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THE GROUP OF SELF-EQUIVALENCES OF CERTAIN COMPLEXES

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The group of self-equivalences (homotopy classes of base point preserving homotopy equivalences) of a certain class of finite CW-complexes is studied. This class includes, in particular, all closed, connected, n-manifolds M with finite fundamental group such that $\pi_i(M) = 0$, $1 < i < n$. Such complexes are easily seen to be the quotient space of a fixed point free action of a finite group on a homotopy n-sphere, and include the Klein-Clifford manifolds.

The main result characterizes this group as a normal subgroup of Aut $(\pi_1(X))$, for X in the above mentioned class, consisting of all θ such that θ induces either the identity map or the inverse map on $H^{n+1}(\pi_1(X); Z) = Z_k$, k being the order of $\pi_1(X)$. This leads to a collection of general results on the algebraic structure of the group of self-equivalences, as well as several explicit calculations, including the recovery of results due to Olum.

This group has been studied by various authors, e.g. Arkowitz-Curjel [1], Olum [7], Rutter [8], and Shih [9]. In general this group is non-abelian and quite often infinite.

Let Σ^n denote a finite n-dimensional CW-complex of the homotopy type of an n-sphere. Let G be a finite group which acts without fixed points on Σ^n and let $X = \Sigma^n/G$ denote the quotient space of Σ^n by the action of G. We study the group of self-equivalences of X.

Several comments are in order at this point. Firstly, if the order of G is one or two the results are well known and although our methods apply we omit discussion of these in order that our results be more succintly stated. Secondly, if the order of G is greater than two (the case we study) n must be odd. Finally, it is easily shown, e.g. [10], that $\pi_1(X)$ acts trivially on $\pi_j(X)$, $j > 1$. We point out that G may often be non-abelian but G must be periodic of period $n+1$ (see [2], p. 260).

We define the group of self-equivalences for completeness.

Let Y be a topological space with base point $y \in Y$. Let $\mathscr{H}(Y)$ denote the collection of all homotopy equivalences of Y preserving base point. The equivalence relation $\sim$ of base point preserving homotopy divides $\mathscr{H}(Y)$ into classes called self-equivalences of Y and we have

DEFINITION. $Eq(Y) = \mathscr{H}(Y)/\sim$. $Eq(Y)$ is a group under the operation induced by composition of mappings.

We note first that

LEMMA 1.1. *$Eq(X)$ is isomorphic to $Eq(X_{n+1})$ where X_{n+1} is the $(n+1)$st stage of the Postnikov system of X.*

Proof. This follows from the fact that $\pi_1(X)$ acts trivially on $\pi_j(X)$, $j > 1$ and a simple obstruction theory argument as in [1] where the result is stated in the simply connected case.

The Postnikov system for X begins as follows,

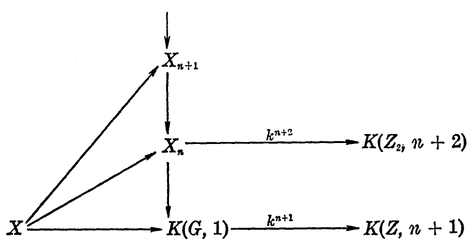

FIGURE 1

As was pointed out above n must be odd so $\pi_{n+1}(X) \approx \pi_{n+1}(\Sigma^n) = Z_2$.

We define $\psi_G = \{\theta \in [K(G, 1), K(G, 1)] \mid \theta^*(k^{n+1}) = \pm k^{n+1}\}$. Then

LEMMA 1.2. $Eq(X_n) \approx \psi_G$.

Proof. By Shih [9] we have the exact sequence

$$1 \longrightarrow H^n(K(G, 1); Z) \longrightarrow Eq(X_n) \longrightarrow \psi_G \longrightarrow 1 .$$

Now since G acts without fixed points on Σ^n and hence is periodic, the odd dimensional cohomology of G vanishes, so the result follows. Now we show that

LEMMA 1.3. $Eq(X_{n+1}) \approx Eq(X_n)$.

