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THE INDEX OF CONVEXITY AND PARALLEL BODIES

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Intuitively, the visibility function for a set C in R^n measures the n-dimensional volume of the star of a variable point of C. Suppose that the visibility function for C is measurable. If the measure of C is positive, normalizing the integral of the function produces a measure of the relative convexity of C, called the Index of convexity of C. The purpose of this paper is to study the relationship between the Index of convexity of a compact set C in R^n and the Indices of its parallel bodies. Continuity properties of the Index are established relative to an appropriate metric on the class of compact sets in R^n.

1. Introduction.

DEFINITION. The visibility function assigns to each point x of a fixed measurable set E in a Euclidean space R^n the Lebesgue outer measure of $\{y: rx + (1 - r)y \in E$ for each r in $[0, 1]\}$ and zero to each point of $R^n \backslash E$.

DEFINITION. Let $E \subset R^n$ be a measurable set with measurable visibility function v_E, and suppose the Lebesgue measure of E, $m(E)$, is finite. If $m(E) > 0$, the Index of convexity of E, $I(E)$, is given by $\int v_E/(m(E)^2)\, dm$. If $m(E) = 0$, we agree to let $I(E)$ *be* 1.

The reader will find a general treatment of the visibility function and the Index of convexity in [3]. Important for the present article are the following results: The visibility function for E is upper-semicontinuous whenever E is compact, and I is upper-semicontinuous on the class of all compact sets in R^n with an appropriate metric; namely, if C and K are compact sets, define $\bar{d}(C, K)$ to be $\sup(d(C, K), m(C \Delta K))$ where d denotes Hausdorff distance.

Let $B_r(x)$ denote the closed r-ball about a point x in R^n.

DEFINITION. Let C be a compact set in R^n. The ε-parallel body of C, denoted by $B_\varepsilon(C)$, is the compact set $\bigcup_{x \in C} B_\varepsilon(x)$.

This paper illustrates the central role played by parallel bodies in the study of the Index of convexity. Using this concept, we can characterize those sequences $\{C_k\}$ of compact sets convergent in the $\bar{d}$ metric to a compact set C of positive measure having the property that $I(C) = \lim_{k\to\infty} I(C_k)$.

We then consider the Index as a function of the radius ε of the parallel body $B_\varepsilon(C)$ of a fixed compact set C in R^n. This function is continuous if C is starshaped. Also, if C is contained in a flat of dimension p, its p dimensional Index of convexity, if it is not "trivially" 1, is determined by the Indices of its parallel bodies in R^n.

With a few exceptions we use the same terminology as in [1]. We denote ordinary Lebesgue measure by m_n or simply by m if only one space is under discussion. Conv ker E and conv E will indicate the convex kernel and convex hull of E, respectively. As usual, bd E, int E, and cl E are the boundary, interior, and closure of E. Finally, xy will denote the line segment joining x to y. If $x \in C$, we say that x sees y via C if $xy \subset C$. The star of x relative to C is simply all those points which x sees via C. As alluded to above, v_C represents the visibility function for a fixed set C, and $I(C)$ is the Index of C.

2. The characterization theorem. The visibility function v_C of a compact set C is continuous on C if and only if the visibility functions for the parallel bodies of C, when restricted to C, converge uniformly to v_C [2]. We establish here an analogous result for the unnormalized integral of the visibility function defined on a compact collection of compact sets in R^n relative to the Hausdorff metric.

If C is a compact set in R^n, denote the unnormalized integral of the visibilility function for C, $\int v_C\, dm$, by $V(C)$.

LEMMA 1. *Let $\{C_l\}$ be a collection of compact sets in R^n convergent in the Hausdorff metric to a compact set C. Then* $V(C) \geqq \limsup_{l\to\infty} V(C_l)$.

Proof. If x is in C, let $S(x)$ denote the star of x relative to C, and let $S_l(x)$ denote star of x relative to C_l. Since $\{C_l\}$ converges to C in the Hausdorff metric, for any fixed $\varepsilon > 0$, $B_\varepsilon(S(x))$ includes $S_l(x)$ for all sufficiently large integers l. This yields $v_C(x) \geqq \sup_{l\to\infty} v_{C_l}(x)$ for each x in C. By Fatou's lemma

$$V(C) = \int v_C dm \geqq \int_C \limsup_{l\to\infty} v_{C_l} dm \geqq \limsup_{l\to\infty} \int_C v_{C_l} dm$$
$$= \limsup_{l\to\infty} \int v_{C_l} dm = \limsup_{l\to\infty} V(C_l)$$

since $m(C_l \backslash C) \to 0$.

