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LINEAR OPERATORS FOR WHICH T^*T AND TT^* COMMUTE (II)

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Let (BN) denote the class of all bounded linear operators on a Hilbert space such that T^*T and TT^* commute. Let $(BN)^+$ be those $T \in (BN)$ which are hyponormal. Embry has observed that if $T \in (BN)$, then $0 \in W(T)$ or T is normal. This is used to show that if $T \in (BN)$, then $(T + \lambda I) \notin (BN)$ unless T is normal. It is also shown that if $T \in (BN)^+$, then T^n is hyponormal for $n \geqq 1$. An example of a $T \in (BN)^+$ such that $T^2 \notin (BN)$ is given. Paranormality of operators in (BN) is shown to be equivalent to hyponormality. The relationship between T being in (BN) and T being centered is discussed. Finally, all 3×3 matrices in (BN) are characterized.

This paper is a continuation of [3]. In that paper we studied bounded linear operators T acting on a separable Hilbert space $\mathscr{h}$ such that T^*T and TT^* commute. Such operators are called bi-normal and the class of all such operators is denoted (BN). This paper will explore some of the properties of hyponormal bi-normal operators. In addition, we will show that no translate of a non-normal bi-normal operator is bi-normal and characterize all 2×2 and 3×3 bi-normal matrices.

It has been pointed out to the author that the term bi-normal has been used earlier by Brown [2]. However, his usage does not appear to be in the current literature so we will continue to use bi-normal for operators in (BN).

1. All shifts, weighted and unweighted, bilateral and unilateral, are in (BN). Further, operators in (BN), if completely nonnormal, have a tendency to be "shift-like". Our first result, due to Embry, is an example of this.

THEOREM 1. *If $T \in (BN)$, then either T is normal or zero is in the interior of the numerical range of T, $W(T)$.*

Proof. Embry has shown that if $T \in (BN)$ and T is not normal, then $0 \in W(T)$ [7, Theorem 1]. She has also shown that if $T \in (BN)$ and $T + T^* \geqq 0$, then T is normal [5, Theorem 2]. Thus if 0 were on the boundary of $W(T)$, by a suitable choice of α, $|\alpha| = 1$, we could consider $T_1 = \alpha T$ where $T_1 \in (BN)$ and $T_1 + T_1^* \geqq 0$. Then T would be normal.

An interesting consequence of Theorem 1 is that no translate of a bi-normal operator can be bi-normal unless the original operator was normal.

For bounded linear operators X and Y let $[X, Y] = XY - YX$.

THEOREM 2. *Suppose that* $T \in (BN)$. *Then* $T + \lambda I \in (BN)$, *some complex* $\lambda \neq 0$, *if and only if* T *is normal.*

Proof. Suppose $T \in (BN)$. Let $\lambda \neq 0$ be real. Then

$$[(T + \lambda I)^*(T + \lambda I), (T + \lambda I)(T + \lambda I)^*] = 0$$

is equivalent to $[[T^*, T], T + T^*] = 0$. Thus if $T + \lambda I \in (BN)$ for some real $\lambda \neq 0$, then $T + \lambda I \in (BN)$ for all real λ. But $0 \notin W(T + \lambda I)$ for λ sufficiently large so T would be normal by Theorem 1. The case when λ is complex easily reduces to the one when λ is real.

2. One reason that the class (BN) is of interest is that it includes many of the weighted translated operators of Parrott [10], and nonanalytic composition operators, such as those studied by Ridge [12]. In particular, (BN) includes the Bishop operator [10, p. 2] for which the question of invariant subspaces is still open.

The Bishop operator actually falls into the following class which is more restrictive than (BN).

DEFINITION 1. A bounded linear operator T is called centered if the set $\{T^n T^{*n}, T^{*n} T^n\}_{n=0}^{\infty}$ consists of pairwise commuting operators.

Centered operators have been studied by Muhly [9] and Morrell [8]. Muhly has shown that centered operators with zero kernels and dense ranges are the direct sums of weighted translation operators [9]. Parrott has asked (in a private communication) whether the same is true for operators in (BN). We answer this in the negative by exhibiting a $T \in (BN)$ such that $T^2 \notin (BN)$, and T is invertible.