Proof. By Kahn [3] we have homomorphism $\rho^n: Eq(X_{n+1}) \rightarrow Eq(X_n)$. By a Serre spectral sequence argument one can easily show that $H^{n+2}(X_n: Z_2) = Z_2$ and hence if $f \in Eq(X_n)$, $f^*: H^{n+2}(X_n, Z_2) \rightarrow H^{n+2}(X_n: Z_2)$ is the identity map. This implies that ρ^n is onto by Lemma 2.1 of [3].

Now in the paper cited it is also shown that $\operatorname{Ker} \rho^n$ is the image of a subgroup of $H^{n+1}(X_{n+1}; Z_2)$. Again using a Serre spectral sequence argument one can show $H^{n+1}(X_{n+1}; Z_2) = 0$. Hence the result follows.

Collecting together the proceeding we have

THEOREM 1.4. *If X is as described above then $Eq(X)$ is isomorphic to $\{\theta \in \mathrm{Aut}(\pi_1(X)) \mid \theta^*$ is either the identity map on $H^{n+1}(\pi_1(X); Z)$ or the map which takes each element to its algebraic inverse$\}$. Here $H^*(\pi_1(X); Z)$ denotes the group cohomology of $\pi_1(X)$, see* [12].

Proof. We note firstly that by the periodicity of G we have $H^{n+1}(\pi_1(X); Z) \approx Z_m$ where m is the order of G. Secondly k^{n+1} is easily seen to be a generator of the group by a spectral sequence argument. Finally it can be shown (see Maclane [4], p. 136) that the cohomology of a $K(G, 1)$ space is "naturally" isomorphic to the "group cohomology" $H^*(G; Z)$. The naturality of this isomorphism means in particular that if h is a map from $K(G, 1)$ to itself inducing θ on $\pi_1(K(G, 1)) = G$ then the following diagram commutes where Ψ is the natural isomorphism.

$$\begin{array}{ccc} H^*(K(G,1); Z) & \xrightarrow[\approx]{\psi} & H^*(G; Z) \\ \downarrow h^* & & \downarrow \theta^* \\ H^*(K(G,1); Z) & \xrightarrow[\approx]{\psi} & H^*(G; Z) \end{array}$$

FIGURE 2

Hence the problem is reduced to a purely (cohomological) algebraic criterion. We see in the remainder of this section and in § 2 that this theorem quite often is a computational tool in determining $Eq(X)$.

First we note immediately that

COROLLARY 1.5. *$Eq(X)$ is a normal subgroup of* $\mathrm{Aut}(\pi_1(X))$.

Proof. Let $\nu \in \mathrm{Aut}(\pi_1(X))$. Then if $\theta \in Eq(X)$ (here we identify $Eq(X)$ with the elements of the automorphism group), we show $\nu\cdot\theta\cdot\nu^{-1} \in Eq(X)$. Let $k \in H^{n+1}(G; Z)$. Then

$$[\nu\cdot\theta\cdot\nu^{-1}]^*k = \nu^{-1*}\theta^*[\nu^*k] = \nu^{-1*}(\pm\nu^*k) = \pm k\ .$$

COROLLARY 1.6. *$Eq(X) \supseteq \mathcal{I}nn(\pi_1(X))$, the group of inner automorphisms of $\pi_1(X)$.*

Proof. All inner automorphisms have the property that they induce the identity map on cohomology ([4], p. 118).

This is particularly useful since a standard group theory fact is that $\mathcal{I}nn(G) \approx G/Z(G)$, $Z(G)$ the center of G.

