CHARACTERIZATION OF A FUNCTION BY CERTAIN INFINITE SERIES IT GENERATES

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Let A be a set of real numbers and F be a class of complex-valued functions defined on the real line such that for each $f \in F$ the infinite series $S(x, f) = \sum_{k=1}^{\infty} f(kx)$ converges for every nonzero x in A. If $0 \in A$, we set $S(0, f) = f(0)$. It seems to be an interesting problem to study the different sets A and function classes F such that each $f \in F$ is uniquely determined by the sums $S(x, f)$ where $x \in A$. Clearly, the larger the class F is studied, the larger set A is needed to guarantee uniqueness. We have positive results for a class of entire functions of exponential type and for fairly large classes of continuous functions. Some examples are also given to show that in general A cannot be too small.

1. **Introduction.** For a function f holomorphic in the open unit disc U of the complex plane and continuous on the closure of U, let

$$s_n(f, \delta) = \frac{1}{n} \sum_{k=1}^{n} f(e^{i2\pi\delta k/n}), \qquad n = 1, 2, \cdots, 0 < \delta \leq 1.$$

Sufficient conditions on the function f were given in [2, 5] to guarantee that f is uniquely determined by the means $s_n(f, 1)$, and in [3] both positive and negative results were given for the case $0 < \delta < 1$. The annulus case was also studied in [4]. In this paper we study a related problem for an unbounded interval.

Let A be a set of real numbers and F be a class of complex-valued functions defined on the real line R such that for each $f \in F$, the infinite series $S(x, f) = \sum_{k=1}^{\infty} f(kx)$ converges for every nonzero $x \in A$. If $0 \in A$, we denote $S(0, f) = f(0)$. We study various sets A and various function classes F such that each $f \in F$ is uniquely determined by the sums $S(x, f)$ where $x \in A$. Some examples will be given to show that in general A cannot be too small.

2. **Results for entire functions.** We start with a fairly small function class. For $r > 0$, let $PW(r\pi, 1)$ be the class of all entire functions $f(z)$ of exponential type at most $r\pi$ such that $f(x) \in L^2(R)$ and that for some $p > 1$, $f(n/r) = 0(|n|^{-p})$ for $n = \pm 1, \pm 2, \cdots$. Let Z be the set of all integers. We have the following theorem of the Carlson type.

THEOREM 1. *Every function f in $PW(\pi, 1)$ is uniquely determined by the sequence $S(n, f)$ where $n \in Z$.*

We remark that the above result is in a sense sharp in that every $S(n, f)$, $n \in Z$, is needed to determine f. That is, we have the following

LEMMA 1. *For each $n \in Z$, there is a unique function $f_n \in PW(\pi, 1)$ such that $S(m, f_n) = \delta_{m,n}$ for all $m \in Z$, where $\delta_{m,n}$ is the Kronecker delta.*

Let $\mu(n)$ denote the Möbius function (cf. [8]), that is,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with distinct primes } p_j \\ 0 & \text{if } p^2 | n \text{ for some } p > 1 . \end{cases}$$

We can determine the functions f_n in the above lemma explicitly as in the following

LEMMA 2.

$$f_0(z) = \frac{\sin \pi z}{\pi z} , \qquad\qquad \text{and for} \quad n = 1, 2, \cdots ,$$

$$f_n(z) = \sum_{k \mid n} \mu\left(\frac{n}{k}\right) \frac{\sin \pi(z - k)}{\pi(z - k)} , \quad \text{and for} \quad n = -1, -2, \cdots ,$$

$$f_n(z) = f_{-n}(-z) .$$

Of course, at the removable singular points these functions are defined so that they are entire. It is then clear that they are in the class $PW(\pi, 1)$. By using the equations

$$\sum_{k \mid n} \mu\left(\frac{n}{k}\right) = \delta_{n,1} ,$$

it is not difficult to show that they satisfy the relations $S(m, f_n) = \delta_{m,n}$ for all $m, n \in Z$. Finally, the uniqueness result in Theorem 1 guarantees that they must be the functions in Lemma 1. By using this sequence of functions f_n, we can actually construct each function $f \in PW(\pi, 1)$ from the corresponding sequence $S(n, f)$, $n \in Z$, as in the following

THEOREM 2. *Let $f \in PW(\pi, 1)$. Then*

$$(1) \qquad\qquad f(z) = \sum_{n=-\infty}^{\infty} S(n, f) f_n(z)$$

where the infinite series converges uniformly on each horizontal strip $|\operatorname{Im} z| \leqq K < \infty$ of the complex plane.