EXAMPLE 1. Let $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. Then $T \in (BN)$, $T^2 \notin (BN)$, and T is invertible.

3. Powers of hyponormal or bi-normal operators need not be hyponormal or bi-normal. Operators which are both hyponormal and bi-normal are somewhat "nicer". Let $(BN)^+$ denote the hyponormal bi-normal operators.

THEOREM 3. *Suppose that* $T \in (BN)^+$. *Then* T^n *is hyponormal for* $n \geqq 1$.

Proof. If C, D are positive operators such that $C \geqq D \geqq 0$, then $TCT^* \geqq TDT^* \geqq 0$ and $T^*CT \geqq T^*DT \geqq 0$ for any bounded operator T. Suppose now that $T \in (BN)^+$. Since $T^*T \geqq TT^*$, we have $T^{*2}T^2 \geqq (T^*T)^2$ and $(TT^*)^2 \geqq T^2T^{*2}$. But $T^*T \geqq TT^*$ and $[T^*T, TT^*] = 0$ implies that $(T^*T)^2 \geqq (TT^*)^2$. Hence $T^{*2}T^2 \geqq (T^*T)^2 \geqq T^2T^{*2}$ and T^2 is hyponormal. Suppose then that $T^{*n}T^n \geqq (T^*T)^n \geqq (TT^*)^n \geqq T^nT^{*n}$ for some integer $n \geqq 2$. Then $T^{*n}T^n \geqq (TT^*)^n$ implies that $T^{*n+1}T^{n+1} \geqq (T^*T)^{n+1}$ and $(T^*T)^n \geqq T^nT^{*n}$ implies that $(TT^*)^{n+1} \geqq T^{n+1}T^{*n+1}$. But $(T^*T)^{n+1} \geqq (TT^*)^{n+1}$. The theorem now follows by induction.

4. The assumption that $T \in (BN)$ is hyponormal can be weakened to $T \in (BN)$ is paranormal but no added generality is achieved as the next result shows. Recall that T is paranormal if $\| T^2\phi \| \cdot \| \phi \| \geqq \| T\phi \|^2$ for all $\phi \in \mathscr{H}$. See for example [1]. Hyponormal operators are paranormal.

THEOREM 4. *Suppose that* $T \in (BN)$. *If* T *is also paranormal, then it is hyponormal.*

Proof. Suppose that T is paranormal. Then $AB^2A - 2\lambda A^2 + \lambda^2 I \geqq 0$ for every $\lambda > 0$ where $A = (TT^*)^{1/2}$ and $B = (T^*T)^{1/2}$ [1]. Suppose that $T \in (BN)$. The condition for paranormality becomes

$$(*) \qquad A^2B^2 - 2\lambda A^2 + \lambda^2 I \geqq 0 \text{ for every } \lambda > 0.$$

Since $[A^2, B^2] = 0$, there exists a spectral measure $E(\cdot)$ such that

$$A^2 = \int f(t)dE(t) \quad \text{and} \quad B^2 = \int g(t)dE(t)\ .$$

Substituting these integrals into $(*)$ gives

$$\int (f(t)g(t) - 2\lambda f(t) + \lambda^2)dE(t) \geqq 0\ .$$

Let $\theta = \{(x, y): x \geqq 0,\ y \geqq 0 \text{ and } xy - 2\lambda x + \lambda^2 \geqq 0 \text{ for all } \lambda > 0\}$. Then $(f(t), g(t)) \in \theta$ almost everywhere dE. We will show now that actually $\theta = \{(x, y): x \geqq 0, y \geqq 0, \text{ and } y \geqq x\}$. Then $g(t) \geqq f(t)$ almost everywhere dE and $T^*T \geqq TT^*$ as desired. To see that $\theta = \{(x, y): x \geqq 0, y \geqq 0 \text{ and } y \geqq x\}$, observe that $xy - 2\lambda x + \lambda^2 = 0$, $\lambda > 0$, defines the curve $y = h_\lambda(x) = 2\lambda - \lambda^2/x$ in the first quadrant. The line $y = x$ is tangent to $h_\lambda(x)$ at $x = \lambda$. Since $h_\lambda(x)$ is everywhere concave down we have that it lies entirely on or below $y = x$. But θ consists of those points in the first quadrant lying above the graph of h_λ for every $\lambda > 0$, that is, above the line $y = x$.

