PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 2, 1974

ON CHARACTERIZING CERTAIN CLASSES OF FIRST COUNTABLE SPACES BY OPEN MAPPINGS

KENNETH ABERNETHY

This paper has three main results. These are characterizations of Nagata spaces, γ-spaces, and semi-metric spaces, respectively, as images of metrizable spaces under certain kinds of continuous open mappings.

1. Introduction. A basic area of research in general topology is the study of how various classes of spaces are related through mappings (see [3] and [5]). More specifically, many important classes of spaces have been characterized as the image of a metrizable space under an open continuous mapping of some sort. For example, Heath [10] has characterized developable spaces in this way and Hanai and Ponomarev independently have given an elegant characterization of first countable spaces (see Theorem 2.1). In recent years considerable attention has been given to the problem of characterizing generalized metrizable spaces in this way. We mention some of these results in § 2. In this paper we characterize Nagata, semi-metric, and γ-spaces as the image of a metrizable space under certain types of open continuous mappings. Definitions and some known results are given in § 2, Nagata spaces are characterized with Theorem 3.3, γ-spaces with Theorem 4.3, and semi-metric spaces with Theorem 5.3. Throughout the paper the set of natural numbers will be denoted by N.

2. Definitions and background results. The spaces which interest us in this paper can be described in terms of sequences of open covers. It should be pointed out that many of the definitions which follow are not the original definitions, but are actually characterizations which were proved, after the particular concept had been introduced, in efforts to unify the various concepts. Consequently, the definitions we give, in terms of a *COC*-function, display some degree of this unification.

Let (X, T) be a topological space and let g be a function from $N \times X$ into T. Then g is called a *COC-function for X* (*COC*= countably many open covers) if it satisfies these two conditions: (1) $x \in \bigcap_{n=1}^{\infty} g(n, x)$ for all $x \in X$; (2) $g(n+1, x) \subseteqq g(n, x)$ for all $n \in N$ and $x \in X$. Note that if g is a *COC*-function for X, we obtain countably many open covers of X, $\langle G_n \rangle$, by taking $G_n = \{g(n, x): x \in X\}$ for each n.

Now let X be a space with *COC*-function g, and consider the following conditions on g:

(A) $y_n \in g(n, x)$ for each $n \in N$ implies that the sequence $\langle y_n \rangle$ has x as a cluster point.

(B) $g(n, x) \cap g(n, y_n) \neq \emptyset$ for each $n \in N$ implies that $\langle y_n \rangle$ has x as a cluster point.

(C) $y_n \in g(n, x)$ and $p_n \in g(n, y_n)$ for each $n \in N$ implies that $\langle p_n \rangle$ has x as a cluster point.

(D) $\{g(n, x): n = 1, 2, \cdots\}$ is a fundamental system of neighborhoods for x, for each x, and $x \in g(n, y_n)$ for each $n \in N$ implies that $\langle y_n \rangle$ has x as a cluster point.

(E) If H is closed and $p \in \overline{\cup \{g(n, x): x \in H\}}$ for each $n \in N$, then $p \in H$.

(F) $x \in g(n, y_n)$ for each $n \in N$ implies that $\langle y_n \rangle$ has x as a cluster point.

If X is a space with a *COC*-function g satisfying (A), X is called a *first countable space* and g a *first countable function for* X; X is called a *Nagata space* and g is called a *Nagata function for* X if g satisfies (B); if g satisfies (C), X is called a γ*-space* and g a γ*-function for* X; if g satisfies (D), X is called a *semi-metric space* and g a *semi-metric function for* X; X is called a *stratifiable space* and g a *stratifiable function for* X *if* g satisfies (E); and finally if g satisfies (F), X is called a *semi-stratifiable space* and g a *semi-stratifiable function for* X.

Ceder [6] first studied stratifiable spaces under the name "M_3-spaces". Borges [4] renamed them "stratifiable" and investigated them in more detail. Creede [7] introduced semi-stratifiable spaces. Our definition of semi-metric spaces is a characterization given by Heath in [9] where he studies semi-metric spaces. Hodel [12] introduced γ-spaces. Ceder [6] also introduced Nagata spaces, but our definition is a characterization due to Heath [9].

It is clear from our definitions that a space is a semi-metric space if and only if it is a first countable semi-stratifiable space. Also, it is true (c.f. [4]) that a space is a Nagata space if and only if it is a first countable stratifiable space.

