5], [10]. Hallstrom [6] has given necessary and sufficient conditions that $R(X)$ admit a bounded point derivation at a point x . Essentially, the complement of X has to be "thin" at x , in terms of analytic capacity.

Let a_n, r_n be two sequences of positive numbers such that

$$1 > a_n + r_n > a_n > a_n - r_n > a_{n+1} + r_{n+1},$$

for $n = 1, 2, 3, \dots$. Let D_n denote the open disc with centre a_n and radius r_n . Let X be the compact set obtained by removing $\bigcup_{n=1}^{+\infty} D_n$ from the closed unit disc D . X is an example of a so-called L -set.

For these L -sets, the point 0 is a peak point for $R(X)$ if and only if $\sum_{n=1}^{+\infty} r_n/a_n = +\infty$ [10], and $R(X)$ admits a p th order bounded point derivation at 0 provided $\sum_{n=1}^{+\infty} r_n/(a_n^{p+1}) < +\infty$. Let E denote the negative real axis. Applying Corollary 1 to X we obtain the following:

THEOREM 4.

(1) Suppose $\sum_{n=1}^{+\infty} r_n/a_n < +\infty$. Then $\lim_{z \rightarrow 0} \lim_{z \in E} (d^0(z, 0)) = 0$.

(2) Suppose $\sum_{n=1}^{+\infty} r_n/(a_n^{p+1}) < +\infty$. Then $\lim_{y \in E} \lim_{z \rightarrow 0} d^p(z, 0) = 0$.

By choosing, say, $a_n = 1/(n + 1)$, $r_n = 1(n + 1)!!$ we can ensure that the hypothesis of (2) is satisfied for every $p \leq 0$, so that $f^{(p)}$ is equiconvergent to $f^{(p)}(0)$ on $R_0(X, 1)$, for every p .

One might wonder whether some kind of Browder density theorem might work for $p > 0$: if $R(X)$ admits a p th order bounded point derivation at x , are there always other bounded derivations at nearby points? The answer is no: in [9] an example is constructed in which $R(X)$ admits a first order bounded point derivation at just one point. Moreover, this example can be modified to produce an example with a bounded point derivation of every order at that certain point, and no other bounded point derivations of any order ≥ 1 anywhere else.

What goes wrong? The following observation may clarify things. If μ represents a first order bounded point derivation on $R(X)$ at x and $\mu^2(y) < +\infty$, set

$$C = \int \frac{(z - x)^2}{z - y} d\mu(z),$$

$$D = \int \frac{(z - x)^2}{(z - y)^2} d\mu(z).$$

Then, provided $C \neq 0$ and $D \neq 0$, the measure

$$V = \left\{ \frac{1}{C} \frac{(z - x)^2}{(z - y)^2} - \frac{1}{CD} \frac{(z - x)^2}{z - y} \right\} \mu$$

represents a first order bounded point derivation on $R(X)$ at y . So this gives a sufficient condition for the existence of other derivations: $\{y \mid \mu^2(y) < +\infty, C \neq 0, D \neq 0\} \neq \emptyset$. Unfortunately μ^2 is the potential associated with harmonic functions in R^t , and the associated capacity, C^2 , vanishes on planar sets. So it is entirely possible, even likely, that $\mu^2(y) \equiv +\infty$ on $\text{spt } \mu$. In fact, $\mu^2(y) < +\infty$ if and only if

$$\sum_{n=1}^{+\infty} 4^n |\mu|(A_n(y)) < +\infty,$$

where $A_n(y) = \{z \mid 1/2^{n+1} \leq |z - y| \leq 1/2^n\}$, $n = 1, 2, 3, \dots$. Thus, for instance, if

$$\overline{\lim}_{r \rightarrow 0} \frac{|\mu|(B(y, r))}{r^2} > 0$$

(i.e., $|\mu|$ has positive area density at y), then $\mu^2(y) = +\infty$.