COROLLARY 1.7. *$Eq(X)$ is solvable if and only if* $\mathrm{Aut}\,(\pi_1(X))$ *is solvable.*

Proof. Since $Eq(X)$ is normal in $\mathrm{Aut}\,(\pi_1(X))$ the result will follow if we can show $\mathrm{Aut}\,(\pi_1(X))/Eq(X)$ is solvable. Let

$$\Psi\colon \mathrm{Aut}\,(\pi_1(X)) \longrightarrow \mathrm{Aut}\,(H^{n+1}(\pi_1(X); Z))$$

be the obvious map $\theta \rightarrow \theta^*$. Now since $H^{n+1}(\pi_1(X); Z)$ is cyclic we have $\mathrm{Aut}\,(H^{n+1}(\pi_1(X); Z))$ is abelian. $Eq(X) \supseteqq \mathrm{Ker}\,\Psi$ so

$$\mathrm{Aut}\,(\pi_1(X))/Eq(X)$$

is abelian and hence solvable.

The homomorphism Ψ is a very useful one in the general problem of determining $Eq(X)$. In fact $Eq(X) = \Psi^{-1}(K)$ where K is the subgroup of $\mathrm{Aut}\,(Z_m)$, $m =$ order of $\pi_1(X)$, consisting of the identity map and the inverse automorphism $(x \rightarrow -x)$.

We now consider what happens if G and H are finite groups of orders at least 2 and $G \times H$ acts on Σ^n without fixed points. The quotient space is a complex of the type considered in this paper. We point out that if $G \times H$ acts on Σ^n without fixed points then $(|G|, |H|) = 1$. This follows from the periodicity of G, H and $G \times H$. We have

THEOREM 1.8. *If $G \times H$ acts on Σ^n without fixed points and $|G| \geqq 2$, $|H| \geqq 2$, then we have the following two cases:*

$$Eq(\Sigma^n/G \times H) = \begin{cases} Eq(\Sigma^n/G) \times Eq(\Sigma^n/H) & |G| > 2 \ \ |H| > 2 \\ Eq(\Sigma^n/H) \quad |G| = 2 & |H| > 2\,. \end{cases}$$

Proof. This follows by looking at the Kunneth theorem in group cohomology and noting that since the orders of G and H are relatively prime $\mathrm{Aut}\,(G \times H) \approx \mathrm{Aut}\,(G) \times \mathrm{Aut}\,(H)$.

The situation of a product of two groups acting on Σ^n occurs frequently (see Milnor [5]). In particular the theorem says that to determine $Eq(X)$ one should look at indecomposable groups.

2. Specific calculations of $Eq(M)$. In this section we calculate explicitly the group $Eq(M)$ for certain manifolds M. In particular we recovers results of Olum for generalized Lens spaces, and show

$Eq(X) \neq 0$ for all X studied in this paper. The methods used in the calculations vary, but fall generally in the domain of group theory and cohomology of groups.

A general reference for this section is Weiss [12].

2A. Generalized lens spaces. We now recover results originally due to Olum in [6] by calculating $Eq(\Sigma^{2n+1}/Z_m)$ $n \geqq 1$ $m > 2$. We point out that a fixed-point-free action of a cyclic group on an odd dimensional sphere is always defined, namely to get the generalized Lens spaces. However, our proof does not make use of any "particular" action of Z_m on Σ^{2n+1}, but in fact shows that $Eq(\Sigma^{2n+1}/Z_m)$ is not dependent on how Z_m acts freely, only that it does act in some fixed-point-free way.

We prove:

THEOREM 2A. 1. *Let* $n \geqq 1$, $m > 2$. *Then* $Eq(\Sigma^{2n+1}/Z_m) \approx \mathscr{U}_m$, *where* $\mathscr{U}_m$ *is the subgroup of* $\mathrm{Aut}(Z_m)$ *consisting of those integers* t, $1 \leqq t < m$ *such that* $t^{n+1} \equiv \pm 1 \pmod m$. *Here we identify* $\mathrm{Aut}(Z_m)$ *with* $\{x \mid 1 \leqq x < m, (x, m) = 1\}$.

Proof. In this proof we calculate directly the induced map on cohomology of an automorphism $\theta\colon Z_m \to Z_m$ by constructing a chain map on a projective resolution of Z as a trivial $Z(Z_m)$-module (see Weiss [12]).

