PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 2, 1974

CONJUGATIONS ON STABLY ALMOST COMPLEX MANIFOLDS

ALLAN L. EDELSON

A stably almost complex structure on a smooth manifold M is an automorphism $J: \tau_M \oplus \theta^k \to \tau_M \oplus \theta^k$ for some $k \geqq 0$, covering the identity map on M, and satisfying $J^2 = -1$. If $k = 0$, J is an almost complex structure. An involution $T: M \to M$ is a conjugation of (M, J) if there exists an involution $\alpha: \theta^k \to \theta^k$ covering T, such that $T_* \oplus \alpha$ is conjugate linear, i.e., $(T_* \oplus \alpha)\circ J = -J\circ(T_* \oplus \alpha)$. The bordism theory of conjugations has been studied by R. Stong. In § 2 of this article it is shown that every closed n-manifold can be realized as the fixed point set of a conjugation on a closed, $2n$-dimensional stably almost complex manifold. This should be compared to the result of Conner and Floyd that the fixed point set of a conjugation on an almost complex $2n$-manifold is n-dimensional, which is false for stably almost complex manifolds. The proof will use the following result:

LEMMA 1. *Every closed manifold is cobordant to the fixed point set of a conjugation on a closed, almost complex manifold.*

Let $H_{m,n}(C) \subset P^m(C) \times P^n(C)$ with $m \leqq n$, be the hypersurface defined as the locus of $w_0z_0 + w_1z_1 + \cdots + w_mz_m = 0$ (in homogeneous coordinates $(w_0, \cdots, w_m)$ and $(z_0, \cdots, z_n)$). Let $H_{m,n}(R)$ be the corresponding real hypersurface. Then generators for the cobordism ring η_* can be taken to be the manifolds $P^{2n}(R)$ and $H_{m,n}(R)$, which are fixed point sets of conjugations on $P^{2n}(C)$ and $H_{m,n}(C)$ respectively. The preceding lemma follows easily.

In § 3, almost complex conjugations on $S^{2q+1} \times S^{2q+1}$ are given, with fixed point set S^{2q+1}. As a consequence, any manifold obtained from $P^{2n}(R)$ or $H_{m,n}(R)$ by surgeries on odd dimensional spheres, is itself the fixed point set of a conjugation on an almost complex manifold.

We will also need the following definition. If T is a free involution on a compact manifold M, a characteristic submanifold for (M, T) is a submanifold $M' \subset M$ of codimension 1, such that $M = W_+ \cup W_-$ (where W_+ and W_- are compact submanifolds of M), $M' = W_+ \cap W_-$, and $T(W_+) = W_-$. M' can always be obtained as the pullback of P^{N-1} by an equivariant map $(M, T) \to (P^N, A)$, where A is the antipodal map.

2. Stably almost complex structures.

LEMMA 2. *The tangent sphere bundle of a manifold is stably almost complex and the bundle involution is a conjugation.*

Proof. Let $D(M)$ denote the tangent disc bundle of M, and $S(M)$ the sphere bundle, with projection map π. There is an isomorphism $\tau_{D(M)} \cong \pi^*\tau_M \oplus \pi^*\tau_M$, and an almost complex structure can be defined by $(x, y) \rightarrow (-y, x)$. The bundle involution acts as -1 in the bundle tangent to the fibres, identified with the second summand, and is a conjugation. Restricting to $S(M)$ gives a conjugation on $\tau_{S(M)} \oplus \nu_{S(M)}$, ν being the normal bundle to the boundary which is θ^1.

This lemma provides an important example of stably almost complex manifolds. We are now ready to state the main result of this section.