Each class of spaces which we characterize below is included in the class of first countable spaces. Consequently we are able to make use of the following theorem proved independently by Hanai [8] and Ponomarev [16].

THEOREM 2.1. *A T_1-space Y is first countable if and only if there is a metrizable space X and a continuous open mapping from X onto Y.*

Let us now mention some characterizations of Nagata spaces and semi-metric spaces. Heath [10] has characterized each of these as follows.

THEOREM 2.2. *A T_1-space is a semi-metric space if and only if there exists a continuous open mapping ψ from some metric space (X, d) onto Y, and a subset X' of X such that* (1) $\psi(X') = Y$ *and* (2) *if $y \in Y$, W an open set containing y, then there exists an $\varepsilon > 0$ such that $\psi(\mathrm{Bd}(\psi^{-1}(y), \varepsilon) \cap X') \subseteq W$.*

THEOREM 2.3. *A T_2-space Y is a Nagata space if and only if there exists an open continuous mapping ψ from some metric space (X, d) onto Y and a subset X' of X such that* (1) $\psi(X') = Y$ *and* (2) *if K is compact in Y, W an open set with $K \subseteq W$, then there exists an $\varepsilon > 0$ such that $\psi(\mathrm{Bd}(\psi^{-1}(K), \varepsilon) \cap X') \subseteq W$.*

Nagata [15] has recently given quite similar characterizations of Nagata spaces and semi-metric spaces using the concept of a q-closed mapping. In comparing these results (of both Heath and Nagata) with results such as Theorem 2.1 and Heath's characterization of developable spaces, we can see that one natural way to try to improve them is to avoid having to consider a subset X' of the metric space X.

3. Nagata spaces.

DEFINITION 3.1. Let X and Y be topological spaces, let $\psi: X \to Y$ be a surjection, and let g be a COC-function for X. Then ψ is an *N-mapping relative to g* (N=Nagata) if given any $y \in Y$ and neighborhood W of y, there is a neighborhood V of y and a positive integer n such that if $g(n, x) \cap \psi^{-1}(V) \neq \varnothing$, then $\psi(x) \in W$. A surjection $\psi: X \to Y$ is an *N-mapping* if there is a COC-function g for X such that ψ is an N-mapping relative to g.

We note that our N-mapping is quite similar to Arhangel'skii's [3] regular mapping. Indeed our definition was suggested by his definition. In [3] he proved a theorem showing that conditions on the range space of a mapping can force the mapping to be regular. This theorem motivated the following proposition on N-mappings.

PROPOSITION 3.2. *Let (X, T) and Y be topological spaces with Y a stratifiable space, and let $\psi: X \to Y$ be a continuous surjection. Then ψ is an N-mapping.*

Proof. Let h be a stratifiable function for Y, and define $g: N \times X \to T$ by $g(n, x) = \psi^{-1}[h(n, \psi(x))]$. Then g is a COC-function for X.

Now let $y \in Y$, and let W be an open set containing y. Then $Y - W$ is closed and $y \notin Y - W$; hence there exists an $n_0 \in N$ such that $y \notin \overline{\bigcup \{h(n_0, p): p \in Y - W\}}$. Let $V = Y - \overline{\bigcup \{h(n_0, p): p \in Y - W\}}$. Now if $g(n_0, x) \cap \psi^{-1}(V) \neq \emptyset$, then $h(n_0, \psi(x)) \cap V \neq \emptyset$. But this means that $\psi(x) \notin Y - W$, i.e., $\psi(x) \in W$.

Using Theorem 2.1 and Proposition 3.2, we are able to characterize Nagata spaces.

THEOREM 3.3. *Let Y be a T_1-space. Then Y is a Nagata space if and only if there is a metrizable space X and an open continuous N-mapping from X onto Y.*

Proof. First assume that Y is a Nagata space. Then Y is first countable; so by Theorem 2.1, there is a metrizable space X and an open continuous surjection from X onto Y. By Proposition 3.2, ψ is an N-mapping.

