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# SYMMETRIC MAXIMAL IDEALS IN $M(G)$

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**Let $G$ be a nondiscrete locally compact abelian group, and $M(G)$ the convolution algebra of bounded regular measures on $G$. In this paper, the following is proved: Let $\{\lambda_k\}_{k=0}^{\infty}$ be a countable subset of $M_c^+(G)$, $0 \neq \lambda_0 \in M_0(G)$, and $\{C_k\}_{k=0}^{\infty}$ a countable family of $\sigma$-compact subsets of $G$ such that $\lambda_k(x + C_k) = 0$ for all $x \in G$ and all $k = 0, 1, 2, \cdots$. Then there exists a nonzero measure $\sigma \in M_0^+$ (supp $\lambda_0$) with compact support such that $\lambda_k[x + C_k + G_p(\text{supp}\,\sigma)] = 0$ for all $x \in G$ and all $k = 0, 1, 2, \cdots$. A consequence of this result is the following: Let $Y$ be the closed ideal in $M(G)$ which is generated by $\bigcup \{L^1(\lambda_k): k = 0, 1, 2, \cdots\}$ for some countable subset $\{\lambda_k\}_{k=0}^{\infty}$ of $M_c(G)$. Then there exist "fairly many" symmetric maximal ideals in $M(G)$ which contain $\bigcup \{L^1(\mu): \mu \in Y\} \cup M_a(G)$ but not $M_0(G)$. Here $L^1(\mu)$ denotes the set of the measures in $M(G)$ which are absolutely continuous with respect to $|\mu|$.**

Throughout the paper, let $G$ be a nondiscrete locally compact abelian group, $\hat{G}$ its dual, and $M(G)$ the convolution algebra of bounded regular measures on $G$. We use the following customary notations:

$$L^1(G) = M_a(G) \subset M_0(G) \subset M_c(G) \subset M(G) .$$

Here $M_0(G)$ denotes the closed ideal of those measures in $M(G)$ whose Fourier transforms vanish at infinity. For the definitions of $M_a(G)$ and $M_c(G)$, see [3: (19.13)]; for the second inclusion, see [8: 5.6.9] or [4]. Given a measure $\mu \in M(G)$, we denote by $L^1(\mu)$ the set of those measures in $M(G)$ which are absolutely continuous with respect to $|\mu|$. For a set $K$ in $G$, define

$$(K)_1 = K \cup (-K) \quad \text{and} \quad (K)_n = (K)_{n-1} + (K)_1 \qquad (n = 2, 3, \cdots) .$$

Thus, the union of all $(K)_n$, denoted by $G_p(K)$, is the subgroup of $G$ generated by $K$.

Our main results are the following.

THEOREM 1. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a countable subset of $M_c^+(G)$, $0 \neq \lambda_0 \in M_0(G)$, and $\{C_k\}_{k=0}^{\infty}$ a countable family of nonempty $\sigma$-compact subsets of $G$ such that*

(a) $$\lambda_k(x + C_k) = 0 \qquad (x \in G;\ k = 0, 1, 2, \cdots) .$$

*Then there exists a nonzero measure $\sigma \in M_0^+(\text{supp}\,\lambda_0)$ with compact support such that*

(b) $\lambda_k[x + C_k + G_p(\text{supp}\,\sigma)] = 0 \qquad (x \in G; k = 0, 1, 2, \cdots)$.

*If, in addition, G is metrizable, such a measure σ can be taken so that* $(\text{supp}\,\sigma) - x_0$ *is independent for some* $x_0 \in G$.

COROLLARY. *Let* $\{\lambda_k\}_{k=0}^{\infty}$ *be a countable subset of* $M_c(G)$, $0 \neq \lambda_0 \in M_0(G)$, *and Y the closed ideal in M(G) which is generated by* $\bigcup \{L^1(\lambda_k): k = 0, 1, 2, \cdots\}$. *Then there exists a nonzero measure* $\sigma \in M_0^+(\text{supp}\,\lambda_0)$ *with compact support such that*

$$|\mu|\,[x + G_p(\text{supp}\,\sigma)] = 0 \qquad (x \in G; \mu \in Y)\,.$$

*If, in addition, G is metrizable, such a measure σ can be taken so that* $(\text{supp}\,\sigma) - x_0$ *is independent for some* $x_0 \in G$.