To prove Theorem 1, we let $f \in PW(\pi, 1)$ such that $S(m, f) = 0$

for all $m \in Z$. That is $f(0) = 0$,

$$(2) \qquad \sum_{k=1}^{\infty} f(km) = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} f(-km) = 0$$

for all $m = 1, 2, \cdots$. Since $f(n) = 0(1/|n|^p)$ for some $p > 1$ and $n = \pm 1, \pm 2, \cdots$, these infinite series converge absolutely. As in [2] we multiply $\mu(m)$ to the first series in (2) and sum it from 1 to N to obtain

$$(3) \qquad f(1) = - \sum_{j=N+1}^{\infty} \left(\sum_{\substack{n|j}}^{1 \le n \le N} \mu(n) \right) f(j) \, .$$

Let $d(m)$ be the number of divisors of m and it is known (cf. [8]) that $d(m) = O(m^{\delta})$ for all positive δ. Pick $\delta = (p - 1)/2$. Then from (3), we get

$$|f(1)| \le \sum_{j=N+1}^{\infty} \sum_{n|j} |\mu(n)| \, |f(j)|$$

$$\le \sum_{j=N+1}^{\infty} d(j)|f(j)| = O(N^{-(p-1)/2}) \, .$$

Hence, $f(1) = 0$. For each fixed $k \ge 1$, we let $a_j = f(jk)$ and note that $\sum_{\nu=1}^{\infty} a_{j\nu} = 0$. This gives $f(k) = a_1 = 0$. Hence, $f(k) = 0$ for $k = 1, 2, \cdots$. Applying the same proof to the second series in (2), we have $f(k) = 0$ for $k = -1, -2, \cdots$. Next, by the Paley-Wiener theorem (cf. [1]) we can write

$$f(z) = \int_{-\pi}^{\pi} e^{izt} g(t) dt$$

where $g \in L^2(-\pi, \pi)$. Since $f(k) = 0$ for all $k \in Z$, all the Fourier coefficients of g vanish, so that $g = 0$ a.e., or f is the zero function. This completes the proof of Theorem 1.

To prove Theorem 2, we let K be any positive number, and by a Phragmen-Lindelof theorem, we note that $|\sin \pi z/\pi z| \le e^{\pi K}/\pi K = C_K$ for all z in the strip $|\operatorname{Im} z| \le K$. Hence, for $|\operatorname{Im} z| \le K$,

$$|f_n(z)| \le C_k \sum_{k|n} \left| \mu\left(\frac{n}{k}\right) \right| \le C_K d(n) \, .$$

Let $f \in PW(\pi, 1)$. Then $f(n) = O(|n|^{-p})$ for some $p > 1$ as $|n| \to \infty$. Hence, for $n = \pm 1, \pm 2, \cdots$

$$|S(n, f)| \le \sum_{k=1}^{\infty} |f(kn)| \le c_1 \sum_{k=1}^{\infty} k^{-p}|n|^{-p} = c_2|n|^{-p} \, .$$

Pick $\delta = (p - 1)/2$. Then for all z in the strip $|\operatorname{Im} z| \le K$, we have $|S(n, f)f_n(z)| \le c_3|n|^{-p}d(|n|) \le C_{K,\delta}|n|^{-(p+1)/2}$, so that the series

$$\sum_{n=-\infty}^{\infty} S(n, f) f_n(z)$$

converges uniformly on every horizontal strip $|\operatorname{Im} z| \leqq K < \infty$ to an entire function $f^*(z)$. Next, we consider the sequence

$$g_n(t) = \frac{1}{2\pi} S(n, f) \sum_{k|n} \mu\left(\frac{n}{k}\right) e^{-ikt} .$$

By the same estimate as above and by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} g_n(t)$ converges uniformly to a continuous function $g(t)$ on $[-\pi, \pi]$. Hence,

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{itz} g(t) dt &= \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} e^{itz} g_n(t) dt \\
&= \sum_{n=1}^{\infty} S(n, f) \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(z-k)t} dt \\
&= \sum_{n=1}^{\infty} S(n, f) \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{\sin \pi(z-k)}{\pi(z-k)} \\
&= \sum_{n=1}^{\infty} S(n, f) f_n(z) .
\end{aligned}$$