Now assume that X is metrizable and ψ is an open continuous N-mapping from X onto Y. Clearly Y is first countable, so it suffices to show that Y is stratifiable. Let g be a COC-function for X relative to which ψ is an N-mapping. Let $y \in Y$, $n \in N$. Then choose any $s \in \psi^{-1}(y)$ and define $h(n, y) = \psi[g(n, s)]$, for every n. We claim that h is a stratifiable function for Y. Let H be closed in Y, and suppose that $p \in \overline{\bigcup \{h(n, z): z \in H\}}$, for each $n \in N$. Suppose $p \notin H$; then $p \in Y - H = W$, which is open. Thus there exist a neighborhood V of p and an $n_0 \in N$ such that if $g(n_0, x) \cap \psi^{-1}(V) \neq \emptyset$ then $\psi(x) \in W$.

Now since V is a neighborhood of p, $V \cap (\bigcup \{h(n, z): z \in H\}) \neq \emptyset$ for each $n \in N$. Thus there is a $z \in H$ such that $h(n_0, z) \cap V \neq \emptyset$. Therefore, if t is such that $h(n_0, z) = \psi[g(n_0, t)]$, we have $g(n_0, t) \cap \psi^{-1}(V) \neq \emptyset$. But this implies that $\psi(t) = z \in W$, an obvious contradiction.

In reference to the remark at the end of § 2, we note that we are able to do away with having to look at a subset X' of the metric space X in our characterization of Nagata spaces. (A similar remark applies to our characterization of semi-metric spaces in § 5.)

4. γ-spaces. This section proceeds almost exactly as § 3. We begin by giving the definition of the kind of mapping we need in order to characterize γ-spaces.

DEFINITION 4.1. Let X and Y be topological spaces, let $\psi: X \rightarrow Y$ be a surjection, and let g be a COC-function for X. Then ψ is a *G-mapping relative to g* (G=gamma) if given any $y \in Y$ and neighborhood W of y, there is a neighborhood V of y and an $n \in N$ such that $\psi[\bigcup \{g(n, x): x \in \psi^{-1}(V)\}] \subseteqq W$. A surjection $\psi: X \to Y$ is a *G-mapping* if there is a *COC*-function g for X such that ψ is a G-mapping relative to g.

PROPOSITION 4.2. *Let (X, T) and Y be topological spaces with Y a γ-space, and let $\psi: X \to Y$ be a continuous surjection. Then ψ is a G-mapping.*

Proof. Let h be a γ-function for Y, and define g as in the proof of Proposition 3.2. Let $y \in Y$ and let W be a neighborhood of y. We claim that there exists an $n_0 \in N$ such that $\bigcup \{h(n_0, z): z \in h(n_0, y)\} \subseteqq W$. For suppose not; then we can choose sequence $\langle z_n \rangle$ and $\langle u_n \rangle$ such that $z_n \in h(n, y)$ and $u_n \in h(n, z_n) - W$ for each $n \in N$. But since h is a γ-function, this means that y is a cluster point of $\langle u_n \rangle$; obviously a contradiction since $u_n \notin W$ for any $n \in N$.

Now let $V = h(n_0, y)$. Then we have $\psi[\bigcup \{g(n_0, x): x \in \psi^{-1}(V)\}] = \bigcup \{h(n_0, z): Z \in h(n_0, y)\} \subseteqq W$.

THEOREM 4.3. *Let Y be a T_1-space. Then Y is a γ-space if and only if there is a metrizable space X and an open continuous G-mapping from X onto Y.*

Proof. Suppose that Y is a γ-space. Then by Theorem 2.1 there is a metrizable space X and an open continuous surjection $\psi: X \to Y$. By Proposition 4.2 ψ is a G-mapping.

On the other hand, suppose that X is metrizable and $\psi: X \to Y$ is an open continuous surjection, which is a G-mapping. Then Y is first countable. Let f be a first countable function for Y, and let $y \in Y$. Choose an $s \in \psi^{-1}(y)$ and define $h(n, y) = \psi[g(n, s)] \cap f(n, y)$. Now suppose $y_n \in h(n, p)$ and $x_n \in h(n, y_n)$ for each $n \in N$. Then we must show that p is a cluster point of $\langle x_n \rangle$. Assume not, and choose a neighborhood U of p and an integer k so that for $n \geqq k$, $x_n \notin U$. Now there exists a neighborhood V of p and an n_0 (which we may choose to be $\geqq k$ since $g(n+1, x) \subseteqq g(n, x)$ for all x and n) such that $\psi[\bigcup \{g(n_0, x): x \in \psi^{-1}(V)\}] \subseteqq U$. But we can choose an $m_0 \geqq n_0$ such that $h(m_0, p) \subseteqq V$. Then we have $\psi[\bigcup \{g(m_0, x): x \in \psi^{-1}(V)\}] \subseteqq U$. Now since $y_{m_0} \in h(m_0, p)$, we get $\psi[g(m_0, s_{m_0})] \subseteqq U$, where $h(m_0, y_{m_0}) = \psi[g(m_0, s_{m_0})] \cap f(m_0, y_{m_0})$ and $s_{m_0} \in \psi^{-1}(y_{m_0})$. But there is a $t_{m_0} \in \psi^{-1}(x_{m_0}) \cap g(m_0, s_{m_0})$ and so $\psi(t_{m_0}) = x_{m_0} \in U$, a contradiction.