THEOREM 2. *Let* $\{\lambda_k\}_{k=0}^{\infty}$ *and Y be as in the Corollary. Then there exists a symmetric maximal ideal Θ in M(G) such that*

$$\bigcup \{L^1(\mu): \mu \in Y\} \cup M_a(G) \subset \Theta \qquad \text{but } M_0(G) \not\subset \Theta\,.$$

*Furthermore, the set of all Θ's with these properties has cardinal number larger than or equal to* $\max\{2^{\omega}, \text{Card}\,\hat{G}\}$. *Here ω denotes the smallest uncountable cardinal.*

Theorem 1 improves the main result in [6] and its Corollary generalizes a theorem of Rudin [7] (see N. Th. Varopoulos [9] in this connection). The idea of our proof is due to T. W. Körner [5: Ch. XIII]. Although the arguments needed are similar to those in [6], we give a detailed proof of Theorem 1.

We need some lemmas.

LEMMA 1. *Let λ be a measure in* $M_c^+(G)$, *and D a compact subset of G such that* $\lambda(x + D) = 0$ *for all* $x \in G$. *Then, for each finite set F in G,* $n \in N$ (the natural numbers) *and* $\varepsilon > 0$, *there exists a neighborhood V of* $0 \in G$ *such that*

$$\lambda[x + D + (F + V)_n] < \varepsilon \qquad (x \in G)\,.$$

*Proof.* Let $F$, $n$, and $\varepsilon$ be as above. Take a compact set $K$ in $G$ so that $\lambda(G\backslash K) < \varepsilon$, and fix any neighborhood $V_0$ of 0 with compact closure. Since $F$ is finite, $\lambda[x + D + (F)_n] = 0$ for all $x \in G$ by hypothesis. Thus, for each $x \in G$, we can find a neighborhood $W_x$ of 0 so that

$$\lambda[x + D + (F)_n + (W_x)_{2n}] < \varepsilon\,.$$

(Note that $D + (F)_n$ is compact.) It follows from compactness of the set $K - [D + (F)_n + (\bar{V}_0)_n]$ that there exist finitely many points $x_1, x_2, \cdots, x_m \in G$ such that

$$K - [D + (F)_n + (V_0)_n] \subset \bigcup_{j=1}^{m} (x_j + W_{x_j}) .$$

Put $V = \bigcap_{j=1}^{m} W_{x_j} \cap V_0$. If $x \in K - [D + (F)_n + (V_0)_n]$, then $x \in x_j + W_{x_j}$ for some $j = j(x)$, and so

$$\begin{aligned}\lambda[x + D + (F + V)_n] &\leqq \lambda[x_j + W_{x_j} + D + (F)_n + (V)_n] \\ &\leqq \lambda[x_j + D + (F)_n + (W_{x_j})_{2n}] < \varepsilon .\end{aligned}$$

If $x \notin K - [D + (F)_n + (V_0)_n]$, then

$$[x + D + (F + V)_n] \cap K \subset [x + D + (F)_n + (V_0)_n] \cap K = \varnothing ,$$

and so $\lambda[x + D + (F + V)_n] \leqq \lambda(G \backslash K) < \varepsilon$. This completes the proof.

LEMMA 2. *Suppose that $G$ is metrizable and $\lambda_0$ a nonzero measure in $M_c^+(G)$. Then there exists a point $x_0 \in G$ and a nonempty, totally disconnected, compact, perfect subset $K_0$ of* supp $\lambda_0$ *with the following three properties.*

(a) *Every nonempty* (relatively) *open subset of $K_0$ has positive $\lambda_0$-measure.*

(b) *The elements of $K_0 - x_0$ have the same order, say $q_0$.*

(c) *If $V_1, V_2, \cdots, V_m$ are $m$ disjoint, nonempty, open subsets of $K_0$, there exist $m$ points $x_j \in V_j$ such that $x_1 - x_0, x_2 - x_0, \cdots, x_m - x_0$ are independent.*

*Proof.* Since $G$ is metrizable and $\lambda_0$ is continuous, we may assume that $\lambda_0$ is carried by a totally disconnected compact set.