We take the particularly nice resolution of Z given by

$$0 \longleftarrow Z \xleftarrow{d_0} \Lambda a_0 \xleftarrow{d_1} \Lambda a_1 \xleftarrow{d_2} \Lambda a_2 \xleftarrow{d_3} \cdots$$

where $\Lambda = Z(Z_m)$, Z_m generated by ν, d_0 is the augmentation and

$$d_{2k}(a_{2k}) = (\nu^{m-1} + \cdots + 1)a_{2k-1}$$
$$d_{2k+1}(a_{2k+1}) = (\nu - 1)a_{2k}$$

(see Cartan-Eilenberg [2], p. 251).

Let $\theta_k\colon Z_m \to Z_m$ be the automorphism given by $\theta_k(\nu) = \nu^k$, $(k, m) = 1$ $1 \leqq k < m$. We calculate the induced map on $H^*(Z_m; Z)$ of θ_k, by constructing a chain map $\Lambda = (\Lambda^i)$ $i \geqq 0$ corresponding to the pair $(\theta_k, 1_Z)$ as described in Weiss [12].

By direct calculation we get the diagram

$$\begin{array}{ccccccccc}
0 \longleftarrow & Z & \xleftarrow{d_0} & \Lambda a_0 & \xleftarrow{d_1} & \Lambda a_1 & \xleftarrow{d_2} & \Lambda a_2 & \longleftarrow \cdots \\
 & \Big\uparrow{\scriptstyle id_Z} & & \Big\uparrow{\scriptstyle \Lambda^0} & & \Big\uparrow{\scriptstyle \Lambda^1} & & \Big\uparrow{\scriptstyle \Lambda^2} & \\
0 \longleftarrow & Z & \xleftarrow{d_0} & \Lambda a_0 & \longleftarrow & \Lambda a_1 & \longleftarrow & \Lambda a_2 & \longleftarrow \cdots
\end{array}$$

FIGURE 3

where $\Lambda^{2i}(a_{2i}) = [(J_2)^i]a_{2i} \quad i \geqq 0$

$$\Lambda^{2i+1}(a_{2i+1}) = [(J_2)^i \cdot J_1]a_{2i+1}$$

where

$$J_1 = \frac{(\nu^k - 1)}{(\nu - 1)}, \qquad J_2 = \frac{(\nu^{mk} - 1)}{(\nu - 1)}$$

each considered as elements in Λ. It can be directly verified that (Λ^i) $i \geqq 0$, is a chain map (i.e., the above diagram commutes), where we assume the upper resolution is a Λ-module via the map θ_k.

We now wish to determine what the induced map in cohomology will be in dimension $2n + 2$ (since Z_m acts on a $2n + 1$-sphere). It is clear that since the coefficients Z in $H^*(Z_m, Z)$ are to be regarded as a trivial Λ-module, that the induced map on cohomology of θ_k is gotten by multiplication by $J_2^{n+1}(1)$, where this means evaluating the element $J_2^{n+1} \in \Lambda$ by replacing ν by 1, for example, if $\sigma \in H^{2n+2}(Z_m, Z)$ then

$$[\theta_k^* \sigma] = \sigma(\Lambda^{2n+2})^* = \sigma(J_2)^{n+1} = [J_2^{n+1}(1)]\sigma .$$

Performing this evaluation we see that:

$$J_2^{n+1}(1) = k^{n+1}$$

so that $\theta_k^*\colon H^{2n+2}(Z_m; Z)$ is multiplication by $[k^{n+1}]$ where $[k^{n+1}]$ denotes the equivalence class of $k^{n+1} \bmod m$ (since $H^{2n+2}(Z_m; Z) \approx Z_m$). So we have $\theta_k \in Eq(\Sigma^{2n+1}/Z_m) \leftrightarrow k^{n+1} \equiv \pm 1 \pmod m$ using Theorem 1.4. This recovers the stated result of Olum.