THEOREM 1. *Every closed n-dimensional manifold is the fixed point set of a conjugation on a closed 2n-dimensional stably almost complex manifold.*

Proof. Choose a cobordism (W^{n+1}, F_1, F_2) with F_2 an arbitrary closed n-manifold. Assume F_1 is the fixed point set of a conjugation on the closed, almost complex manifold M_1. We will construct a closed, stably almost complex $2n$-manifold M_2, with conjugation having fixed point set F_2. Let B denote the tangent sphere bundle to W. Then bB is the unit sphere bundle in $\tau_{bW} \oplus \nu_{bW}$, and the normal bundle of bB in B is trivial. There is then an induced stable almost complex structure and conjugation on bB. Note that throughout this paper, bM will denote the boundary of the manifold M.

LEMMA 3. *The tangent sphere bundle to bW is a stably almost complex submanifold of bB, invariant under the conjugation.*

Proof. Over bW the bundle τ_W splits as $\tau_{bW} \oplus \nu_{bW}$ and $\pi^*\nu_{bW}$ can be identified with the normal bundle in bB, of the tangent sphere bundle to bW. This normal bundle is trivial, so there is an induced stable almost complex structure. Now let S denote the tangent sphere bundle to bW.

LEMMA 4. *There is a stably almost complex submanifold $V \subset B$, invariant under the conjugation, with $bV = V \cap bB = S$.*

Proof. The involution T on B is free, and S is a characteristic submanifold for the restriction $T|_{bB}$. There is a map $f: bB/T \rightarrow P^N$, for N sufficiently large, that is transverse regular on P^{N-1} and with $S/T = f^{-1}(P^{N-1})$. f extends to a map $F\colon B/T \to P^N$, transverse regular on P^{N-1}. Pulling back P^{N-1} under F and lifting to the two-fold covering gives the desired submanifold V. Notice that S is the disjoint union of the tangent sphere bundles to F_1 and F_2.

There is a submanifold, V', of the tangent disc bundle to W consisting of V and the tangent disc bundle of bW. This has trivial normal bundle and is invariant under T. There are corners along S, which can be rounded off preserving the triviality of the normal bundle, and we obtain a smooth, stably almost complex manifold with conjugation. The fixed point set of the conjugation is $F_1 \cup F_2$.

Choose a neighborhood N' of F_1 in V', equivariantly diffeomorphic to the tangent bundle of F_1. Similarly choose a neighborhood N of F_1 in M_1. Define a diffeomorphism from $N\backslash F_1 \to N'\backslash F_1$ by sending $(x, v) \to (x, -v/\|v\|^2)$, where v is a tangent vector at x. This is smooth, and preserves the almost complex structure along the unit sphere bundle. Form a smooth manifold M_2 from $V'\backslash F_1 \cup M_1\backslash F_1$ by identifying the above submanifolds. There are almost complex structures on $V'\backslash N_1'$ and $M_1\backslash N_1$, where N_1' and N_2 are the vectors of length $\leqq 1$. These agree on sphere bundles, and hence M_2 has a stable almost complex structure, provided we add to $\tau_{V'}$ a trivial complex line bundle. The involution on $M_1\backslash F_1$ is free and hence the fixed point set is F_2. This completes the proof of Theorem 1.

3. Conjugations on $S^{2q+1} \times S^{2q+1}$. In [1], Calabi and Eckmann have described almost complex structures on $S^{2q+1} \times S^{2q+1}$. In this section we will describe a conjugation having fixed point set S^{2q+1}. We begin with a description of the principal bundles involved.

Let $\{U_i\}_{0 \leqq i \leqq q}$ be the standard open covering of $P^q(C)$ by coordinate neighborhoods. Then $\{U_i \times U_\alpha\}_{0 \leqq i, \alpha \leqq q}$ is an open covering of $P^q(C) \times P^q(C)$ by coordinate neighborhoods. Let $U_{i\alpha} = U_i \times U_\alpha$. As in [4, Ch. 9], define a principal bundle B over $P^q(C) \times P^q(C)$ with group $G = S^1 \times S^1$ and transition functions $\Psi_{i\alpha,j\beta}\colon U_{i\alpha} \cap U_{j\beta} \to G$ given by

$$\Psi_{i\alpha,j\beta}([Z], [W]) = \left(\frac{z_i \,|\, z_j\,|}{z_j \,|\, z_i\,|}, \frac{w_\alpha \,|\, w_\beta\,|}{w_\beta \,|\, w_\alpha\,|}\right).$$