Suppose first that there exist a natural number $q$ and an element $y \in G$ such that

$$E(q, y) = \{x \in \operatorname{supp} \lambda_0 : qx = y\}$$

has positive $\lambda_0$-measure. Let $q_0$ be the smallest natural number such that $\lambda_0[E(q_0, y_0)] > 0$ for some $y_0 \in G$. Fix any element $x_0 \in E(q_0, y_0)$. Then $\lambda_0[E(q, qx_0)] = 0$ for all $q \in N$ with $1 \leqq q < q_0$, so that there exists a compact subset $K_0$ of $E(q_0, y_0) \backslash \{\bigcup_{q=1}^{q_0-1} E(q, qx_0)\}$ with $\lambda_0(K_0) > 0$. Replacing $K_0$ by the support of $\lambda_0 | K_0$, we may assume that $K_0$ is perfect and satisfies (a). Evidently (b) holds. Suppose now that (c) holds for some $m \in N$ (note that (c) is trivial for $m = 1$). Let $V_1, \cdots, V_m, V_{m+1}$ be $m + 1$ disjoint, nonempty, open subsets of $K_0$. There are $m$ points $x_1 \in V_1, \cdots, x_m \in V_m$ such that $x_1 - x_0, \cdots, x_m - x_0$ are independent. Let $H$ be the subgroup of $G$ which is generated by $x_0, x_1, \cdots, x_m$. By minimality of $q_0$, we have $\lambda_0[E(q, y)] = 0$ for all $q \in N$

with $1 \leqq q < q_0$ and all $y \in H$. Since $\lambda_0(V_{m+1}) > 0$ by (a) and $H$ is at most countable, we can find an element $x_{m+1} \in V_{m+1}$ so that

$$x_{m+1} \notin E(q, y) \qquad (q = 1, 2, \cdots, q_0 - 1; y \in H) .$$

It is now easy to prove that the elements $x_1 - x_0, \cdots, x_m - x_0, x_{m+1} - x_0$ are independent. By induction on $m$, we obtain (c).

Suppose next that $\lambda_0[E(q, y)] = 0$ for all $q \in N$ and all $y \in G$. Then $F = \{x \in \operatorname{supp} \lambda_0 : \operatorname{ord} x < \infty\}$ has $\lambda_0$-measure zero, so that there exists a nonempty compact, perfect subset $K_0$ of $(\operatorname{supp} \lambda_0)\backslash F$ which satisfies (a). It is now easy to prove that (b) and (c) hold for $x_0 = 0$.

LEMMA 3. *Let $K$ be a totally disconnected, compact subset of $G$, and $\sigma$ a nonzero measure in $M_c^+(G)$ with* $\operatorname{supp} \sigma = K$. *Then, for each compact subset $\hat{F}$ of $\hat{G}$ and $\varepsilon > 0$, there exists a finite partition $\{K_j\}_{j=1}^n$ of $K$ into disjoint clopen subsets such that:*

(i) $0 < \sigma(K_j) < \varepsilon \qquad (j = 1, 2, \cdots, n)$;

(ii) $\left|\sum_{j=1}^n \sigma(K_j)\hat{\nu}_j(\chi) - \hat{\sigma}(\chi)\right| < \varepsilon \qquad (\chi \in \hat{F})$

*whenever $\nu_j \in M^+(K_j)$ and $\|\nu_j\|_M = 1$ for all $j = 1, 2, \cdots, n$.*

*Proof.* Since $\hat{F}$ is compact while $K$ is totally disconnected and compact, there is a finite partition $\{K_j\}_{j=1}^n$ of $K$ into disjoint clopen subsets which satisfies (i) and

$$\sup\{|\chi(x) - \chi(y)| : x, y \in K_j\} < (3\|\sigma\|_M)^{-1}\varepsilon$$

for all $\chi \in \hat{F}$ and all $j = 1, 2, \cdots, n$. If $\nu_j \in M^+(K_j)$ and $\|\nu_j\|_M = 1$ for $j = 1, 2, \cdots, n$, then we have

$$|\sigma(K_j)\hat{\nu}_j(\chi) - \widehat{\sigma | K_j}(\chi)| < \|\sigma\|_M^{-1}\sigma(K_j)\varepsilon \qquad (\chi \in \hat{F}) .$$

To see this, take any $x_j \in K_j$. Then $\chi \in \hat{F}$ implies

$$\begin{aligned} &|\sigma(K_j)\hat{\nu}_j(\chi) - \widehat{\sigma | K_j}(\chi)| \\ &= \left|\sigma(K_j)\int_{K_j} \bar{\chi} d\nu_j - \int_{K_j} \bar{\chi} d\sigma\right| \\ &\leqq \sigma(K_j)\int_{K_j} |\bar{\chi} - \bar{\chi}(x_j)| d\nu_j + \int_{K_j} |\bar{\chi} - \bar{\chi}(x_j)| d\sigma \\ &\leqq 2\sigma(K_j)(3\|\sigma\|_M)^{-1}\varepsilon < \|\sigma\|_M^{-1}\sigma(K_j)\varepsilon . \end{aligned}$$

Adding these inequalities for all $j$'s, we obtain (ii).

