

## NONZERO SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR NONLINEAR SYSTEMS

ATHANASSIOS G. KARTSATOS

We are mainly concerned here with solutions of

$$(*) \quad x' = A(t, x)x + F(t, x),$$

which satisfy the following conditions

$$(1.1) \quad x \in B, \quad x(t) \neq 0.$$

Here  $A(t, u)$  is a real  $n \times n$  matrix defined and continuous on  $J \times R^n$ , where  $J$  is a subinterval of  $R = (-\infty, \infty)$ . The real  $n$ -vector  $F(t, u)$  is also defined and continuous on  $J \times R^n$ . In (1.1)  $B$  is a Banach space of continuous functions on  $J$ .

Two theorems are given concerning the solution to the above problem in the case of a finite interval  $J$ . The first theorem (Th. 3.1) deals with the homogeneous system

$$(1.2) \quad x' = A(t, x)x,$$

and the second (Th. 4.1) is concerned with the system (\*) with a *small* perturbation  $F(t, u)$ . The third result of this paper (Th. 5.1) extends to rather heavily nonlinear systems a result of Medvedev [12] dealing with the existence of nontrivial, nonnegative solutions of (\*) on  $[0, \infty)$ . Medvedev considered the case of a perturbed linear system. The method employed here is a comparison technique. In other words, for each function  $f \in M$  (a certain compact, convex set of functions in  $B$ ) we assume the existence of solutions in  $M$  of the linear system

$$(1.u) \quad x' = A(t, u(t))x + F(t, u(t)),$$

and then we apply a fixed point theorem for multi-valued mappings, due to Eilenberg and Montgomery [4], in order to ensure the existence of solutions in  $M$  of the equation (\*). As far as the author knows, the first application of the above fixed point theorem was given by Schmitt [14], who considered the case  $B = \{x \in C([0, \omega]); x(0) = x(\omega)\}$ , and a linear second order scalar equation with deviating arguments.

For results related to the contents of this paper, the reader is referred to Lasota, Opial [11], Opial [13], Avramescu [1], [2], Cor-duneanu [3] and Kartsatos [7], [8].

2. Preliminaries. Let  $J$  be a subinterval of  $R = (-\infty, \infty)$ .

By  $C[J, R^n]$  we denote the space of all bounded, continuous,  $R^n$ -valued functions on  $J$ , associated with the norm

$$\|f\|_J = \sup_{t \in J} \|f(t)\|.$$

By  $B$  we shall always denote a closed subspace of the Banach space  $C[J, R^n]$ . By  $\|\cdot\|$  we denote the norm  $\|x\| = \sum_{i=1}^n |x_i|$ ,  $x = (x_1, x_2, \dots, x_n) \in R^n$ . We also put  $\|A\| = \sup_j \sum_{i=1}^n |\alpha_{ij}|$  for a real  $n \times n$  matrix  $A = [\alpha_{ij}]$ ,  $i, j = 1, 2, \dots, n$ .  $K$  will denote the cone of nonnegative vectors in  $R^n$ .

The following lemma is an elementary result in algebraic topology, but it is crucial for the proofs of the theorems of this paper. For a proof of it the reader is referred to Hu [6, p. 96].

LEMMA 2.1. *Compact convex subsets of a Banach space are acyclic absolute neighborhood retracts.*

3. The problem ((1.2), (1.1)). In what follows  $J = [a, b] \subset R$ .

THEOREM 3.1. *Assume that the real  $n \times n$  matrix  $A(t, u)$  is defined and continuous on  $J \times S_\mu$ , where  $S_\mu = \{x \in R^n; \|x\| \leq \mu\}$  and  $\mu > 1$  is a constant. Let  $\|A(t, u)\| \leq p(t)$ ,  $(t, u) \in J \times S_\mu$ , where  $p \in C[J, R]$  and*

$$\exp \left\{ \int_a^b p(t) dt \right\} \leq \mu.$$

*Assume that for every  $u \in C[J, R]$  with  $u(a) \in K$  and  $\|u\|_J \leq \mu$ , the linear system*

$$(3.u) \quad x' = A(t, u(t))x$$

*has at least one solution  $x(t)$ ,  $t \in [a, b]$  satisfying (1.1) and  $x(a) \in K$ . Then the problem ((1.2), (1.1)) has at least one nontrivial solution.*

