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A QUASI ORDER CHARACTERIZATION OF SMOOTH CONTINUA

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L. E. Ward, Jr. characterized a generalized tree as a compact Hausdorff space which admits a partial order satisfying certain conditions. An analogous characterization of smooth continua, in terms of quasi ordered topological spaces, is obtained.

A *quasi order* on a topological space X is a reflexive and transitive binary relation $\leqq$. If this relation is also antisymmetric it is called a *partial order*. The quasi order $\leqq$ is *closed* if $\{(x, y) \in X \times X \mid x \leqq y\}$ is a closed subset of the product space $X \times X$.

For each $x \in X$, the set $L(x) = \{y \in X \mid y \leqq x\}$ (respectively, $M(x) = \{y \in X \mid x \leqq y\}$) is called the *set of predecessors* (respectively, *successors*) of x. Let $E(x) = L(x) \cap M(x)$ and note that $\leqq$ is a partial order if and only if each $E(x)$ is a singleton. In case $\leqq$ is closed, the sets $L(x)$, $M(x)$, and $E(x)$ are closed subsets of X.

If $x \leqq y$ and $x \notin E(y)$ we write $x < y$. The quasi order $\leqq$ is *order dense* if whenever $x < y$, there exists $z \in X$ such that $x < z < y$.

Let S be a subset of X. An element $z \in S$ is a *zero* of S if $z \leqq x$ for each $x \in S$. If $x \leqq y$ or $y \leqq x$ for all $x, y \in S$, then S is called a *chain*.

We define the equivalence relation ρ on X by

$$(x, y) \in \rho \text{ if and only if } E(x) = E(y)\ .$$

Let ϕ: $X \to X/\rho$ denote the natural quotient map.

A continuum (= compact connected Hausdorff space) X is *hereditarily unicoherent* at the point p [2] if for each $x \in X$, there exists a unique subcontinuum of X, denoted $[p, x]$, irreducible between p and x. We say X is *hereditarily unicoherent* if it is hereditarily unicoherent at each of its points.

If the continuum X is hereditarily unicoherent at p then X admits a very natural quasi order $\leqq_p$, called the *weak cut point order with respect to* p:

$$x \leqq_p y \text{ if and only if } x \in [p, y]\ .$$

Note that for each $x \in X$, $L(x) = [p, x]$.

The continuum X is *smooth* if there exists a point $p \in X$ such that X is hereditarily unicoherent at p and the quasi order $\leqq_p$ is closed. By [1], Theorem 3.1, p. 65, this definition is equivalent to

Gordh's original definition [2]. To emphasize the point p we will often write "X is smooth at p". A *generalized tree* is a hereditarily unicoherent, arcwise connected[1] smooth continuum. Ward's original definition [6] is stated here as Theorem 1. According to [4] the definitions are equivalent.

THEOREM 1. *The compact Hausdorff space X is a generalized tree if and only if X admits a partial order $\leqq$ such that*

(1) *$\leqq$ is closed;*

(2) *$\leqq$ is order dense;*

(3) *if $x, y \in X$, then $L(x) \cap L(y)$ is a nonempty chain;*

(4) *if Y is a closed and connected subset of X, then Y contains a zero.*

It follows that $\leqq$ is the weak cut point order with respect to p where $\{p\} = \bigcap \{L(x) \mid x \in X\}$ and $L(x) = [p, x]$.

It is the purpose of this paper to establish an analogous characterization for smooth continua.

Consider the following properties that a quasi order $\leqq$ on a space X may possess:

(i) $\leqq$ is closed;

(ii) $\leqq$ is order dense;

(iii) there exists $p \in \bigcap \{L(x) \mid x \in X\}$ and each $L(x)$ is a chain;

(iv) if Y is a closed connected subset of X, then Y contains a zero;

(v) $E(x)$ is connected for each $x \in X$;

(vi) if Y is a closed connected subset of X and $p \in Y$, then $E(y) \subseteqq Y$ for each $y \in Y$.

THEOREM 2. *Let X be a compact Hausdorff space which admits a quasi order $\leqq$ satisfying* (i)–(vi). *Then X is a continuum which is smooth at p.*

The theorem will be proved via a series of lemmas. Unless otherwise stated assume X, $\leqq$, and p are as above. Observe that (vi) implies p is the unique zero of X.