Note that $Z = (z_0, \cdots, z_q)$, $W = (w_0, \cdots, w_q)$, and $Z \in S^{2q+1}$, $W \in S^{2q+1}$. Then $\Psi_{i\alpha,j\beta}\Psi_{j\beta,k\gamma} = \Psi_{i\alpha,k\gamma}$. Now let $B_{i\alpha} = U_{i\alpha} \times G$ and define $\tilde{T}_{i\alpha}\colon B_{i\alpha} \to B_{\alpha i}$ by $\tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) = ([\bar{W}], [\bar{Z}], \bar{\mu}, \bar{\lambda})$.

LEMMA 5. *The map $\tilde{T}\colon B \to B$ defined by $\tilde{T}|_{B_{i\alpha}} = \tilde{T}_{i\alpha}$ is a well-defined involution covering $T([Z], [W]) = ([\bar{W}], [\bar{Z}])$.*

Proof. We need to show that the diagram

$$\begin{array}{ccc} B_{i\alpha} & \xrightarrow{\tilde{T}_{i\alpha}} & B_{\alpha i} \\ \downarrow & & \downarrow \\ B_{j\beta} & \xrightarrow{\tilde{T}_{j\beta}} & B_{\beta j} \end{array}$$

in which the vertical maps are the identifications defined on the appropriate intersections, is commutative. We have

$$\begin{aligned} \Psi_{\alpha i,\beta j}\tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) &= \left([W], [Z], \frac{\bar{w}_\alpha \,|\, \bar{w}_\beta \,|}{\bar{w}_\beta \,|\, \bar{w}_\alpha|}\bar{\mu}, \frac{\bar{z}_i \,|\, \bar{z}_j \,|}{\bar{z}_j \,|\, \bar{z}_i \,|}\bar{\lambda}\right) \\ &= \tilde{T}_{j\beta}\Psi_{i\alpha,j\beta}([Z], [W], \lambda, \mu) \end{aligned}$$

and so the diagram commutes. The remainder of the lemma is clear. Note the use of the symbol $\Psi_{i\alpha,j\beta}$ to denote the map $B_{i\alpha} \rightarrow \beta_{j\alpha}$ defined on the appropriate intersection.

Define a map $h_{i\alpha}\colon B_{i\alpha} \rightarrow S^{2q+1} \times S^{2q+1}$ by

$$h_{i\alpha}([Z], [W], \lambda, \mu) = \left(\lambda \frac{z_i}{|\, z_i \,|} Z, \mu \frac{w_\alpha}{|\, w_\alpha \,|} W\right).$$

Then $h_{j\beta}\Psi_{i\alpha,j\beta} = h_{i\alpha}$ so that there is a well-defined diffeomorphism $h\colon B \rightarrow S^{2q+1} \times S^{2q+1}$.

LEMMA 6. *The involution*

$$h\tilde{T}h^{-1}\colon S^{2q+1} \times S^{2q+1} \longrightarrow S^{2q+1} \times S^{2q+1}$$

is given by $(Z, W) \rightarrow (\bar{W}, \bar{Z})$.

Proof. We have

$$h_{\alpha i}\tilde{T}_{i\alpha}([Z], [W], \lambda, \mu) = \left(\bar{\mu}\frac{\bar{w}_\alpha}{|\, \bar{w}_\alpha \,|}\bar{W}, \bar{\lambda}\frac{\bar{z}_i}{|\, \bar{z}_i \,|}\bar{Z}\right),$$

and the lemma follows.