*Proof.* Let  $S^0 = \{u \in C[J, R]; u(a) \in K \text{ and } \|u\|_J \leq \mu\}$ , and let  $u \in S^0$ . Since for this function  $u(t)$  the system (3.u) has at least one nontrivial solution  $x_u \in B$  with  $x_u(a) \in K$ , it follows that  $\|x_u(a)\| = \lambda > 0$ . Consequently, the function  $y(t) = (1/\lambda) x_u(t)$  belongs to the space  $B$  and is a solution of (3.u) with  $\|y(a)\| = (1/\lambda) \|x_u(a)\| = 1$ . Thus, we have

$$(3.1) \quad y(t) = y(a) + \int_a^t A(s, u(s))y(s) ds,$$

which, applying Gronwall's inequality, yields

$$(3.2) \quad \|y(t)\| \leq \|y(a)\| \exp \left\{ \int_a^b p(t) dt \right\} \leq \mu .$$

This inequality shows that for each  $u \in S^0$  there exists at least one nontrivial solution of (3.u) which belongs to  $B \cap S^0$  and has initial value  $y(a)$  with  $\|y(a)\| = 1$ . Now let  $q = \sup_{t \in J} p(t)$  and

$$(3.3) \quad S = \{u \in B \cap S^0; \|u'\|_J \leq \mu q, \|u(a)\| = 1\} .$$

Then  $S$  is a convex set consisting of equicontinuous and uniformly bounded functions. This implies that  $Q \equiv \bar{S}$  (where the closure is taken w.r.t. the sup-norm) is a compact and convex subset of  $C[J, R]$ . Consider now the operator  $U$ , which maps the function  $u \in Q$  into the set  $U(u)$  consisting of all solutions  $y(t)$  of (3.u) with  $y \in B \cap S^0$  and  $\|y(a)\| = 1$ . Then it is easy to see that  $U(u)$  is a compact, convex subset of  $Q$ . In order to apply the Eilenberg-Montgomery Theorem [4] since, by Lemma 2.1,  $Q, U(u)$  are acyclic absolute neighborhood retracts, it remains to show that  $U$  is continuous in the following sense:  $\|x_n - x\|_J \rightarrow 0, \|u_n - u\|_J \rightarrow 0$  and  $x_n \in U(u_n)$  imply  $x \in U(u)$ . At first we have

$$(3.4) \quad x_n(t) = x_n(a) + \int_a^t A(s, u_n(s))x_n(s) ds .$$

Let

$$(3.5) \quad y(t) = x(a) + \int_a^t A(s, u(s))x(s) ds .$$

Then

$$(3.6) \quad \begin{aligned} \|x_n - y\|_J &\leq \|x_n(a) - x(a)\| \\ &+ \int_a^b \|A(t, u_n(t))x_n(t) - A(t, u(t))x(t)\| dt . \end{aligned}$$

Since the integrand in (3.6) converges uniformly to zero, it follows that

$$(3.7) \quad \lim_{n \rightarrow \infty} \|x_n - y\|_J = 0 ,$$

i.e.,  $y(t) \equiv x(t), t \in J$ . Letting  $x(t) = y(t)$  in (3.5), we see that  $x(t)$  satisfies (3.u). It also follows easily that  $x \in B, \|x(a)\| = 1, \|x\|_J \leq \mu$  and  $\|x'\|_J \leq \mu q$ .

Consequently,  $x \in U(u)$ . Thus, the operator  $U$  has a fixed point  $x \in Q$ , which is a nontrivial solution to the problem ((1.2), (1.1)).

**EXAMPLE 3.1.** Let  $B$  consist of all functions  $u \in C[[a, b], R^n]$  such that

$$(3.8) \quad Mu(a) - Nu(b) = 0,$$

where  $M = [m_{ij}]$ ,  $N = [n_{ij}]$  are real, constant  $n \times n$  matrices. Now consider the system (3.u) with  $A(t, u)$  satisfying the assumptions of Th. 3.1. Let  $X_u(t)$  be the fundamental matrix of solutions of (3.u) with  $X_u(a) = I$  (the identity  $n \times n$  matrix). Assume that the system

$$(3.9) \quad [M - NX_u(b)]c = 0$$

has at least one nontrivial solution  $c_u \in K$  for every  $u \in S_\mu$ . Then the solution  $x_u$  of (3.u) with  $x_u(a) = c_u \neq 0$  satisfies

$$(3.10) \quad Mx_u(a) - Nx_u(b) = [M - NX_u(b)]c_u = 0.$$

Consequently, according to Th. 3.1, there exists at least one solution to the problem ((1.2), (1.1)).