Therefore, we have

$$f^*(z) = \int_{-\pi}^{\pi} e^{itz} (g(t) + g(-t)) dt + f(0) .$$

By the Paley-Wiener theorem (cf. [1], p. 103) we can conclude that f^* is an entire function of exponential type at most π and belongs to $L^2(R)$ on the real axis. For positive integers n, it is not difficult to show that

$$\begin{aligned}
|f^*(n)| &= \left| \sum_{k=1, n|k}^{\infty} \mu\left(\frac{k}{n}\right) S(k, f) \right| \\
&= \left| \sum_{k=1}^{\infty} \mu(k) S(kn, f) \right| \leqq \sum_{k=1}^{\infty} |S(kn, f)| \\
&\leqq C_1 \sum_{k=1}^{\infty} k^{-p} n^{-p} = O(n^{-p}) .
\end{aligned}$$

This also holds for negative integers n. That is, we have proved that $f^* \in PW(\pi, 1)$. Now, since $S(n, f^*) = \sum_{k=-\infty}^{\infty} S(k, f) S(n, f_k) = S(n, f)$ for all $n \in Z$, we have $f = f^*$ by Theorem 1, and thus, the proof of Theorem 2 is completed.

3. **Examples and counterexamples.** In this section we will present some examples which will be used to indicate the sharpness of results to follow. The first proposition deals with entire functions of exponential type.

PROPOSITION 1.   *Every function $f \in PW(r\pi, 1)$, $r > 0$, is uniquely determined by the sequence $S(x, f)$ where $x \in r^{-1}Z$; furthermore $r^{-1}Z$ cannot be reduced.*

The next three propositions deal with more general sets A and functions defined on the interval $[0, \infty) = R^+$. If A is bounded below, we have

PROPOSITION 2.   *If $0 < \delta = \inf \{x : x \in A\}$ then every f with*

$$(\text{supp } f) \subset [0, \delta)$$

satisfies $S(x, f) = 0$ for all $x \in A$.

If A is bounded above we obtain

PROPOSITION 3.   *If $\Delta = \sup \{x : x \in A\} < \infty$ then there is a nontrivial function $f \in PW(2\pi/\Delta, 1)$ satisfying $S(x, f) = 0$ for all $x \in A$.*

Finally we show that A may contain an interval about the origin and also be unbounded or A may be a neighborhood of infinity and contain a sequence converging to zero. Let S be the space of complex-valued functions f which are in C^∞ and for any $n > 0$, $|f(x)| |x|^n = O(1)$ as $|x| \to \infty$, and let Z^+ be the set of positive integers. We have

PROPOSITION 4.   *There is a nontrivial function $f \in S$ whose cosine transform f_c is also in S satisfying $S(x, f) = 0$ for all $x \in A = [0, 1/2] \cup Z^+$ and $S(x, f_c) = 0$ for all $2\pi/x \in A$.*

We turn now to the proof of Proposition 1. Actually Proposition 1 requires little work since it can be seen that this is just a restatement of Theorem 1 and Lemma 1 with a change of variable. The important point in Proposition 1 is that as $A = r^{-1}Z$ becomes larger, i.e., r gets larger, the sums characterize a larger function class.

Proposition 2 is clear since all the sums vanish identically. We will soon show that this proposition is in a sense dual to Proposition 3. The duality is due to the Poisson summation formula. Let $f \in L^1(R^+)$ and following [10] define f_c the cosine transform of f to be

$$f_c(t) = \sqrt{\frac{2}{\pi}} \int_{R^+} f(x) \cos xt \, dx \ .$$

If f, in addition, is continuous and $\{f(n\alpha)\}_{n=0}^\infty \in l^1$ for $\alpha > 0$ then

$$\beta^{1/2}\left[\frac{1}{2}f_c(0) + \sum_{n=1}^\infty f_c(n\beta)\right] = \alpha^{1/2}\left[\frac{1}{2}f(0) + \sum_{n=1}^\infty f(n\alpha)\right]$$

where $\alpha\beta = 2\pi$. This formula is known as the Poisson summation

formula.

If f has compact support in

$$(0, 2\pi/\varDelta), \int_{R^+} f = 0, \quad \text{and} \quad A \cap [0, 2\pi/\varDelta) = \varnothing ,$$

then Proposition 1 tells us that $S(x, f) = 0$. Suppose f is smooth enough (say C^2) so that $f_c \in PW(2\pi/\varDelta, 1)$. Then, by the Poisson summation formula we have $S(2\pi/x, f_c) = 0$ for all $x \in A$. Note that $\sup \{2\pi/x: x \in A\} \leqq \varDelta$, and this proves Proposition 3.