Again following [4], consider the principal bundle B' over $P^q(C) \times P^q(C)$ with group $G' = C/D$ where D is the subgroup of C generated by the complex numbers $\{1, i\}$. Define transition functions $\Psi'_{i\alpha,j\beta}\colon U_{i\alpha} \cap U_{j\beta} \rightarrow G'$ by

$$\begin{aligned} \Psi'_{i\alpha,j\beta}([Z], [W]) = &-\frac{1}{2\pi i}(\log |\, z_i \,| + i \log |\, w_\alpha \,|) + \frac{1}{2\pi i}\left(\log \frac{z_i}{z_j} + i \log \frac{w_\alpha}{w_\beta}\right) \\ &+ \frac{1}{2\pi i}(\log |\, z_j \,| + i \log |\, w_\beta \,|). \end{aligned}$$

We wish to define a bundle equivalence $f\colon B \to B'$. First define an isomorphism $g\colon G \to G'$ by

$$g(\lambda, \mu) = \left(\frac{1}{2\pi i}\log \lambda\right) + i\left(\frac{1}{2\pi i}\log \mu\right).$$

It follows that $g\Psi_{i\alpha\, j\beta} = \Psi'_{i\alpha,\, j\beta}\colon U_{i\alpha} \cap U_{j\beta} \to G'$, and hence that f can be defined by defining $f_{i\alpha} = 1 \times g\colon B_{i\alpha} \to B'_{i\alpha}$. There is an induced involution $\tilde{T}'_{i\alpha} = (1 \times g)\tilde{T}_{i\alpha}(1 \times g^{-1})\colon B'_{i\alpha} \to B'_{\alpha i}$ given by

$$\tilde{T}'_{i\alpha}([Z], [W], [v]) = ([\bar{W}], [\bar{Z}], [i\bar{v}]),$$

and an involution $\tilde{T}'\colon B' \to B'$. Here $[v]$ denotes the class in G' of the complex number v.

LEMMA 6. *$\tilde{T}'$ is a conjugation of the complex manifold B'.*

Proof. In local coordinates, $\tilde{T}'$ is given by $\tilde{T}'([Z], [W], [v]) = ([\bar{W}], [\bar{Z}], [i\bar{v}])$. We need only verify that the map $[v] \to [i\bar{v}]$ is a conjugation of the complex manifold $G' = C/D$. Since this map sends $[iv]$ to $[(-i)i\bar{v}]$, the lemma follows.

THEOREM 2. *S^{2q+1} is the fixed point set of a conjugation $S^{2q+1} \times S^{2q+1}$.*

Proof. The diffeomorphisms $f\colon (B, \tilde{T}) \to (B', \tilde{T}')$ and $h\colon (B, \tilde{T}) \to (S^{2q+1} \times S^{2q+1}, \bar{T})$ are equivariant with respect to the given involutions, and commute with the projections onto $P^q(C) \times P^q(C)$. Note that $\bar{T}$ is defined by $\bar{T}(Z, W) = (\bar{W}, \bar{Z})$. Then theorem follows since the fixed point set of $\bar{T}$ is diffeomorphic to S^{2q+1}.

REFERENCES

1. E. Calabi and B. Eckmann, *A class of compact, complex manifolds which are not algebraic*, Annals of Math., **58** (1953), 494-500.
2. P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer-Verlag, Berlin, 1969.
3. ———, *Differentiable Periodic Maps*, Springer-Verlag, Berlin, 1964.
4. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2, Interscience Publishers, New York, 1969.
5. P. S. Landweber, *Fixed point free conjugations on complex manifolds*, Annals of Math., **86** (967), 491-502.
6. J. Milnor, *Stiefel-Whitney numbers of complex manifolds and of spin manifolds*, Topology v. III, (1965), 223-230.
7. R. E. Stong, *Notes on Cobordism Theory*, Princeton Univ. Press, 1968.

Received April 6, 1973 and in revised form August 7, 1973.

UNIVERSITY OF CALIFORNIA, DAVIS