4. **Perturbed systems on  $[a, b]$ .** We wish to point out that there appear to be some severe limitations to the method employed in Th. 3.1 if we consider a perturbed nonlinear system of the form (\*). These limitations are due to the fact that constant multiples of solutions of (1.u) are not, in general, solutions of the same system. However, it is possible to know a priori the existence of solutions of (1.u), which belong to  $B$  and are uniformly bounded with respect to  $u \in M$  (a suitable subset of  $B$ ). Under such an assumption we provide the following:

**THEOREM 4.1.** *Assume that for every function  $u \in B$  there exists at least one solution  $x_u(t)$  of the system (1.u), which belongs to  $B$  and satisfies  $\|x_u(a)\| \leq \mu$  (a positive constant). Let also  $\|A(t, u)\| \leq p(t)$  for every  $(t, u) \in J \times R^n$ , where  $p \in C[J, R]$ . Furthermore, assume that*

$$(4.1) \quad \liminf_{n \rightarrow \infty} (1/n) \int_a^b \sup_{\|u\| \leq n} \|F(t, u)\| dt = 0.$$

*Then there exists at least one solution of (\*), which belongs to  $B$ .*

*Proof.* Fix  $u \in B$  and suppose that  $x_u \in B$  is a solution of (1.u) with  $\|x_u(a)\| \leq \mu$ . Then it follows from Gronwall's inequality that

$$(4.2) \quad \begin{aligned} \|x_u\|_J &\leq \left[ \mu + \int_a^b \|F(s, u(s))\| ds \right] \exp \left\{ \int_a^b \|A(s, u(s))\| ds \right\} \\ &\leq \left[ \mu + \int_a^b \|F(s, u(s))\| ds \right] \exp \left\{ \int_a^b p(s) ds \right\} \\ &= \lambda \left[ \mu + \int_a^b \|F(s, u(s))\| ds \right]. \end{aligned}$$

To show that there is some positive integer  $n_0$  such that  $\|u\| \leq n_0$  implies  $\|x_u\| \leq n_0$  for any solution  $x_u \in B$  of (1.u) with  $\|x_u(a)\| \leq \mu$ , assume on the contrary that there are sequences  $\{u_n\}$ ,  $\{x_n\}$ , with  $x_n$  a solution of

$$(4.1_n) \quad x' = A(t, u_n(t))x + F(t, u_n(t)), \quad \|x(a)\| \leq \mu,$$

such that  $\|u_n\|_J \leq n$ ,  $\|x_n\|_J \geq n$ . Then we have

$$(4.3) \quad \begin{aligned} 1 &\leq \|x_n\|_J/n \leq \lambda \left[ \frac{\mu}{n} + \frac{1}{n} \int_a^b \|F(s, u_n(s))\| ds \right] \\ &\leq \lambda \left[ \frac{\mu}{n} + \frac{1}{n} \int_a^b \sup_{\|u\| \leq n} \|F(s, u)\| ds \right], \end{aligned}$$

which implies a contradiction, because the inferior limit of the right-hand side of (4.3) is zero. Actually there is an infinity of such  $n_0$ 's. Now let  $S = \{u \in B; \|u\|_J \leq n_0, \|u'\|_J \leq d\}$ , where

$$(4.4) \quad d = n_0 \sup_{t \in J} p(t) + \sup_{\substack{t \in J \\ \|u\| \leq n_0}} \|F(t, u)\|.$$

The set  $S$  is equicontinuous, uniformly bounded and convex. The closure  $Q = \bar{S}$  is a compact and convex subset of  $C[J, R^n]$ . It is easy to see now that the operator  $U$ , which maps every function  $u \in Q$  into the set  $U_u$  of all solutions of (1.u) which belongs to  $B$  and satisfy  $\|x_u\|_J \leq \mu$ , satisfies  $Uu \subset Q$ . Moreover,  $Uu$  is a compact, convex subset of  $Q$ . The continuity of  $U$  follows as in Th. 3.1 as well as the rest of the proof.

It is evident that the above method cannot be applied in case  $F(t, 0) \equiv 0$ , because the fixed point of the operator  $U$  above could be the zero solution of (\*). As an application, we give the following example, in which the existence of solutions to the associated linear system has been shown by Schmitt [14].