5. Semi-metric spaces.

Since the proofs of Proposition 5.2 and Theorem 5.3 are very similar to the proofs of Proposition 3.2 and Theorem 3.3 respectively, we omit them here.

DEFINITION 5.1. Let X and Y be topological spaces, let $\psi: X \to Y$ be a surjection, and let g be a COC-function for X. Then ψ is an *SM-mapping relative to g* (SM=semi-metric) if given any $y \in Y$ and neighborhood W of y, there is an $n \in N$ such that $g(n, x) \cap \psi^{-1}(y) \neq \varnothing$ implies that $\psi(x) \in W$. A surjection $\psi: X \to Y$ is an *SM-mapping* if there is a COC-function g for X such that ψ is an SM-mapping relative to g.

We remark that our SM-mapping is a generalization of Ponomarev's [16] π-mapping. In fact if (X, d) is a metric space and $g(n, x) = \mathrm{Bd}(x, 1/n)$ then a π-mapping and an SM-mapping relative to g are identical concepts. Notice, however, that in Theorem 5.3 we get an SM-mapping relative to a COC-function g which is not directly related to the metric on X. Consequently, that SM-mapping is not necessarily a π-mapping. In connection with this remark, the question arises as to whether we can find characterizations similar to the ones we give, but with mappings which are directly related to the metric on X. Such characterizations, if they exist, would in a sense be improvements of our theorems.

PROPOSITION 5.2. *Let (X, T) and Y be topological spaces with Y a semi-stratifiable space, and let $\psi: X \to Y$ be a continuous surjection. Then ψ is an SM-mapping.*

THEOREM 5.3. *Let Y be a T_1-space. Then Y is a semi-metric space if and only if there is a metrizable space X and an open continuous SM-mapping from X onto Y.*

In addition to comparing this characterization with those of Heath and Nagata mentioned above, the reader should compare it with results by Alexander [1] and Burke [5].

6. Some properties of the mappings. We note that each of the kinds of mappings we have defined in this paper (N-mapping, G-mapping, and SM-mapping) is countably productive. We can use this property to get relatively simple proofs that Nagata spaces, γ-spaces, and semi-metric spaces are countably productive as follows (the results are known, with the possible exception of the γ-space case).

THEOREM 6.1. *Let $\{Y_n: n = 1, 2, \cdots\}$ be a sequence of T_1-spaces.*
(1) *If each Y_n is a Nagata space, so is $\prod_{n=1}^{\infty} Y_n$.*
(2) *If each Y_n is a γ-space, so is $\prod_{n=1}^{\infty} Y_n$.*
(3) *If each Y_n is a semi-metric space, so is $\prod_{n=1}^{\infty} Y_n$.*

Proof. (1) By Theorem 3.3 for each n, there is a metric space X_n and an open continuous N-mapping $\psi_n: X_n \to Y_n$. Then $\prod_{n=1}^{\infty} X_n$ is a metric space and $\prod_{n=1}^{\infty} \psi_n$ is an open continuous mapping from $\prod_{n=1}^{\infty} X_n$ onto $\prod_{n=1}^{\infty} Y_n$. It is not difficult to show that this mapping is in fact an N-mapping. Consequently, again by Theorem 3.3, $\prod_{n=1}^{\infty} Y_n$ is a Nagata space.

The proofs of (2) and (3) are similar.

Now we ask when finite-to-one and compact mappings are SM-mappings, and derive several corollaries concerning images under these mappings.