Hence, V is an upper-semicontinuous function on the metric space of compact sets in R^n with the Hausdorff metric. Since $\{x\colon \|x\| \leqq 1\}$ in R^n is the Hausdorff limit of a sequence of compact sets of measure zero, V fails to be globally continuous.

In establishing our main result we use the following famous theorem of Dini: Let $\{f_n\}$ be a sequence of upper-semicontinuous nonnegative functions defined on a compact metric space $\mathscr{A}$. Suppose for each x in $\mathscr{A}$, $\{f_n(x)\}$ converges monotonically to zero. Then $\{f_n\}$ converges uniformly to the zero function on $\mathscr{A}$. For simplicity of notation, let $V_k(C)$ denote $V(B_{1/k}(C))$ for each $k \in Z^+$.

THEOREM 1. *Let $\mathscr{A}$ be a compact collection of compact sets relative to the Hausdorff metric. The function $V: \mathscr{A} \to R$ is continuous on $\mathscr{A}$ if and only if $\{V_k\}$ converges uniformly to V on $\mathscr{A}$ as $k \to \infty$.*

Proof. Suppose first that V is continuous on $\mathscr{A}$. The previous lemma implies that $\{V_k - V\}$ is a sequence of upper-semicontinuous functions. Clearly, the sequence converges monotonically to zero on $\mathscr{A}$. Since $\mathscr{A}$ is compact, Dini's theorem now applies.

Conversely, suppose V is discontinuous at some compact set C in $\mathscr{A}$. Since V is upper-semicontinuous, there must then exist a sequence $\{C_k\}$ in $\mathscr{A}$ and an $\varepsilon > 0$ satisfying $V(C) > V(C_k) + \varepsilon$ and $d(C_k, C) < 1/k$ for all k. By the definition of Hausdorff distance, $B_{1/k}(C_k) \supset C$ so that $V_k(C_k) > V(C_k) + \varepsilon$. Thus, the convergence cannot be uniform on $\mathscr{A}$.

Suppose that $\{C_l\}$ is a sequence of compact sets in R^n convergent to C in the $\bar{d}$ metric (not merely in the Hausdorff metric), and $m(C) > 0$. The following are necessary and sufficient conditions for $\lim_{l\to\infty} I(C_l)$ to exist and to equal $I(C)$.

THEOREM 2. *Let $\{C_l\}$ be a collection of compact sets convergent in the metric $\bar{d}$ to a compact set C of positive measure. Then $\lim_{l\to\infty} I(C_l) = I(C)$ if and only if $I(B_{1/k}(C_l)) \to I(C_l)$ uniformly on $\{C_l: l \in Z^+\}$ as $k \to \infty$.*

Proof. Let $\mathscr{A}$ denote $\{C\} \cup \{C_l: l \in Z\}$. $\mathscr{A}$ is compact relative to the $\bar{d}$ metric and has only one limit point, the set C. Since Lebesgue measure is an upper-semicontinuous function on the metric space of compact sets in R^n with the Hausdorff metric, an application of Dini's theorem yields the uniform convergence of $\{m(B_{1/k}(F))\}$ to $m(F)$ for each $F \in \mathscr{A}$ as $k \to \infty$. It follows that $\lim_{k\to\infty} m(B_{1/k}(C_k)) = m(C)$. A slight modification of the technique used in the preceding theorem now yields the sufficiency of the conditions as the Index of convexity is upper-semicontinuous with respect to the metric $\bar{d}$. Conversely, if $\lim_{l\to\infty} I(C_l) = I(C)$, then $\lim_{l\to\infty} V(C_l) = V(C)$. Our result now follows from Theorem 1, as any uniformly convergent sequence of functions on $\mathscr{A}$ will automatically be uniformly convergent on

$\{C_l: l \in Z^+\}$.