To prove the following lemma, we need a definition. Let $K$ be a subset of $G$ whose elements have the same order $q_0 (2 \leqq q_0 \leqq \infty)$. Let also $L_1, L_2, \cdots, L_n$ be finitely many subsets of $K$, and $M$ any natural number. We say that $L_1, L_2, \cdots, L_n$ are $M$-independent if and only if $\sum_{j=1}^n m_j x_j \neq 0$ whenever $m_j \in Z$ (the integers), $|m_j| < q_0$,

$x_j \in L_j$ $(j = 1, 2, \cdots, n)$ and $0 \neq \sum_{j=1}^{k} |m_j| < M$.

LEMMA 4. *Suppose that $G$ is metrizable, and that $\{\lambda_k\}_{k=0}^{\infty}$ and $\{C_k\}_{k=0}^{\infty}$ are as in Theorem 1. Let also $x_0 \in G$ and $K_0 \subset \operatorname{supp} \lambda_0$ be as in Lemma 2. Then there exists a nonzero measure $\sigma \in M_0^+(K_0)$ such that* $(\operatorname{supp} \sigma) - x_0$ *is independent and*

$$(\mathrm{P}_1) \qquad \lambda_k[x + C_k + (\operatorname{supp} \sigma)_1] = 0 \qquad (x \in G; k = 0, 1, 2, \cdots) .$$

*Proof.* Write

$$\hat{G} = \bigcup_{n=1}^{\infty} \hat{E}_n \quad \text{and} \quad C_k = \bigcup_{n=1}^{\infty} C_{kn} \qquad (k = 0, 1, 2, \cdots) ,$$

where the $\hat{E}_n$ are compact subsets of $\hat{G}$ while the $C_{kn}$ are compact subsets of $G$ such that $C_{kn} \subset C_{k(n+1)}$ for all $k$ and $n$. (It is well-known that $G$ is metrizable if and only if $\hat{G}$ is $\sigma$-compact. See, for example, [4].) Let $\lambda$ be the measure in $M(G)$ defined by $\lambda(E) = \lambda_0[(E + x_0) \cap K_0]$ for all Borel subsets $E$ of $G$. Then, $0 \neq \lambda \in M_0^+(G)$ and the elements in $\operatorname{supp} \lambda = K_0 - x_0$ have the same order $q_0$.

We shall now construct a sequence $(n_p)_{p=1}^{\infty}$ of natural numbers, a sequence $(\mathscr{I}_n)_{n=1}^{\infty}$ of finite collections of disjoint clopen subsets of $K_0 - x_0$, a sequence $(\sigma_n)_{n=1}^{\infty}$ of probability measures in $L^1(\lambda)$, and a sequence $(\hat{F}_n)_{n=1}^{\infty}$ of compact subsets of $\hat{G}$. They will satisfy the following three conditions. Every $\sigma_n$ has the form

$$(\mathrm{i}) \qquad \sigma_n = \sum_{I \in \mathscr{I}_n} a_I \lambda_I$$

where each $a_I$ is a positive real number, $\lambda_I = \lambda \mid I$ the restriction of $\lambda$ to $I$, and

$$\|\sigma_n\|_M = \sum_{I \in \mathscr{I}_n} a_I \lambda(I) = 1 .$$

$$(\mathrm{ii}) \qquad \sup \{|\widehat{\sigma_n \mid I}(\chi)| : \chi \in \hat{G} \backslash \hat{F}_n\} < 2^{-n} \sigma_n(I) \qquad \forall I \in \mathscr{I}_n .$$

$$(\mathrm{iii}) \qquad \hat{E}_n \subset \hat{F}_n .$$

For $n = 1$, such $\mathscr{I}_1$, $\sigma_1$, and $\hat{F}_1$ may be quite arbitrary. We set $n_1 = 1$, and suppose that $n_p$, $\mathscr{I}_{n_p}$, $\sigma_{n_p}$, and $\hat{F}_{n_p}$ have been constructed for some $p \in N$. Let $l_p = \operatorname{Card} \mathscr{I}_{n_p}$, and write

$$\mathscr{I}_{n_p} = \{I_i\}_{i=1}^{l_p} = \{I_i^p\}_{i=1}^{l_p} .$$

Let $M_p$ be the largest natural number such that

$$(1) \qquad \max \{\sigma_{n_p}(I) : I \in \mathscr{I}_{n_p}\} \leqq M_p^{-2} ,$$

and set

$$(2) \qquad T_p = \{A \subset \mathscr{I}_{n_p} : 1 \leqq \operatorname{Card} A \leqq M_p\} = \{A_r\}_{r=1}^{s_p} .$$