EXAMPLE 4.1. Consider the scalar equation

$$(4.5) \quad x'' + p(t, x, x')x' + q(t, x, x')x = r(t, x, x'),$$

where  $p, q, r$  are continuous, periodic in  $t$  of period  $T > 0$ , and  $|p(t, u, v)| \leq p_1(t), |q(t, u, v)| \leq q_1(t), |r(t, u, v)| \leq r_1(t)$  for all  $(t, u, v) \in [0, T] \times R \times R$ . Assume that for each continuously differentiable  $T$ -periodic function  $u(t)$  with  $\alpha \leq u \leq \beta$  ( $\alpha, \beta$  fixed positive constants with  $\alpha < \beta$ ), we have

$$q(t, u(t), u'(t))\beta \leq r(t, u(t), u'(t)) \leq q(t, u(t), u'(t))\alpha$$

for all  $t \in [0, T]$ . Then the equation

$$(4.6) \quad x'' + p(t, u(t), u'(t))x' + q(t, u(t), u'(t))x = r(t, u(t), u'(t))$$

has at least one  $T$ -periodic solution  $x(t)$  such that  $\alpha \leq x(t) \leq \beta$ ,  $t \in (-\infty, \infty)$ . Now let

$$(4.7) \quad \lambda = \sup_{t \in [0, T]} |p_1(t)|, \quad \mu = \beta \sup_{t \in [0, T]} |q_1(t)| + \sup_{t \in [0, T]} |r_1(t)|.$$

Then

$$(4.8) \quad |x''(t)| \leq \lambda|x'(t)| + \mu.$$

Since the function  $g(s) = \lambda s + \mu$  satisfies

$$\int_0^\infty \frac{s ds}{\lambda s + \mu} = +\infty,$$

it follows from Lemma 5.1, p. 428, of [5] that there exists a number  $M > 0$ , depending only on  $g, \alpha, \beta$  such that  $|x'(t)| \leq M, t \in R$ . Let  $B$  be the Banach space of all functions  $f = (f_1, f_2) \in C[[0, T], R^2]$  such that  $f_1(0) = f_1(T)$ . Then the problem of the existence of nontrivial  $T$ -periodic solutions of (4.5) is equivalent to the problem of finding nontrivial solutions  $x \in B$  of the system

$$(4.9) \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

It is evident that the operator  $U$  of the proof of Th. 4.1 will be defined on a suitable compact, convex subset of the set  $\{u \in B; \|u\|_T = |u_1| + |u_2| \leq \beta + M\}$ . A particular case of (4.5) is the following equation:

$$(4.10) \quad x'' + [1 + x/(1 + x^2)]x' - [3 - \sin x']x = -[5 + \cos t].$$

Here we have  $\alpha = 1, \beta = 3$ . Thus, (4.10) has at least one  $2\pi$ -periodic solution  $x(t)$ , such that  $1 \leq x(t) \leq 3, t \in (-\infty, \infty)$ .

5. Positive solutions on infinite intervals. The theorem of this section extends to the case (\*) a result of Medvedev [12], who considered perturbed linear systems. By  $K$  we denote the cone  $\{\xi \in R^n; \xi_i \geq 0, i = 1, 2, \dots, n\}$ .

**THEOREM 5.1.** *For the system (\*) assume the following:*

(i) *the real  $n \times n$  matrix  $A(t, u)$  is defined and continuous on the set  $R \times K$ ;*

(ii)  $\| [I + hA(t, \xi)](x_1 - x_2) \| \leq (1 - ph)\|x_1 - x_2\|$  for every  $(t, \xi, x_1, x_2) \in R \times K \times R^n \times R^n, h \in (0, H)$ , where  $p, H$  are positive constants with  $pH < 1$ ;

(iii)  $[I + hA(t, \xi)]x \in K$  for every  $x \in K$  and  $(t, \xi, h) \in R \times K \times$

$(0, H)$ ;

(iv)  $F: R \times K \rightarrow R^n$ , continuous, and  $\alpha \leq F_i(t, \xi) \leq \beta$ ,  $(t, \xi) \in R \times K$ , where  $\alpha, \beta$  are constants with  $0 < \alpha < \beta$ .

Then the system (\*) has at least one solution  $x(t)$ , which is defined and bounded on  $R$ , and satisfies  $x_i(t) \geq \alpha$ ,  $t \in R$ ,  $i = 1, 2, \dots, n$ .