LEMMA 1. *The space X/ρ is compact Hausdorff and the map $\phi: X \rightarrow X/\rho$ is monotone.*

Proof. First note that $\{E(x) \mid x \in X\}$ is a pairwise disjoint closed covering of X. From Theorem 2, [7], p. 147, and [3], p. 132, we infer $\{E(x) \mid x \in X\}$ is an upper semicontinuous decomposition of X.

[1] An *arc* is a continuum (not necessarily metrizable) with exactly two noncut points.

By Theorem 3–33, [3], p. 133, X/ρ is compact Hausdorff. Finally, it follows from (i) and (v) that $\phi^{-1}(\phi(x)) = E(x)$ is closed and connected; hence $\phi: X \rightarrow X/\rho$ is monotone.

The quasi order $\leqq$ on X induces a relation $\leqq'$ on X/ρ defined by

$$\phi(x) \leqq' \phi(y) \text{ if and only if } x \leqq y .$$

For the sake of clarity let $L'(\phi(x))$ denote the set of predecessors of $\phi(x)$ in X/ρ.

LEMMA 2. *The space X/ρ is a generalized tree which is smooth at $\phi(p)$. Moreover, $\leqq'$ is the weak cut point order with respect to $\phi(p)$ and $L'(\phi(x))$ is the unique subcontinuum of X/ρ irreducible between $\phi(p)$ and $\phi(x)$.*

Proof. It is straightforward to verify that $\leqq'$ is a partial order satisfying the hypotheses of Theorem 1.

LEMMA 3. *The space X is a continuum. In particular, $L(x)$ is closed and connected for each $x \in X$.*

Proof. Since $L(x)$ is the inverse image of $L'(\phi(x)) \subseteqq X/\rho$ under the monotone map $\phi: X \rightarrow X/\rho$ it follows from Theorem 9, [5], p. 131, that $L(x)$ is closed and connected. Since $p \in \bigcap \{L(x) \mid x \in X\}$ and $X = \bigcup \{L(x) \mid x \in X\}$, the lemma is proved.

LEMMA 4. *If Y is a subcontinuum of X and $p \in Y$, then $\phi^{-1}(\phi(Y)) = Y$.*

Proof. We show only $\phi^{-1}(\phi(Y)) \subseteqq Y$. If $z \in \phi^{-1}(\phi(Y))$ there exists $y \in Y$ such that $\phi(y) = \phi(z)$. By (vi)

$$z \in E(z) = E(y) \subseteqq Y .$$

LEMMA 5. *The continuum X is hereditarily unicoherent at p.*

Proof. Let x be a fixed, but arbitrary, point in X and let $Y \subseteqq X$ be a subcontinuum irreducible between p and x. Then $\phi(Y) \subseteqq X/\rho$ is a subcontinuum containing $\phi(p)$ and $\phi(x)$. Since X/ρ is a generalized tree, $L'(\phi(x)) \subseteqq \phi(Y)$. It follows from

$$L(x) = \phi^{-1}(L'(\phi(x)) \subseteqq \phi^{-1}(\phi(Y)) = Y$$

and Lemma 3 that $L(x) = Y$. That is, $L(x)$ is the unique subcontinuum of X irreducible between p and x.

We have shown that the space X is a continuum which is hereditarily unicoherent at p. Moreover, $[p, x] = L(x)$ for each $x \in X$. It follows immediately that $\leqq$ is the weak cut point order with respect to p. Since $\leqq$ is closed by hypothesis, the proof of Theorem 2 is complete.

The converse of Theorem 2 is also true. Before proceeding, however, we need a few results about smooth continua. The reader is referred to [2] for the details.

THEOREM 3. *If the continuum X is smooth at p then X/ρ is a generalized tree which is smooth at $\phi(p)$, the map $\phi: X \to X/\rho$ is monotone, and* $\operatorname{int}_X E(x) = \square$.[2]

LEMMA 6. *If the continuum X is smooth at p then $x \leqq_p y$ (respectively, $x <_p y$) if and only if $\phi(x) \leqq_{\phi(p)} \phi(y)$ (respectively, $\phi(x) <_{\phi(p)} \phi(y)$). Moreover, if Y is a subcontinuum of X and $p \in Y$, then $\phi^{-1}(\phi(Y)) = Y$.*

THEOREM 4. *If the continuum X is smooth at p then $\leqq_p$ satisfies* (i)–(vi).