2B. $Eq(X) \neq 0$. In studying the general problem of $Eq(X)$, where X is a topological space, it appears to be unknown exactly when $Eq(X) = 0$ or in fact what this condition would imply about the space X. It is known for example that if X is contractible or $X = K(Z_2, n)$ then $Eq(X) = 0$. However, it appears that no other general conditions are known. However, for the spaces in this paper we have:

THEOREM 2B. 1. *$Eq(X) \neq 0$ for all X satisfying the hypotheses in* § 1.

Proof. If $\pi_1(X)$ is non-abelian then since $Eq(X) \supseteqq \vartheta_{nn}(\pi_1(X))$ by Corollary 1.6 and $\vartheta_{nn}(\pi_1(X)) \neq 0$ we get the desired result.

If $\pi_1(X)$ is abelian then $\pi_1(X)$ is a direct sum of cyclic groups of prime power order. Using Theorem 1.8 we can conclude that $Eq(X)$ is either the direct sum of the corresponding self-equivalence groups for the summands (as in Case 1 of Theorem 8) or the direct sum of the self-equivalence groups corresponding to the summands which are two-torsion free. So we must in fact show if we look at any of these cyclic summands of $\pi_1(X)$, the corresponding group of self-equivalences is nonzero.

Let p be a prime dividing the order of $\pi_1(X)$ such that Z_{p^i} is a direct summand of $\pi_1(X)$ where either p is an odd prime or $i > 2$. We can find such a summand since the order of $\pi_1(X)$ is greater than two. Now by the results of $Eq(\Sigma^{2k+1}/Z_{p^i})$ in Theorem 2A. 1 we know $Eq(\Sigma^{2k+1}/Z_{p^i})$ is always nonzero since there are always at least two distinct solutions to:

$$x^{k+1} \equiv \pm 1 \pmod{p^i}$$

namely $x = 1$ and $x = p^i - 1$. Hence $Eq(X)$ must be nontrivial since it is the direct sum of nontrivial abelian groups.

2C. The quaternionic group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Let Q denote the eight element quaternionic group. Q acts on S^3 without fixed points as a subgroup of S^3 considered as quaternions of unit norm. We have,

THEOREM 2C. 1. *$Eq(S^3/Q) \approx S_4$ the group of permutation of 4 symbols.*

Proof. Firstly we note Aut $(Q) \approx S_4$ (Zassenhaus [13]). In fact in the cited reference for this fact, Aut (Q) can be represented as a group of permutations of the cosets of a subgroup of order 6 contained in it. Using this representation and a projective resolution (Cartan-Eilenberg [2], p. 25) we directly calculate the induced map on cohomology of an automorphism. The calculation, which is straight forward but tedious may be found in [10]. There it is shown that in fact each automorphism induces the identity map on cohomology in the appropriate dimension, so $Eq(S^3/Q) \approx S_4$.

2D. The group D_{12}. Let D_{12} be the 12-element-group generated by elements x, y subject to the relations $x^3 = y^2 = (xy)^2$ and $xyx = y$. D_{12} is a generalized quaternionic group and is seen to act on S^3 without fixed points (Milnor [4]). In a similar manner as in 2C (see [10]) one can show:

THEOREM 2D. 1. *$Eq(S^3/D_{12}) \approx Z_2 \times S_3$.*

2.E. The symmetric group S_3. It is known, [5], that S_3, the symmetric group on three symbols, cannot act on any n-sphere without fixed points. However, it can be shown that it can act on a homotopy 3-sphere without fixed points [11]. We have,

THEOREM 2E. 1. $Eq(\Sigma^3/S_3) \approx S_3$.

Proof. Since $\vartheta_{nn}(S_3) \approx \mathrm{Aut}\,(S_3) \approx S_3$ the result follows from Corollary 1.6.

In fact any periodic group can act on some homotopy sphere without fixed points [11], and so this furnishes a whole collection of examples of the type of complexes considered in this paper, which are not manifolds.

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