An immediate corollary to Theorem 4 which might save time in the construction of examples is the following.

COROLLARY 1. *There are no weighted shifts which are paranormal and not hyponormal.*

5. Under certain conditions T being in (BN) does imply T is centered. We give two.

THEOREM 5. *Suppose that* $\|T\| \leqq 1$. *If* $T^*T = f(TT^*)$ *and* $TT^* = g(T^*T)$ *where* f *and* g *are continuous functions from* $[0, 1]$ *into* $[0, 1]$, *then* T *is centered.*

Proof. If $T^*T = f(TT^*)$, then

$$(*) \quad T^{*2}T^2 = T^*f(TT^*)T = f(T^*T)T^*T = f(f(TT^*))f(TT^*) = f_2(TT^*)$$

where f_2 is a continuous function from $[0, 1]$ into $[0, 1]$. The second equality of $(*)$ is trivially valid if f is a polynomial. By taking uniform limits of polynomials it can be seen that it is true for all continuous functions f. From $(*)$ and an induction argument, we get that $T^{*n}T^n = f_n(TT^*)$ and $T^nT^{*n} = g_n(T^*T)$ for continuous functions f_n, g_n mapping $[0, 1]$ into $[0, 1]$, $n \geqq 1$. Hence $[T^{*j}T^j, T^iT^{*i}] = 0$ for all integers $i, j \geqq 0$.

The assumption that f, g are continuous can be considerably weakened. If h, k are bounded measurable functions from $[0, 1]$ into $[0, 1]$, then let $(h \odot k)(x) = h(k(x))k(x)$. Set $h_1 = h$ and define $h_n = (h_{n-1} \odot h)$ for $n \geqq 2$. Then the theorem is true if f_n and g_n are well-defined dE measurable functions for every integer $n \geqq 1$. dE is the spectral measure of the $*$-algebra generated by I, T^*T and TT^*. Clearly the assumption $\|T\| \leqq 1$ is not restrictive.

S. K. Parrott has proven the following result (private communication).

THEOREM 6. *If* $T \in (BN)$ *and* T^*T *has a cyclic vector, then* T *is unitarily equivalent to a weighted translation operator.*

6. The operator $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ acting on C^2 shows that Theorem 6 is not valid for an arbitrary $T \in (BN)$. Our next example shows it is also not true for $T \in (BN)^+$.

EXAMPLE 2. Let

$$T_n = \begin{bmatrix} 0 & 0 & \sqrt{2}\, g(n+1) \\ g(n) & g(n) & 0 \\ g(n) & -g(n) & 0 \end{bmatrix}, \ n \geqq 1,$$

where $g(n)$ is a strictly increasing sequence of positive numbers converging to 1. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot \\ T_1 & 0 & 0 & \cdot \\ 0 & T_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

acting on $\mathscr{k}$ where $\mathscr{k}$ is a countable number of copies of C^3. Then $A \in (BN)^+$, but $A^2 \notin (BN)$. $A \in (BN)$ since A^*A and AA^* are diagonal. $A \in (BN)^+$ since $T_{n+1}^* T_{n+1} \geqq T_n T_n^*$, $n \geqq 1$. So show $A^2 \notin (BN)$, one need only show that $[(T_{n+1}T_n)(T_{n+1}T_n)^*, (T_{n+3}T_{n+2})^*(T_{n+3}T_{n+2})] \neq 0$ for some $n \geqq 1$. Picking $n = 1$ and $g(1) = 0$ makes the calculation easier.

It is easy to modify Example 2 to get an invertible A such that $A \in (BN)^+$ and $A^2 \notin (BN)$. This is done by picking a sequence $\{g(n)\}_{n=-\infty}^{\infty}$ such that $g(n) < g(n+1)$, $\lim_{n\to\infty} g(n) = 1$, and $\lim_{n\to-\infty} g(n) = c > 0$. Define A to be a matrix weighted bilateral shift with weights T_n, T_n as in Example 2.