Thus, if we wish to recapture a large class of functions from their sums we must have a set A which is unbounded and contains a sequence converging to 0. Proposition 4 shows that we even need much more than that. Let f have a cosine transform f_c which is in C^∞ and has compact support in $(0, 4\pi)$ (hence $f_c \in S$). Suppose $\int_{R^+} f_c = 0$ and f_c is odd about 2π, then $S(x, f_c) = 0$ for all $2\pi/x \in A$ and then by the Poisson summation formula $S(x, f) = 0$ for all $x \in A$. Furthermore, $f \in S$ since the Fourier transform is an $L^2(R)$ isomorphism of S onto S.

4. Results for larger classes. Let $W^{1,1}(R^+)$ be the Sobolev space of functions in $L^1(R^+)$ which are absolutely continuous and whose derivatives are in $L^1(R^+)$. In [9] it was shown that for any $f \in W^{1,1}(R)$, $S(x, f)$ is absolutely convergent for any $x \neq 0$. Of course, a similar statement is true for $f \in W^{1,1}(R^+)$ and $x > 0$. We have the following result.

THEOREM 3. *Let* $A = \{L_j/n: n = 1, 2, \cdots, 0 = L_1 < L_2 < \cdots\}$ *then the set of functionals* $\{S(x, \cdot): x \in A\}$ *is total over* $W^{1,1}(R^+)$ *if and only if the sequence* $\{L_j\}$ *is unbounded.*

If the L_i are bounded above, then Proposition 3 tells us that $\{S(x, \cdot): x \in A\}$ is not total. The converse is a bit more difficult. Suppose $\{L_i\}_{i=1}^\infty$ is unbounded, and suppose f is in $W^{1,1}(R^+)$ and satisfies $S(x, f) = 0$ for all $x \in A$. We must show that $f \equiv 0$. We first observe that $\int_{R^+} f = 0$. Indeed, for any $h > 0$ we have

$$
\begin{aligned}
\left| \int_0^\infty f(t)dt - hS(h, f) \right| &= \left| \int_0^\infty \Big(f(t) - \sum_{k=1}^\infty \chi_{[(k-1)h, kh]}(t)f(kh) \Big) dt \right| \\
&\leqq \sum_{k=1}^\infty \int_{(k-1)h}^{kh} |f(t) - f(kh)| \, dt \\
&= \sum_{k=1}^\infty \int_{(k-1)h}^{kh} \left| \int_{kh}^t f'(\tau)d\tau \right| dt \\
&\leqq |h \int_0^\infty |f'| \, .
\end{aligned}
$$

(4)

Let $\{h_j\}$ be a sequence in A with $h_j \to 0$. Setting $h = h_j$ in (4) yields the result.

Let $\nu(t)$ be the "saw-toothed" function defined by

$$\nu(t) = \begin{cases} [t] - t + \dfrac{1}{2} & t \notin Z \\ \\ 0 & t \in Z . \end{cases}$$

For any $x \in A$, $x \neq 0$, and f as above, we have as in [5]

$$0 = S(x, f) = \sum_{k=1}^{\infty} f(kx)$$

(5)
$$= - \int_0^{\infty} \nu(t/x)f'(t)dt + \frac{1}{x} \int_0^{\infty} f(t)dt - \frac{1}{2}f(0)$$

$$= - \int_0^{\infty} \nu(t/x)f'(t)dt .$$

In [6, 7] Davenport established

$$\sin 2\pi n t = \lim_{N \to \infty} \sum_{k=1}^{N} \frac{\mu(k)}{k} \pi \nu(nkt)$$

where the convergence is uniform. Now for any $x \in A$, $x/k \in A$. Hence, if $x \neq 0$, $x \in A$, we have by (5)

$$0 = \lim_{N \to \infty} \sum_{k=1}^{N} \frac{\mu(k)}{k} \pi \int_0^{\infty} \nu(kt/x)f'(t)dt$$

$$= \int_0^{\infty} [\sin (2\pi \, t/x)]f'(t)dt .$$

But this means that the Fourier sine transform of f' vanishes at all the points $2\pi/x$, $x \in A$. Since this set is dense on R^+ and $f' \in L^1(R^+)$ we conclude that $f' \equiv 0$. Hence, $f \equiv 0$ which was to be shown.