*Proof.* Consider the system

$$(5.f) \quad x' = A(t, f(t))x + F(t, f(t)), \quad t \in R,$$

where  $f \in C[R, R^n]$  and  $f(t) \in K$ ,  $t \in R$ . Then it follows from Th. 2 in [12] that there exists a unique bounded solution  $x_f(t)$ ,  $t \in R$  of (5.f) such that  $\|x_f\|_R \leq \|F\|/p = \lambda$ , where  $\|F\| = \sup \|F(t, u)\|$ ,  $(t, u) \in R \times K$ . Moreover, the components  $x_{f,i}(t)$ ,  $i = 1, 2, \dots, n$ , satisfy  $x_{f,i}(t) \geq \alpha$ ,  $t \in R$ . We note that  $x_f(t)$  is not necessarily the only solution of (5.f) with the property  $x_{f,i}(t) \geq \alpha$ . Now let  $\{t_m\}$ ,  $m = 1, 2, \dots$ ,  $0 < t_m < t_{m+1}$ , be a sequence of points such that  $\lim_{m \rightarrow \infty} t_m = +\infty$ , and consider the system

$$(5.m) \quad x' = A(t, f(t))x + F(t, f(t)), \quad t \in I_m = [-t_m, t_m],$$

where  $f \in C[-t_m, t_m, R^n]$ ,  $f_i(t) \geq \alpha$ ,  $i = 1, 2, \dots, n$ . If we fix  $m$  and  $f$ , then the system (5.m) has at least one solution  $x_f$ , which satisfies  $x_{f,i}(t) \geq \alpha$ ,  $t \in I_m$  and  $\|x_f\|_m \leq \lambda$  (here  $\|\cdot\|_m$  denotes the sup-norm on  $I_m$ ). In fact,  $f$  is the restriction on  $I_m$  of a function  $\bar{f} \in C[R, R^n]$  with  $f(t) \in K$ , and  $x_f$  is the restriction on  $I_m$  of the unique bounded solution of (5.f), which corresponds to  $\bar{f}$ . Now let

$$(5.6) \quad S = \{f \in C[I_m, R^n]; f(t) \in K, t \in I_m, \|f\|_m \leq \lambda, \|f'\|_m \leq \mu\},$$

where  $\mu = 4\|F\|/pH$ . Then  $\bar{S} = Q$  is a compact and convex set. Now let the operator  $U$  map the function  $f \in Q$  into the set of solutions of (5.m), which satisfy  $x_i(t) \geq \alpha$ ,  $t \in I_m$  and  $\|x\|_m \leq \lambda$ . Then every  $x \in Uf$  satisfies

$$\|x'\|_m \leq \left(\frac{4}{H} - p\right)\|F\|/p + \|F\| = \mu.$$

The above inequality follows from the fact that the vector  $A(t, \xi)x$  satisfies a Lipschitz condition w.r.t.  $x$  in  $R^n \times R^n$  with Lipschitz constant  $4/H - p$ , uniformly in  $(t, u) \in R \times R^n$  (cf. Medvedev [12]). Thus, as before, the set  $Uf$  is a compact, convex subset of  $Q$ . It remains to show that  $U$  is continuous, but this follows as before and the proof is omitted. Consequently, there exists at least one solution  $x_m(t)$  of (\*), which is defined on  $I_m$ , and satisfies  $x_{m,i}(t) \geq \alpha$ ,  $i = 1, 2, \dots, n$  and  $\|x_m\|_m \leq \lambda$ . Since the choice of  $m$  was arbitrary,

an application of the proof of Lemma 8.1, p. 149 of [9] proves the existence of a bounded solution of (\*), which, in our case, has to satisfy  $x_i(t) \geq \alpha > 0$ ,  $i = 1, 2, \dots, n$ ,  $t \in R$ , as a uniform limit, on finite intervals, of a diagonal sequence of functions with the same property.

EXAMPLE 5.1. Consider the system (\*) with

$$A(t, x) = \begin{bmatrix} -[1 + \sin^2 x_1] & 0 \\ \frac{1}{2} \cos x_2 & -[e^{-|tx_1|} + 1] \end{bmatrix}$$

and

$$F(t, x) = \begin{bmatrix} 2 + \sin(tx_1) \\ 2 + \cos(x_2^2 + 1) \end{bmatrix}.$$