Since the Index of convexity is not upper-semicontinuous with respect to the Hausdorff metric [3], the necessity of using the $\bar{d}$ metric in the previous theorem is not surprising. To verify this, given $l \in Z^+$, let C_l be the planar set $\{(x, y): 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\} \cup \{(x, y): 1 \leqq x \leqq 2,\ y = k/l,\ k = 0, 1, \cdots l\}$. The sequence $\{C_l\}$ converges to $C = \{(x, y): 0 \leqq x \leqq 2,\ 0 \leqq y \leqq 1\}$, and $I(C_l) = I(C)$ for all l. However, $I(B_{1/k}(C_l))$ does not converge uniformly to $I(C_l)$ on $\{C_l: l \in Z^+\}$. One can also see that both Theorems 1 and 2 fail if the metric only reflects convergence in measure.

3. Parallel bodies of a fixed set. Let C be a fixed compact set in R^n. Define $I^*: (0, \infty) \to R$ by $I^*(r) = I(B_r(C))$. It is obvious that $I^*(r) > 0$ for $r > 0$ and that $\lim_{r\to\infty} I^*(r) = 1$.

THEOREM 3. *Let C be a compact set in R^n. Then $I^*(r-) \leqq I^*(r) = I^*(r+)$ for each $r > 0$.*

Proof. Let $\{s_k\}$ be a decreasing sequence of positive numbers convergent to r. The visibility function for C is the pointwise limit of the sequence of visibility functions corresponding to $\{B_{s_k}(C)\}$. Hence $I^*(r) = I^*(r+)$. If $\{r_k\}$ is an increasing sequence of positive numbers convergent to r, then the visibility function for $\bigcup_{k=1}^{\infty} B_{r_k}(C)$ is the pointwise limit of the sequence of visibility function corresponding to $\{B_{r_k}(C)\}$. An application of the Dominated Convergence Theorem yields the existence of $I^*(r-)$. Since $\{B_{r_k}(C)\}$ converges in the $\bar{d}$ metric to $B_r(C)$, we have $I^*(r-) \leqq I^*(r)$.

A compact planar set for which I^* has infinitely many discontinuities is

$$(0, 0) \cup \bigcup_{k=1}^{\infty} \left\{(x, y): \frac{1}{2k+1} \leqq x^2 + y^2 \leqq \frac{1}{2k}\right\}.$$

THEOREM 4. *If C is a compact starshaped set, I^* corresponding to C is continuous.*

Proof. We need only show that I^* is left continuous at each point of $(0, \infty)$. Fix r in $(0, \infty)$, and let $\{r_k\}$ be an increasing sequence of positive reals convergent to r. As we have seen, $\{B_{r_k}(C)\}$ converges in the $\bar{d}$ metric to $B_r(C)$. Moreover, since C is starshaped, $\bigcup_{k=1}^{\infty} B_{r_k}(C)$ is precisely $\operatorname{int} B_r(C)$. The assertion now easily follows from the Dominated Convergence Theorem upon verifying that $v_{B_{r_k}(C)}(x) \to v_{B_r(C)}(x)$ for each x in $\operatorname{int} B_r(C)$.

Suppose such a point x sees a point z via $\operatorname{int} B_r(C)$. The compactness of xz forces the segment to be totally contained in $B_{r_k}(C)$ for k sufficiently large. However, x may see some points of $B_r(C)$ via line segments not wholly contained in the interior of $B_r(C)$. Fortunately, the arduous task of establishing that the set of such points has measure zero has been executed in [2] in proving that the visibility function for a compact starshaped set whose convex kernel has dimension exceeding $n - 2$ is continuous on the interior of the set. It follows that almost every point in the star of x relative to $B_r(C)$ is in the star of x relative to $B_{r_k}(C)$ for k sufficiently large so that $v_{B_{r_k}(C)}(x) \to v_{B_r(C)}(x)$.

A compact set that is the closure of an open set and that has Index 1 must be convex [3]. Hence, if $I^*(r) = 1$ for some $r > 0$, then $I^*(s) = 1$ for all $s > r$.

THEOREM 5. *Let C be a compact set in R^n. The number* 1 *is a value of I^* corresponding to C if and only if* $\mathrm{bd}\,(\mathrm{conv}\, C) \subset C$.