Returning to the problem posed in § 1, we note that for $x \in \partial(\hat{X})$,

without some condition on \dot{X} , we cannot ensure that there will be a sequence $x_n \rightarrow x$ with $x_n \in \dot{X}$ and $f'(x_n)$ equiconvergent to $f'(x)$ on $R_0(X, 1)$, even when $s^1(x) < +\infty$. For let X be the example of [9], with a bounded point derivation just at 0, and select any sequence $\{x_n\}$ of distinct points of X , tending to 0. For each n ($n = 1, 2, 3, \dots$) there is a function $f_n \in R_0(X, 1)$ such that $f'_n(x_n) > 4n$. Inductively, choose a closed disc D_n centered at x_n such that f_n is analytic in a neighborhood of D_n , $\|f_n\|_{D_n} \leq 2$, $|f'_n(z)| > 2n$ for $z \in D_n$, $D_n \cap D_m = \emptyset$ for $m < n$, $x_m \notin D_n$ for $m > n$. Form a new compact set $Y = X \cup (\bigcup_{n=1}^{\infty} D_n)$. Then $R(Y)$ still admits a bounded point derivation at 0. The only other bounded point derivations are at points of the D_n . For $z \in D_n$, $s^1(z) > n$. So there is no sequence of points of $\dot{Y} = \bigcup_{n=1}^{\infty} \dot{D}_n$ along which f' is equiconvergent to $f'(0)$ on $R_0(Y, 1)$.

7. Let X be a compact subset of the plane. Let A be an algebra of functions on C which contains the polynomial and all of whose functions are analytic on \dot{X} . Suppose A , regarded as a subset of $C(X)$, forms a function algebra. Suppose A enjoys the *Arens property*: For each $x \in X$,

$$A_x = \{f \in A \mid f \text{ is analytic on a neighborhood of } x\}$$

is dense in A in the uniform norm on X . (A sufficient condition for this is that A contains a dense subset B which is " T_ϕ -invariant", i.e., the function $T_\phi f$, given by

$$T_\phi f(z) = \frac{1}{\pi} \int \frac{f(z) - f(\zeta)}{z - \zeta} \frac{\partial \phi(\zeta)}{\partial \bar{\zeta}} d\mathcal{L}^2(\zeta)$$

belongs to B whenever f belongs to B and ϕ is a continuously differentiable function with compact support. An example is $A = A(X)$, the algebra of all continuous functions on C which are analytic on X ; another example is $A = A^\alpha(X)$, the uniform closure on X of those functions in $A(X)$ which satisfy a condition $\text{Lip } \alpha$ on C .) Then most of what we have done for $R(X)$ goes through for A . New functions d^p, s^p, d^p_0 may be defined analogously, for instance:

$$d^p(x, y) = \sup \{ |f^{(p)}(x) - f^{(p)}(y)| \mid f \in A, \|f\|_X \leq 1, \\ f \text{ is analytic on a neighborhood of } \{x, y\} \}.$$

For any $x \in C$ we can form A_x . So given any compact set $Y \subset C$ we may form a new algebra

$$Y(A) = \bigcap_{x \in Y} (\text{Uniform closure on } Y \text{ of } A_x \cap A(Y)).$$

$Y(A)$ is clearly a uniform algebra on Y , contains the polynomials, and all its functions are analytic on \dot{Y} . Moreover, by its definition,

it has the Arens property.

Replacing $R(X)$ by A , Theorem 1 will go through, except that (7) will have to be changed:

(7') if $x \in \partial X$, V_n is a decreasing sequence of compact neighborhoods of x , whose intersection is $\{x\}$, and $X_n = X \cup V_n$, then $s_n^p(x) \uparrow s^p(x)$, and $d_n^p(x, \cdot) \uparrow d^p(x, \cdot)$, where s_n^p and d_n^p are the s^p and d^p functions associated with the algebras $X_n(A)$.

Lemma 1 goes through, using the Arens property.

The maximal ideal space of A is X (cf. [1], its Šilov boundary is a subset of ∂X , so Theorems 2 and 3 work for A in place of $R(X)$).