Then we have

$$(5.7) \quad A(t, f(t))_{\xi}^{\xi} = \begin{bmatrix} -[1 + \sin^2 f_1(t)]\xi_1 \\ \frac{\xi_1}{2} \cos f_2(t) - [e^{-|tf_1(t)|} + 1]\xi_2 \end{bmatrix}$$

for every  $f \in C[R, R^2]$  with  $f(t) \in K$ ,  $t \in R$  and every  $\xi \in R^2$ . Thus, if the components of the vector (5.7) are  $Q_1$ ,  $Q_2$ , we have

$$\begin{aligned} -2 &\leq \frac{\partial Q_1}{\partial \xi_1} = -[1 + \sin^2 f_1(t)] \leq -1, \\ -2 &\leq \frac{\partial Q_2}{\partial \xi_2} = -[e^{-|tf_1(t)|} + 1] \leq -1, \\ \frac{\partial Q_1}{\partial \xi_2} &= 0, \quad \left| \frac{\partial Q_2}{\partial \xi_1} \right| \leq \frac{1}{2}. \end{aligned}$$

It follows from the remark (c) of Medvedev [12] that the condition (ii) of Th. 5.1 is satisfied with  $p = 1/2$  and  $h \in (0, 1)$ . Moreover, an  $H < 1$  can be chosen so that  $I + hA(t, \xi)$ ,  $h \in (0, H)$ , has nonnegative entries. It follows from Th. 5.1 that there exists at least one solution  $x(t)$  of (\*) such that  $x_i(t) \geq 1$  for  $t \in R$  and  $\|x\|_{R^2} \leq \|F\|/p = 12$ .

6. Discussion. A condition of the type (4.1) was first given by Opial [13], who considered the system (\*) along with the generalized boundary conditions  $Tx = r$ , where  $T$  is a bounded linear operator on  $C[[a, b], R^n]$  with values in  $R^n$ , and  $r \in R^n$  is fixed. In the case studied by Opial,  $T$  was invertible on  $R^n$ , and this reduces the problem to finding solutions to a system similar to (\*) but with homogeneous boundary conditions  $Tx = 0$ . Thus  $B = \{x \in C[[a, b], R^n];$



$Tx = 0$ }, and Schauder's fixed point theorem can be applied instead of one for multi-valued mappings. The considerations in Thms. 3.1 and 4.1 can be extended to infinite intervals, provided that enough information is available for the associated linear systems. We should note that every system of the form

$$(6.1) \quad x' = D(t, x) + E(t, x),$$

can be written in the form (\*) if the  $n$ -vector  $D(t, x)$  is continuously differentiable with respect to  $x$ . For a proof of this statement the reader is referred to [10, p. 73].

The author wishes to express his thanks to the referee for his helpful suggestions.

#### REFERENCES

1. C. Avramescu, *Problèmes aux limites non-linéaires*, Ann. Math. Pura ed Appl., **80** (1968), 167-176.
2. ———, *Sur l'existence des solutions pour un problème aux limites général*, Ann. Math. Pura ed Appl., **82** (1969), 69-81.
3. C. Corduneanu, *Sur les systèmes différentiels de la forme  $y' = A(x, y)y + b(x, y)$* , Anal. Stiint. Univ. "Al I Cusa" Iasi, (N.S.) **4** (1958), 45-52.
4. S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. J. Math., **68** (1946), 214-222.
5. P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1964.
6. S. T. Hu, *Theory of Retracts*, Wayne State Univ. Press, Detroit, 1965.
7. A. G. Kartsatos, *Convergence in perturbed nonlinear systems*, Tôhoku Math. J., **24** (1972), 539-546.
8. ———, *Bounded solutions to perturbed nonlinear systems and asymptotic relationships*, J. Reine Angewandte Math., to appear.
9. M. A. Krasnoselskii, *Translation along trajectories of differential equations*, Amer. Math. Soc. Transl., No. **18** (1968).
10. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. **1**, Academic Press, New York, 1969.
11. A. Lasota and Z. Opial, *Sur les solutions périodiques des équations différentielles ordinaires*, Ann. Polon. Math., **16** (1964), 69-94.
12. N. V. Medvedev, *Certain tests for the existence of bounded solutions of systems of ordinary differential equations*, Diff. Uravneniya, **4** (1968), 1258-1264.
13. Z. Opial, *Linear problems for systems of nonlinear differential equations*, J. Diff. Equations, **3** (1967), 580-594.
14. K. Schmitt, *Periodic solutions of linear second order differential equations with deviating argument*, Proc. Amer. Math. Soc., **26** (1970), 282-285.

Received March 16, 1973

UNIVERSITY OF SOUTH FLORIDA