It is interesting to note the algebraic nature of the set A in Theorem 3. In order to use Davenport's result, we needed to have $kA^{-1} \subset A^{-1}$ for every positive integer k. Along these same lines we have

LEMMA 3. *Let f be a bounded function from R^+ into R satisfying $f(x) = O(x^{-p})$ for some $p > 1$, as $x \to \infty$. Let $A = (\delta, \infty)$, $\delta \geq 0$, then the values $S(x, f)$, $x \in A$, uniquely determine f on (δ, ∞).*

For suppose that $S(x, f) = 0$, $x \in A$, then we obtain the infinite homogeneous linear system

$$0 = S(nx, f) , \qquad\qquad n = 1, 2, \cdots .$$

As seen in the proof of Theorem 1, this system has only the trivial solution, hence $f(x) = 0$ for all $x \in A = (\delta, \infty)$. This lemma is remarkable in that very few restrictions were put on the functions f. In fact, f may be nonmeasurable. Of course, as a corollary we obtain

COROLLARY 1. *If f is as in Lemma 3 and f is continuous, then $S(x, f)$, $x \in A$, uniquely determines f on (δ, ∞) where A need only be a dense subset of (δ, ∞) satisfying $kA \subset A$ for all $k \in Z^+$.*

In all the above results of this section we have needed the density of the determining set A^{-1} in R^+ (as in Theorem 3) or the whole interval (δ, ∞) when we determine the functions on (δ, ∞). Furthermore, Propositions 2, 3, and 4 seem to indicate that A must be large in order to characterize large function classes. However, we have the following

LEMMA 4. *If $f \in C^2(R^+)$, $f(x) = O(x^{-p})$ for some $p > 1$, as $x \to \infty$, and $f'(0) = 0$, then f is uniquely determined by the sums $S(x, f)$ for all x in*

$$A = [0, \delta_1) \cup (\delta_2, \infty)$$

where $0 < \delta_1 < \delta_2$.

Suppose $S(x, f) = 0$, $x \in A$. Then by Lemma 3 we know that $(\operatorname{supp} f) \subset [0, \delta_2]$. Let, as before, f_c be the cosine transform of f. Due to the regularity of f we see that f_c also satisfies the hypotheses of Lemma 3. Furthermore, the Poisson summation formula yields

$$0 = S(x, f) = S\left(\frac{2\pi}{x}, f_c\right)$$

for all $x \in (0, \delta_1)$. Lemma 3 then implies that f_c vanishes identically in $(2\pi/\delta_1, \infty)$. But f_c is entire so $f_c \equiv 0$. Thus $f \equiv 0$ which was to be proved.

Comparing Lemma 4 to Proposition 4 it is clear that the hypotheses can be relaxed only slightly since the functions in Proposition 4 can be chosen to satisfy the hypotheses of Theorem 4. We do have the following

THEOREM 4. *Let f be as in Lemma 4, and for $\delta > 0$ let*

(6) $$A = (\delta, \infty) \cup \{2\delta/k \colon k \in Z^+\} .$$

Then f is uniquely determined by the sums $S(x, f)$, $x \in A$.

As usual we assume that $S(x, f) = 0$, $x \in A$, and conclude by Lemma 3 that $(\text{supp } f) \subset [0, \delta]$. It follows that f_c is entire of exponential type at most δ. By the Poisson summation formula

$$(7) \qquad\qquad S(x, f_c) = 0$$

for $x \in \{k\pi/\delta\}$, $k = 1, 2, \cdots$. Integration by parts shows that $f_c(x) = O(x^{-2})$ as $x \longrightarrow \infty$, hence $f_c \in PW(\delta, 1)$. Since f_c is even we have by (7) $S(k\pi/\delta, f_c) = 0$ for $k = \pm 1, \pm 2, \cdots$. By a change of variable as in Proposition 1 we conclude by Theorem 2 that $f_c(z) = a \sin \delta z/\delta z$. The constant a must be zero since $f_c(x) = O(x^{-2})$ as $x \longrightarrow \infty$. It follows that $f \equiv 0$.

We finally remark that by Proposition 4 the set A in (6) cannot be replaced by

$$(\delta, \infty) \cup \{2\delta/k\colon k \in 4Z^+\} \ .$$

However, if we assume in addition $f \in C^\infty$ we can replace A in (6) by the set

$$(\delta, \infty) \cup \{2\delta/k\colon k = N, N + 1, \cdots\}$$

for any positive integer N.

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