8. Now we turn to $H^\infty(U)$, the Banach algebra of bounded analytic functions (with L^∞ norm) on the bounded open set $U \subset C$. First, we look at $H^\infty(U)$ itself. There is a natural projection map from the maximal ideal space \mathcal{M} of $H^\infty(U)$ to \bar{U} , given by $\phi \rightarrow \phi(z)$ (recall that z denotes the identity map of C). The fiber \mathcal{M}_x over a point $x \in U$ consists of one point $\phi_x =$ evaluation at x . The fiber \mathcal{M}_x over a point $x \in \partial U$ is usually very large. Gamelin and Garnett [5] showed that a necessary and sufficient condition for \mathcal{M}_x to be a peak set for $H^\infty(U)$ is that

$$(4) \quad \sum_{n=1}^{+\infty} 2^n \gamma(A_n(x) \setminus U) = +\infty.$$

Here γ denotes the analytic capacity:

$$\gamma(K) = \sup \{ \|f'(\infty)\| \mid f \text{ is analytic off } K, \|f\| \leq 1, f(\infty) = 0 \}.$$

When \mathcal{M}_x is not a peak set, they showed that it contains a distinguished homomorphism, ϕ_x , characterized by the property that it has a representing measure on \mathcal{M} with no mass on \mathcal{M}_x .

We say that an element $D \in H^\infty(U)^*$, a continuous linear map of $H^\infty(U)$ to C , is a first order bounded point derivation at a point $\phi \in \mathcal{M}$ if

$$D(fg) = \phi(f)Dg + \phi(g)Df$$

whenever $f, g \in H^\infty(U)$. D is called regular if $Dz \neq 0$, and a regular D is normalised if $Dz = 1$. We shall be concerned with regular derivations only, but we note that there are usually many derivations on $H^\infty(U)$ which annihilate z . For instance, let U be the open unit disc. Then Hoffman [7] has shown that the fiber \mathcal{M}_1 over the point $1 \in \partial U$ contains many homeomorphic images of the unit disc, on each

of which all the functions in $H^\infty(U)$ are analytic. So there is a superabundance of bounded point derivations at points of \mathcal{M}_1 , and each of these derivations annihilates z .

Inductively, we say $H^\infty(U)$ admits a regular normalized p th order bounded point derivation at $\phi \in \mathcal{M}$ if the following hold:

- (1) For each ν , $1 \leq \nu \leq p - 1$, D^ν is a ν th order regular normalized bounded point derivation at ϕ .
- (2) There is an element $D^p \in H^\infty(U)^*$ such that

$$D^p(fg) = \sum_{\nu=0}^p \binom{p}{\nu} D^\nu f D^{p-\nu} g,$$

for all $f, g \in H^\infty(U)$, where $D^0 f$ means $\phi(f)$.

- (3) $D^p z^p = p!$

We observe that for $p \geq 1$ there cannot be any regular p th order bounded point derivation at a point $\phi \in \mathcal{M}_x \setminus \{\phi_x\}$. For such a derivation would have a representing measure μ on \mathcal{M} , and then $((z-x)^p/p!)\mu$ would be a representing measure for ϕ with no mass on \mathcal{M}_x , which is impossible.

THEOREM 5. *Let $x \in U$, $p \geq 1$. Then $H^\infty(U)$ admits a regular bounded p th order point derivation at the distinguished homomorphism ϕ_x in the fiber over x if and only if*

$$(5) \quad \sum_{n=1}^{+\infty} 2^{(p+1)n} \gamma(A_n(x) \setminus U) < +\infty.$$

Proof. If (5) holds, then certainly (4) fails, so \mathcal{M}_x is not a peak fiber and ϕ_x exists. By a device in Gamelin and Garnett's proof of the peak set criterion [5, p. 459, third paragraph], U can be shrunk a little to produce a compact set X with the properties:

- (1) $X \subset U \cup \{x\}$,
- (2) $x \in X$,
- (3) $\sum_{n=1}^{+\infty} 2^{(p+1)n} \gamma(A_n(x) \setminus X) < +\infty$.