Proof. If $\mathrm{bd}\,(\mathrm{conv}\, C) \subset C$, it is easy to show that $B_r(C)$ is convex for sufficiently large r. Conversely, suppose $\mathrm{bd}\,(\mathrm{conv}\, C) \cap C^c \neq \emptyset$. Given a fixed p in $\mathrm{bd}\,(\mathrm{conv}\, C) \cap C^c$, choose an outer unit normal x to a hyperplane of support of $\mathrm{conv}\, C$ at p. Evidently, for each $r > 0$, $p + rx \notin B_r(C)$. Since p is a convex combination of points in C, say $\{x_1, x_2, \cdots, x_k\}$, the point $p + rx$ is a convex combination of $\{x_1 + rx, \cdots, x_k + rx\} \subset B_r(C)$. Hence $B_r(C)$ is never convex, so that $I^*(r)$ is never one.

COROLLARY. *Let C be a compact nonconvex starshaped set in R^n. Then I^* corresponding to C never assumes the value* 1.

Even when C is starshaped, it is not necessarily true that I^* corresponding to C be a monotone increasing function. It would be useful to characterize those compact sets having this property, for if I^* were bounded, monotone increasing, and right continuous, a normalization of I^* yields a probability distribution function.

If $\{C_k\}$ is a sequence of compact sets converging in the $\bar{d}$ metric to C, it may well occur that C is contained in a hyperplane H. Relative to H, C has an $n - 1$ dimensional Index of convexity which we denote by $I^H(C)$. One might guess that if the projections of $\{C_k\}$ onto H converged to C in terms of $n - 1$ dimensional measure, then $\limsup I(C_k) \leqq I^H(C)$ would follow. Unfortunately, the inequality is invalid. Let $H = \{(x, y, z): z = 0\}$. Let $C_k = \{(x, y, z): x^2 + y^2 \leqq 1,\ \ 0 \leqq z \leqq 1/k\} \cup \{(x, y, z): (x - 2)^2 + (y - 2)^2 \leqq 1,\ \ 0 \leqq z \leqq 2/k\}$. If $C = C_1 \cap H$, then $\{C_k\} \to C$. However, $I^H(C) = 1/2$ while $I(C_k) = 5/9$ for all k. Clearly the projection of each C_k is just C.

We now show that the $n-1$ dimensional Index of such a set C is the limit of the Indices of its parallel bodies in R^n.

THEOREM 6. *Let H be a hyperplane in R^n. Suppose C is a compact subset of H and $m_{n-1}(C) > 0$. Then $\lim_{k\to\infty} I(B_{1/k}(C)) = I^H(C)$, the Index of convexity of C with respect to H.*

Proof. We may assume that $H = \{(x_1, \cdots, x_n): x_n = 0\}$. Let $\pi: R^n \to R^{n-1}$ be defined by $\pi(x_1, \cdots, x_n) = (x_1, \cdots, x_{n-1})$. Two facts are obvious: For each $k \in Z^+$ we have $B_{1/k}(C) \supset \pi(C) \times [-1/k, 1/k]$, and if x belongs to $\pi(C) \times [-1/k, 1/k]$ then the star of x relative to $B_{1/k}(C)$ includes the Cartesian product of the star of $\pi(x)$ relative to $\pi(C)$ with $[-1/k, 1/k]$.

Suppose $p \in R^{n-1}$ does not belong to the star of $\pi(x)$ relative to $\pi(C)$. If k is sufficiently large, no point of $\{p\} \times [-1/k, 1/k]$ can be seen by any point in $\{\pi(x)\} \times [-1/k, 1/k]$ via $B_{1/k}(C)$ or else $\pi(x)p \subset \pi(C)$ by the compactness of C.