THEOREM 6.2. *Let $\psi: X \to Y$ be a continuous finite-to-one surjection, and let g be a semi-stratifiable function for X. Then ψ is a SM-mapping relative to g.*

Proof. Let $y \in Y$, W an open set in Y containing y. Suppose that $\psi^{-1}(y) = \{x_1, \cdots, x_k\}$. For each i, $1 \leqq i \leqq k$, there exists an n_i such that $x_i \notin \cup \{g(n_i, p): p \in X - \psi^{-1}(W)\}$. Let $n_0 = \max\{n_i: i=1, \cdots, k\}$. Then (recall that g is decreasing) if $g(n_0, x) \cap \psi^{-1}(y) \neq \emptyset$, we know $x \notin X - \psi^{-1}(W)$, i.e., $\psi(x) \in W$.

THEOREM 6.3. *Let $\psi: X \to Y$ be a continuous compact surjection, and let g be a K-semi-stratifiable function for X. Then ψ is a SM-mapping relative to g.*

The proof of Theorem 6.3 is very similar to the proof of Theorem 6.2. For the definition and a discussion of K-semi-stratifiable spaces, the reader should see Lutzer [13].

Next we note that we can weaken one implication of Theorem 5.3 to get the following.

THEOREM 6.4. *Let X be a space with a COC-function g, let $\psi: X \to Y$ be an almost-open SM-mapping relative to g. Then Y is semi-stratifiable.*

Proof. (Sketch). Let $y \in Y$, $n \in N$. Then there is an $s \in \psi^{-1}(y)$ with a system of neighborhoods $\{N_\alpha: \alpha \in A\}$ such that each $\psi(N_\alpha)$ is open. Now choose $N_{\alpha_n} \subseteqq g(n, s)$ and define $h(n, y) = \psi(N_{\alpha_n})$. Then h can be shown to be a semi-stratifiable function for Y.

COROLLARY 6.5. *An almost-open continuous finite-to-one image of a semi-stratifiable space is semi-stratifiable.*

Proof. Combine Theorems 6.2. and 6.4.

It should be noted that Henry [11] has proved a slightly stronger result than Corollary 6.5 (pseudo-open rather than almost-open).

COROLLARY 6.6. *An almost-open continuous compact image of a K-semi-stratifiable space is semi-stratifiable.*

Proof. Combine Theorems 6.3 and 6.4.

REFERENCES

1. Charles C. Alexander, *Semi-developable spaces and quotient images of metric spaces*, Pacific J. Math., **37** (1971), 277-293.
2. A. V. Arhangel'skii, *On mappings of metric spaces*, Soviet Math. Doklady, **3** (1962), 953-956.
3. ———, *Mappings and spaces*, Russian Math. Surveys, **21** (1966), 115-162.
4. Carlos J. Borges, *On stratifiable spaces*, Pacific J. Math., **17** (1966), 1-16.
5. Dennis K. Burke, *Cauchy sequences in semi-metric spaces*, Proc. Amer. Math. Soc., **33** (1972), 161-165.
6. Jack Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **11** (1961), 105-125.
7. Geoffrey C. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math., **32** (1970), 47-54.
8. S. Hanai, *On open mappings, II*, Proc. Japan Acad., **37** (1961), 233-238.
9. R. W. Heath, *Arc-wise connectedness in semi-metric spaces*, Pacific J. Math., **12** (1962), 1301-1319.
10. ———, *On open mappings and certain spaces satisfying the first countability axiom*, Fund. Math., **57** (1965), 91-96.
11. Michael Henry, *Stratifiable spaces, semi-stratifiable spaces and their relation through mappings*, Pacific J. Math., **37** (1971), 697-700.
12. R. E. Hodel, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J., **39** (1972), 253-265.
13. David Lutzer, *Semi-metrizable and stratifiable spaces*, General Topology and Appl., **1** (1971), 43-48.
14. E. A. Michael, *On representing spaces as images of metrizable and related spaces*, General Topology and Appl., **1** (1971), 329-344.
15. Jun-iti Nagata, *On generalized metric spaces and q-closed mappings*, to appear.
16. V. I. Ponomarev, *Axioms of countability and continuous mappings*, Bull. Pol. Akad. Nauk., **8** (1960), 127-134.

Received September 24, 1973 and in revised form December 1, 1973. The results in this paper are taken from the author's Ph.D. dissertation, done at Duke University under the supervision of R. E. Hodel. The author wishes to express his gratitude for Professor Hodel's help and encouragement.

WAYNESBURG COLLEGE