There remains then the problem of determining what types of operators are in $(BN)^+$.

In the process of proving Theorem 1 of [3] we proved the following result which could be helpful.

If C is self-adjoint, let $E_C(\cdot)$ be the spectral measure of C.

PROPOSITION 1. *If* $T \in (BN)^+$, *then* $E_{T^*T}([b, \|T\|])\mathscr{k}$ *is an invariant subspace of* T *for every* $b > 0$. *Furthermore,* $E_{T^*T}([0, b)) \leqq E_{TT^*}([0, b))$ *for every* $b > 0$.

By considering weighted shifts in $(BN)^+$ it is easy to see that the subspaces need never be reducing and $[b, \|T\|]$ cannot be replaced by a noninterval or by an interval without $\|T\|$ as an end point.

7. The presence of a large number of examples is useful both in making conjectures and in finding counterexamples. There has also been some interest in the condition (BN) when $\dim \mathscr{k} < \infty$ [4]. For these reasons we will now characterize all operators in (BN) when $\dim \mathscr{k} = 2$ and $\dim \mathscr{k} = 3$.

DEFINITION 2. If $\{\phi_i\}$ is an orthonormal basis, D is a diagonal matrix with respect to this basis, and U is a permutation of the basis, then $T = UD$ is called a weighted permutation.

We say that a matrix A is a form for T if T is unitarily equivalent to a scalar multiple of either A or A^*.

THEOREM 7. *If $T \in (BN)$ and $\dim \mathscr{K} = 2$, then the possible forms are:*

(I1) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$, *a an arbitrary complex number.*

(I2) $\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$, $b > 0$.

(I3) $\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$, *an arbitrary.*

THEOREM 8. *If $T \in (BN)$ and $\dim \mathscr{K} = 3$, then the possible forms are:*

(II1) $\begin{bmatrix} c & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{X}} \\ 0 & & \end{bmatrix}$ *where X is* (I2), *c an arbitrary complex number.*

(II2) *A weighted permutation.*

(II3) $\begin{bmatrix} 0 & b & -1 \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$, $b > 0$.

(II4) $\begin{bmatrix} 0 & 0 & a \\ u_{21} & u_{22} & 0 \\ u_{31} & u_{32} & 0 \end{bmatrix}$ *where $a > 0$ and $\begin{bmatrix} u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix}$ is unitary.*

Proof. Theorem 7 is easy. Form (II3) is best developed from the form developed in [4] for matrices T such that $[T^\dagger T, TT^\dagger] = 0$ where $T^\dagger$ is the generalized inverse of T. If $T \in (BN)$, then $[T^\dagger T, TT^\dagger] = 0$. Form (II4) is best developed by looking at the polar form and determining possible unitary parts of T.

Example 1 was found by considering an operator of form (II4). The blocks in Example 2 are also (II4) forms.

In looking for (BN) matrices the following matrix version of Theorem 6 is useful.

THEOREM 9. *Suppose that $T \in (BN)$ and that $\dim \mathscr{K} = n < \infty$. If T^*T has n different eigenvalues, then T is a weighted permutation.*

Theorem 9 can be given a simple matrix proof by observing that if $T = U(T^*T)^{1/2}$ and $T \in (BN)$, then $U(T^*T) = (TT^*)U$ and T^*T and TT^* may be simultaneously diagonalized. Furthermore, T^*T and TT^* have the same spectrum. It is then easy to see that the only possible U are permutations of the basis that diagonalizes T^*T and TT^*.

It is easy to verify that in all of the forms in Theorem 7 and Theorem 8, except possibly (II4), that zero is in the convex hull of $\sigma(T)$. Is this always true when $n = \dim \mathscr{H} < \infty$? Is it true when $\dim \mathscr{H}$ is infinite? If it is not always true, for what dimensions is it true?

8. All of the two-dimensional bi-normal operators have a square which is normal. Such operators are automatically bi-normal (though never nontrivially hyponormal). This result was proved in [4] and observed independently by Embry in a private communication.

Operators such that T^2 is normal have been studied by Embry [6] and completely characterized by Radjavi and Rosenthal [11].

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