For each $k \in Z^+$ and y in $\pi(C)$, let $r_k(y)$ be the $n-1$ dimensional measure of the projection onto H of the union of the stars of all points in $\{y\} \times [-1/k, 1/k]$ relative to $B_{1/k}(C)$ minus $v_{\pi(C)}(y)$. Thus for fixed k if $T(y)$ denotes $\bigcup_{x \in \{y\}\times[-1/k,1/k]} \{z: xz \subset B_{1/k}(C)\}$ we have

$$r_k(y) = m_{n-1}(\pi(T(y))) - v_{\pi(C)}(y) . \tag{1}$$

The compactness of $[-1/k, 1/k]$ and $B_{1/k}(C)$ imply (i) $T(y)$ is compact so that $\pi(T(y))$ is compact for each $y \in \pi(C)$ (ii) if $\{y_i\}$ is a sequence in $\pi(C)$ with limit y, then

$$\pi(T(y)) \supset \bigcap_{l=1}^{\infty} \bigcup_{i=l}^{\infty} \pi(T(y_i)) . \tag{2}$$

By (1) and (2), r is the difference of two upper-semicontinuous functions. Evidently $\lim_{k\to\infty} r_k(y) = 0$ for each y in $\pi(C)$. Since $B_{1/k}(C) \subset B_{1/k}(\pi(C)) \times [-1/k, 1/k]$ we conclude that for each x in $\pi(C) \times [-1/k, 1/k]$

$$v_{B_{1/k}(C)}(x) \leqq \frac{2}{k} (v_{\pi(C)} \circ \pi(x) + r_k \circ \pi(x)) . \tag{3}$$

Choosing M to be the $n-1$ dimensional measure of $\pi(B_1(C))$ we also see that

$$v_{\pi(C)} \circ \pi(x) + r_k \circ \pi(x) \leqq M \tag{4}$$

for $x \in \pi(C) \times [-1/k, 1/k]$ and all $k \in Z^+$.

To show $\lim_{k\to\infty} I(B_{1/k}(C)) = I^H(C)$, it suffices to show

$$\lim_{k\to\infty} \frac{k}{2} m(B_{1/k}(C)) = m_{n-1}(\pi(C)) \tag{5}$$

and

$$\lim_{k\to\infty} \frac{k^2}{4} \int v_{B_{1/k}(C)} dx_1 \cdots dx_n = \int v_{\pi(C)} dx_1 \cdots dx_{n-1} . \tag{6}$$

To establish (5) we again use the fact that $\pi(C) \times [-1/k, 1/k] \subset B_{1/k}(C) \subset B_{1/k}(\pi(C)) \times [-1/k, 1/k]$ to conclude

$$\frac{2}{k} m_{n-1}(\pi(C)) \leqq m_n(B_{1/k}(C)) \leqq \frac{2}{k} m_{n-1}(B_{1/k}(\pi(C))) .$$

The compactness of $\pi(C)$ implies that $m_{n-1}(B_{1/k}(\pi(C))) \to m_{n-1}(\pi(C))$ as $k \to \infty$ and (5) easily follows.

Recalling that the star of each x in $\pi(C) \times [-1/k, 1/k]$ relative to $B_{1/k}(C)$ includes the Cartesian product of the star of $\pi(x)$ relative to $\pi(C)$ with $[-1/k, 1/k]$, we have

$$\begin{aligned} \frac{k^2}{4} \int v_{B_{1/k}(C)} dx_1 \cdots dx_n &\geqq \frac{k^2}{4} \int_{\pi(C)} \left(\int_{-1/k}^{1/k} v_{B_{1/k}(C)} dx_n \right) dx_1 \cdots dx_{n-1} \\ \geqq \frac{k^2}{4} \int_{\pi(C)} \left(\int_{-1/k}^{1/k} \frac{2}{k} v_{\pi(C)} \circ \pi dx_n \right) dx_1 \cdots dx_{n-1} & \\ = \frac{k^2}{4} \int_{\pi(C)} v_{\pi(C)} \cdot \frac{4}{k^2} dx_1 \cdots dx_{n-1} = \int v_{\pi(C)} dx_1 \cdots dx_{n-1} & \end{aligned}$$

since $v_{\pi(C)} \circ \pi$ is constant on any vertical line.

The reverse inequality does not follow so easily. We make a preliminary decomposition:

$$\begin{aligned} &\frac{k^2}{4} \int v_{B_{1/k}(C)} dx_1 \cdots dx_n \\ &= \frac{k^2}{4} \int_{\pi(C)} \left(\int_{-1/k}^{1/k} v_{B_{1/k}(C)} dx_n \right) dx_1 \cdots dx_{n-1} \\ &\quad + \frac{k^2}{4} \int_{B_{1/k}(\pi(C))/\pi(C)} \left(\int_{-1/k}^{1/k} v_{B_{1/k}(C)} dx_n \right) dx_1 \cdots dx_{n-1} . \end{aligned} \tag{7}$$

Since

$$\frac{k^2}{4} \int_{-1/k}^{1/k} v_{B_{1/k}(C)} dx_n \leqq M \tag{8}$$

for all k, the second integral in (7) can be made less than $\varepsilon/3$ if k is sufficiently large.

