PACIFIC JOURNAL OF MATHEMATICS
Vol. 54, No. 1, 1974

ON THE RADICALS OF LATTICE-ORDERED RINGS

H. J. SHYR AND T. M. VISWANATHAN

In this note, it is shown that for several classes of lattice-ordered rings, the l-radical $L(A)$ and the prime radical $P(A)$ coincide and that A modulo the l-radical is an f-ring. In particular, this is true for the class of positive square rings satisfying the identity $a_+a_- = 0$.

The most well-behaved lattice-ordered rings are the f-rings satisfying the identities $xa_+ \wedge a_- = 0$ where x is an arbitrary positive element and a an arbitrary element of the l-ring A. All other rings are then studied by dissecting the ring into parts — one part called the radical where the idiosyncracies of the ring play a role and the other is the ring modulo the radical where the ring is expected to behave more like an f-ring. The radicals are themselves varied: There is the l-radical $L(A)$ of Birkhoff and Pierce which is the union of nilpotent l-ideals of A and the P-radical $\mathscr{P}(A)$, being the intersection of all the prime l-ideals of A. It is known that $L(A) \subseteqq P(A)$. The object of this note is to show that equality holds and that the radicals behave well for several classes of l-rings.

2. Square-archimedean rings. A *square-archimedean* ring A is an l-ring satisfying the following: Given x, y in the positive cone A_+, there exists a positive integer $n = n(x, y)$ such that $xy + yx \leqq n(x^2 + y^2)$. The positive square l-rings, having square elements positive or zero are indeed square-archimedean. The following is an example of a commutative l-ring with identity which is square-archimedean but not positive square: The ring A has the additive group of two copies of the ordered group Z of integers with multiplication defined by $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_1 + a_1b_2)$ and order provided by (a_1, a_2) in A^+ if $a_2 \geqq a_1 \geqq 0$ in Z. Notice also that the bound $n(x, y)$ may not be uniform.

It is appropriate at this point to introduce the upper l-radical $U(A)$ which is the union of all nil l-ideals of A. $U(A)$ is an l-ideal whereas the set $H(A)$ of all absolutely nilpotent elements need not be an ideal. We have the containment relation $L(A) \subseteqq P(A) \subseteqq U(A) \subseteqq H(A)$. Throughout the remaining part of this section A is assumed to be a square-archimedean ring.

PROPOSITION 1. *If x and y are elements of A^+ and m a positive integer, then there exist positive integers λ_m and μ_m such that $(x + y)^{2^m} \leqq \lambda_m(x^{2^m} + y^{2^m})$ and $(xy)^{2^m} \leqq \mu_m(x^{2^{m+1}} + y^{2^{m+1}})$.*

Proof. Use induction on m. For the second inequality, $xy \leqq xy + yx \leqq n(x^2 + y^2)$ and so $(xy)^{2m} \leqq n^{2m}(x^2 + y^2)^{2m}$ and now use the first.

PROPOSITION 2. *The set $H(A)$ is a sublattice subring of A which is also square-archimedean.*

Proof. This is a consequence of Proposition 1 and the following identity in A: $a + b = (a \vee b) + (a \wedge b)$.

THEOREM 1. *If A is a square-archimedean ring, then $L(A) = P(A) = U(A)$. In particular, the three radicals coincide for positive square l-rings.*

Proof. We shall obtain a reduction to the case when A itself will be a nil ring. For this, $U(A)$ is an l-ideal of A and so by (2.18) of [2], the l-radical of $U(A)$ is equal to $L(A)$. Since $U(A)$ is a nil ring, the theorem will be proved if we show that the l-radical of a nil ring is the whole ring. This is the next lemma.

LEMMA 1. *For every integer $m \geqq 1$, let $p(m) = 2^m$. If A is a nil ring then the set $I_m = \{x \in A: | x |^{p(m)} = 0\}$ is a nil potent l-ideal. Hence $L(A) = A$.*

Proof. It is enough to prove the result for $m = 1$, since the general case would then follow by induction by passing to the quotient say A/I_{m-1}. For $m = 1$, we already know from Proposition 1 that I_1 is a sublattice subring of A. Given $x \geqq 0$ in I_1 and a in A^+, we have $xax = xax + ax^2 \leqq n(ax)^2$ for some positive integer n and by iteration, $xax \leqq n^s a^s xax$ for every $s \geqq 2$ and so $xax = 0$, making the square of both ax and xa vanish. Thus I_1 is a nilpotent l-ideal of index 2.

REMARK 1. The question naturally arises whether there exists a positive square l-ring for which $U(A) \neq H(A)$. This is another form of a question of Diem. (See p. 79 of [2].)