Proof. It is immediate that (i) and (vi) hold. Since $E(x)$ is the inverse image of the point $\phi(x)$ under the monotone map $\phi: X \to X/\rho$, (v) holds. Conditions (ii) and (iii) follow from Lemma 6 and the fact that $L(x) = \phi^{-1}(L'(\phi(x))$. Finally to show (iv) holds, let Y be a subcontinuum of X. Then $\phi(Y)$ is a subcontinuum of the generalized tree X/ρ. Let $z \in X$ be such that $\phi(z)$ is a zero of $\phi(Y)$. Choose any

$$y \in \phi^{-1}(\phi(z)) \cap Y = E(z) \cap Y .$$

It follows from Lemma 6 that y is a zero of Y.

Observe that condition (iii) is equivalent to condition (3) of Ward's theorem. The paraphrase was inserted as a matter of convenience, since the point p appears in condition (vi).

We remark that each of conditions (i)–(vi) is independent of the remaining five. We include here examples to clarify the necessity of the last two conditions. The omitted details are left to the reader. Let $\leqq_0$ denote the natural partial order on the real numbers.

EXAMPLE 1. (Due to J. Ladwig.) Let X denote the Cantor Set and let $\{(a_n, b_n) \mid n = 1, 2, \cdots\}$ be the collection of "deleted intervals"; i.e.,

$$X = [0, 1] - \bigcup_{n=1}^{\infty} (a_n, b_n)$$

[2] "int_X" denotes interior in the space X and "$\square$" denotes the empty set.

and for $n = 1, 2, \cdots$

$$[a_n, b_n] \cap X = \{a_n, b_n\} .$$

Define $x \leqq y$ if and only if $x \leqq_0 y$ or x and y are endpoints of a common deleted interval. The quasi order $\leqq$ on X satisfies (i)–(iv) and (vi) but not (v).

EXAMPLE 2. In the plane let X be the triangle with vertices $p = (0, 0)$, $(1, 0)$, and $(1, 1)$. Define $(x, y) \leqq (u, v)$ if and only if $x \leqq_0 u$. Then $\leqq$ on X satisfies (i)–(v) but not (vi); e.g., take $Y = [0, 1] \times \{0\}$.

COROLLARY 1. *Let X be a continuum which is smooth at p. Then $\leqq_p$ is a partial order if and only if X is a generalized tree which is smooth at p.*

Proof. If $\leqq_p$ is a partial order then each $E(x)$ is degenerate and conditions (i)–(vi) reduce to (1)–(4) of Theorem 1. The converse is trivial since each $L(x)$ is an arc for each $x \in X$.

It is necessary that the continuum X in Corollary 1 be smooth at p as the example below shows.

EXAMPLE 3. In the plane let

$$\begin{aligned} A &= \left\{\left(x, \sin\frac{1}{x}\right) \mid 0 < x \leqq 1\right\} , \\ B &= \{0\} \times [-1, 1] , \\ C &= [-1, 0] \times \{-1\} . \end{aligned}$$

The continuum $X = A \cup B \cup C$ is clearly not a generalized tree. However, X is hereditarily unicoherent and $\leqq_p$ is a partial order for $p = (-1, 1)$.

Finally observe that in the presence of conditions (i) and (iii)–(vi), condition (ii) is equivalent to

$$\text{(ii}'\text{)} \ \operatorname{int}_{L(x)} E(x) = \square \text{ for each } x \in X - \{p\} .$$

For if X is smooth at p then so is $L(x)$; thus (ii′) is a consequence of Theorem 3. Conversely, we show (i), (ii′), and (iii) imply (ii). Suppose $x, y \in X$ are such that $x < y$ and $x < z < y$ for no $z \in X$. Then $L(y) - L(x)$ is a nonempty open (in $L(y)$) subset of $E(y)$, contradicting (ii′).

REFERENCES

1. G. R. Gordh, Jr., *Concerning closed quasi-orders on hereditarily unicoherent continua*, Fund. Math., **78** (1973), 61–73.

2. G. R. Gordh, Jr., *On decompositions of smooth continua*, Fund. Math., **75** (1972), 51-60.
3. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.
4. R. J. Koch and I. S. Krule, *Weak cut-point ordering on hereditarily unicoherent continua*, Proc. Amer. Math. Soc., **11** (1960), 679-681.
5. K. Kurtowski, *Topology* II, PWN-Academic Press, Warsaw-New York, 1968.
6. L. E. Ward, Jr., *Mobs, trees, and fixed points*, Proc. Amer. Math. Soc., **8** (1957), 798-804.
7. ———, *Partially ordered topological spaces*, Proc. Amer. Math. Soc., **5** (1954), 144-161.

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