Since r_k is measurable for each k, by Egoroff's theorem there is a subset F of $\pi(C)$ such that $m_{n-1}(F) < \varepsilon/3M$ and r_k converges to zero uniformly on $\pi(C)/F$. For any $k \in Z^+$, we now have using (8)

$$\frac{k^2}{4} \int_F \left(\int_{-1/k}^{1/k} v_{B_{1/k}(C)} dx_n \right) dx_1 \cdots dx_{n-1} < \frac{\varepsilon}{3M} \cdot M = \frac{\varepsilon}{3} .$$

Pick k so large that $|r_k(y)| < \varepsilon/3M$ uniformly on $\pi(C)/F$. We can now conclude from (3) and (4) that

$$\begin{aligned}
&\frac{k^2}{4}\int_{\pi(C)/F}\left(\int_{-1/k}^{1/k} v_{B_{1/k}(C)}dx_n\right)dx_1 \cdots dx_{n-1} \\
&\leqq \frac{k^2}{4}\int_{\pi(C)/F}\left(\int_{-1/k}^{1/k} \frac{2}{k}(v_{\pi(C)}\circ\pi + r_k\circ\pi)dx_n\right)dx_1 \cdots dx_{n-1} \\
&\leqq \int_{\pi(C)/F} v_{\pi(C)} + \frac{\varepsilon}{3M}dx_1 \cdots dx_{n-1} \leqq \int_{\pi(C)} v_{\pi(C)}dx_1 \cdots dx_{n-1} + \frac{\varepsilon}{3}\,.
\end{aligned}$$

Combining our three integrals over $B_{1/k}(\pi(C))/\pi(C)$, F and $\pi(C)/F$ we have

$$\frac{k^2}{4}\int v_{B_{1/k}(C)}dx_1 \cdots dx_n \leqq \int v_{\pi(C)}dx_1 \cdots dx_{n-1} + \varepsilon$$

when k is sufficiently large so that (6) is established.

Theorem 6 has the following obvious generalization.

THEOREM 7. *Let C be a compact set in R^n contained in a flat F of dimension p, and suppose $m_p(C) > 0$. If $I^F(C)$ denotes the Index of convexity of C with respect to F, then* $\lim_{k\to\infty} I(B_{1/k}(C)) = I^F(C)$.

The proofs of Theorems 6 and 7 are identical modulo replacing $[-1/k, 1/k]$ by an $n - p$ dimensional sphere of radius $1/k$. Of course, the notation required to establish Theorem 3 would be horrendous, and for this reason we confined ourselves to $p = n - 1$.

Can we *define* the Index of convexity of a compact C of measure zero to be the limit of the Indices of its parallel bodies? If C is a compact set of positive measure, then $\lim_{r\to 0^+} I^*(r) = I(C)$, but if $m(C) = 0$, then the limit need not even exist. To see that I^* may behave poorly, we construct a peculiar compact set C in the plane consisting of a countable collection of vertical segments. Each segment has length one and lies in the strip $\{(x, y): 0 \leqq y \leqq 1\}$. After placing a segment at the origin, we construct the others in an iterative manner. Let $\delta_1 = 1/8$. Initially place two vertical segments at $x = 1 - \delta_1$ and at $x = 1 + \delta_1$. Adjoin to this set 4 equally spaced segments between $x = 1/4\delta_1$ and $x = 3/4\delta_1$. Denote half the distance between such segments by δ_2. Having chosen $\delta_1, \delta_2, \cdots, \delta_n$, construct 2^{2^n} equally spaced vertical segments between $x = 1/4\delta_n$ and $x = 3/4\delta_n$, and denote half the distance between adjacent new segments by δ_{n+1}. Let C be the union of the segments so constructed. Although $I^*(\delta_n -) < 1/2$ for all n, $\lim_{n\to\infty} I^*(\delta_n) = 1$. This example also indicates that I^* need not be of bounded variation.

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