By Hallstrom's Theorem [6, p. 156], $R(X)$ admits a (normalized) bounded point derivation of order p at x . Choose a representing measure μ for this derivation with support on X and no mass at x . Then, for $\nu = 0, 1, \dots, p$ the measure $\mu_\nu = (\nu!(z-x)^{p-\nu}/p!)\mu$ represents a (normalized) ν th order bounded point derivation on $R(X)$ at x , if $\nu \geq 1$, and μ_0 represents x and has no mass at x . Now any function in $H^\infty(U)$ which extends analytically to a neighborhood of x belongs to $R(X)$, so for any two such functions, f and g , we have

$$(6) \quad \int fgd\mu = \sum_{\nu=0}^p \binom{p}{\nu} \int fd\mu_\nu \int gd\mu_{p-\nu}.$$

Since, as is well-known [5, Cor. 2.2], the set of all such functions is pointwise boundedly dense in $H^\infty(U)$, the dominated convergence theorem implies that (6) holds for any $f, g \in H^\infty(U)$. Thus μ represents a regular bounded p th order point derivation on $H^\infty(U)$ at ϕ_x .

For the other direction, assume (5) fails. If \mathcal{M}_x is a peak set there is no distinguished homomorphism, and nothing to prove. Otherwise, (4) fails, and we may, just as in Hallstrom's proof of his Theorem 1' [6, pp. 163-164], construct a sequence of functions g_n , each one in $H^\infty(U)$ and analytic in a neighborhood of x such that $|g_n^{(p)}(x)| > n \|g\|_\infty$. Thus $H^\infty(U)$ cannot admit a p th order bounded point derivation at ϕ_x . This proves the theorem.

We remark that there is at most one regular normalised bounded p th order point derivation at a distinguished homomorphism ϕ_x . For, from the proof of Theorem 5, any two agree on a dense subset of $H^\infty(U)$, and have representing measures with no mass on \mathcal{M}_x . Thus, by dominated convergence, they coincide.

9. The zero order Gleason metric d^0 on the maximal ideal space of $H^\infty(U)$ is given by

$$d^0(\phi, \psi) = \sup \{ |\phi(f) - \psi(f)| \mid f \in H^\infty(U), \|f\|_\infty \leq 1 \}.$$

To define the higher order metrics, we take first the case where ϕ and ψ are distinguished homomorphisms at each of which $H^\infty(U)$ admits normalised regular bounded p th order point derivations D_ϕ^p and D_ψ^p . Then

$$d^p(\phi, \psi) = \sup \{ |D_\phi^p f - D_\psi^p f| \mid f \in H^\infty(U), \|f\|_\infty \leq 1 \}.$$

In all other cases, we set $d^p(\phi, \psi) = +\infty$. Let $s^p(\phi)$ be the norm of D_ϕ^p , if this exists, otherwise $s^p(\phi) = +\infty$. For points $y \in U$ we will write y for "evaluation at y ".

THEOREM 6. *Let $p \geq 1$. Suppose there is a constant $K > 0$ and a sequence of points $y_n \in U$, $|y_n - x| \rightarrow 0$ as $n \rightarrow +\infty$, such that*

$$\text{dist}[y_n, \partial U] \geq K |y_n - x|.$$

Suppose $H^\infty(U)$ admits a regular p th order bounded point derivation at the distinguished homomorphism ϕ_x over x . Then $d^p(y_n, \phi_x) \rightarrow 0$ as $n \rightarrow +\infty$.

Proof. We shall deduce this from Theorem 3. As in Theorem 5, we may shrink U to a compact set X which satisfies the hypotheses of Theorem 3, with a smaller K . Thus there are representing meas-

ures μ_n for the $D_{y_n}^p$, and μ for $D_{\phi_x}^p$, with closed support in $U \cup \mathcal{M}_x$ and no mass on \mathcal{M}_x such that

$$\int f d\mu_n \longrightarrow \int f d\mu$$

uniformly for $f \in R(X)$. Again, since $R_0(X)$ is pointwise boundedly dense in $H^\infty(U)$, this means that $D_{y_n}^p f = \int f d\mu_n$ is equiconvergent to $D_{\phi_x}^p f = \int f d\mu$ for all $f \in H^\infty(U)$.

The analogous result when $p = 0$ (also a corollary of Theorem 3) is due to Gamelin and Garnett [5, 5.1].

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Received April 12, 1973. This research was supported in part by the National Science Foundation under grant GP-28574(X01).

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