3. Rings with well-behaved radicals. We shall now complete the work of Diem by showing that for several classes of rings satisfying specific l-ring identities, the l-radical equals the set N of nilpotents so that all the radicals coincide. A basic tool is the notion of an f-ideal, which is an l-ideal I such that A/I is an f-ring. Thus an l-ideal I is ad f-ideal if and only if it contains all elements of the form $xa^+ \wedge a^-$ and $a^+x \wedge a^-$ for all $x \geqq 0$ and for all a in A. We observe that if the l-ring A has a nilpotent f-ideal, then $L(A) = N$, making all the radicals coincide and in this case the l-radical indeed behaves well since $A/L(A)$ is an f-ring without nilpotent elements.

THEOREM 2. *Let A be an l-ring which satisfies one of the following identities*:

(i) $xa^+ \wedge xa^- = 0$ *and* $a^+x \wedge a^-x = 0$ *for all* $x \geqq 0$ *and* a *in* A.
(ii) $xa^+x \wedge xa^-x = 0$ *for all* $x \geqq 0$ *and* a *in* A.
(iii) $a^+xa^- = 0$ *for all* $x \geqq 0$ *and* a *in* A.
(iv) $xa^+xa^-x = 0$ *for all* $x \geqq 0$ *and* a *in* A.
(v) $a^+a^- = 0$ *for all* a *in* A. *Then* $L(A) = N$.

Proof. We shall produce a nilpotent $\underline{f}$-ideal in all cases except (v).

(i) and (ii). Let $I = \{x \in A: AxA = 0\}$. Let us show that I is an $\underline{f}$-ideal in the case of (ii). A similar proof works for (i). If c, d, and $x \geqq 0$ in A and a an element of A, then $c(xa^+ \wedge a^-)d \leqq cxa^+d \wedge ca^-d \leqq ea^+e \wedge ea^-e$ where e is any upper bound of c, cx, and d and this last element is 0. Since any element is the difference of two positive elements, this shows that $xa^+ \wedge a^-$ belongs in I. Similarly $a^+x \wedge a^-$ belongs in I. Clearly I is a nilpotent $\underline{l}$-ideal.

(iii) and (iv). It is clearly enough to prove (iv). Notice that for every $x \geqq 0$ and a in A, the element $(xa)^2x \geqq 0$. Using this, it is easy to show that the set $J = \{a \in A: (x \mid a \mid)^2 x = 0 \forall x \in A^+\}$ is a nilpotent $\underline{f}$-ideal.

(v) Since A in this case is a positive square ring, by Theorem 1, $L(A) = P(A)$ and by Corollary 4.6 of [2], $P(A) = N$.

COROLLARY. *Let A be an $\underline{l}$-ring. Suppose the upper radical is square-archimedean or satisfies one of the identities above, then* $L(A) = P(A) = U(A)$.

REMARK 2. The l-ring satisfying the identity $a^+a^- = 0$ also has a nilpotent $\underline{f}$-ideal. The proof however requires that $H(A)$ be an $\underline{l}$-ideal, a consequence of Corollary 3.8 of [2]. Since the existence of a nilpotent $\underline{f}$-ideal implies that only a part of the $\underline{l}$-radical behaves undesirably, it may be useful to describe this $\underline{f}$-ideal.

From Lemma 1, if a and s are elements of A^+ and if $a^2 = 0$ and s nilpotent, then $asa = 0$. Now if $r \in A^+$ and $a \in A^+$ an element such that $a^2 = 0$, then rar is nilpotent, since $H(A)$ is an $\underline{l}$-ideal. Hence for every r in A^+ we have $arara = 0$.

Now if $a \in A$ and $r \in A^+$ then $(ra^+ \wedge a^-)^2 \leqq ra^+a^- = 0$. Hence $(ra^+ \wedge a^-)^2 = 0$. Similarly $(a^+r \wedge a^-)^2 = 0$.

Let $Z_1(A) = \{a \in A: (x \mid a \mid)^2 x = 0 \forall x \in A^+\}$. Since A is a positive square ring, $Z_1(A)$ is a nilpotent $\underline{l}$-ideal. Since it may not contain $ra^+ \wedge a^-$, we construct $Z_2(A)$ as the inverse image of $Z_1(A/Z_1(A))$, using the natural epimorphism $A \to A/Z_1(A)$. $Z_2(A)$ is a nilpotent $\underline{f}$-ideal of A.

REFERENCES

1. G. Birkhoff and R. S. Pierce, *Lattice-ordered rings*, An. Acad. Brasil Ci., **28** (1956), 41-69.
2. J. E. Diem, *A radical for lattice-ordered rings*, Pacific J. Math., **25** (1968), 71-82.
3. L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, New York, 1963.
4. I. N. Herstein, *Noncommutative Rings*, The Carus Mathematical Monographs, Number Fifteen, The Mathematical Association of America, 1968.
5. D. G. Johnson, *A structure theory for a class of lattice-ordered rings*, Acta Math., **104** (1960), 163-215.
6. P. Ribenboim, *Rings and Modules*, Interscience Publishers, 1967.

Received July 14, 1972 and in revised form January 8, 1974.

UNIVERSITY OF WESTERN ONTARIO, LONDON
AND
IMPA, RIO DE JANEIRO, BRAZIL