We may assume

$$(3) \qquad A_r = \{I_r\} = \{I_r^p\} \qquad (r = 1, 2, \cdots, l_p) .$$

We shall inductively construct the $\mathscr{I}_n$, $\sigma_n$, and $\hat{F}_n$ for all $n \in N$ with $n_p < n \leqq n_p + s_p$ as follows. Suppose that $\mathscr{I}_n$, $\sigma_n$ and $\hat{F}_n$ have been constructed for some $n = n_p + r - 1$ ($r = 1, 2, \cdots, s_p$), and put

$$(4) \qquad \mathscr{K}_n = \{I \in \mathscr{I}_n : I \subset J \text{ for some } J \in A_r\} .$$

We can find (finite) collections $\{b_j^K\}_j$ of real numbers and collections $\{L_j^K\}_j$ of disjoint clopen subsets of $K \in \mathscr{K}_n$ which satisfy the following six conditions:

$$(5) \qquad 0 < b_j^K \sigma_n(L_j^K) < 2^{-1}\sigma_n(K) \qquad \forall K \in \mathscr{K}_n \text{ and } \forall_j ;$$

$$(6) \qquad \sum_j b_j^K \sigma_n(L_j^K) = \sigma_n(K) \qquad \forall K \in \mathscr{K}_n ;$$

$$(7) \qquad \left| \sum_j b_j^K \widehat{\sigma_n \mid L_j^K}(\chi) - \widehat{\sigma_n \mid K}(\chi) \right| < 2^{-n}\sigma_n(K) \quad \forall K \in \mathscr{K}_n \text{ and } \forall \chi \in \hat{F}_n ;$$

$$(8) \qquad \sum_j \operatorname{dia}(L_j^K) < n^{-1} \qquad \forall K \in \mathscr{K}_n ;$$

(9) The sets $\{L_j^K\}_{K,j}$ are $M_p$-independent ;

$$(10) \qquad \sup_{x \in G} \lambda_k \left[ x + C_{kn} + \left( \bigcup_j L_j^K \right)_1 \right] < (nl_p)^{-1} \qquad \forall K \in \mathscr{K}_n \text{ and } \forall k = 0, 1, \cdots, n .$$

The above conditions are met as follows: For each $K \in \mathscr{K}_n$, apply Lemma 3 to $\sigma = \sigma_n \mid K$, $\varepsilon = 2^{-n}\sigma_n(K)$ and $\hat{F} = \hat{F}_n$. Let $\{K_j\}_j$ be a finite partition of $K$ as in Lemma 3. Using property (c) in Lemma 2, we can find $x_j^K \in K_j$ so that $\bigcup \{\{x_j^K\}_j : K \in \mathscr{K}_n\}$ is independent. If we choose $L_j^K \subset K_j$ so that $x_j^K \in L_j^K$ and the diameter of each $L_j^K$ is "sufficiently small", then (8) and (9) hold and so does (10) by Lemma 1. Finally, it suffices to set $b_j^K = \sigma_n(K_j)/\sigma_n(L_j^K)$.

We now define

$$(11) \qquad \mathscr{L}_n = \mathscr{I}_n \backslash \mathscr{K}_n,\ \mathscr{I}_{n+1} = \mathscr{L}_n \cup \left( \bigcup_{K \in \mathscr{K}_n} \{L_j^K\}_j \right);$$

$$(12) \qquad \theta_{n+1} = \sum_{I \in \mathscr{L}_n} a_I \lambda_I + \sum_{K \in \mathscr{K}_n} \sum_j b_j^K \sigma_n \mid L_j^K ,$$

and take a compact subset $\hat{F}_{n+1}$ of $\hat{G}$, with $\hat{F}_{n+1} \supset \hat{E}_{n+1} \cup \hat{F}_n$, so that (ii) holds with $n$ replaced by $n + 1$.

We repeat the above process with $n_p$ replaced by $n_{p+1} = n_p + s_p$, which completes our induction. Let $\sigma_\infty$ be a weak-* cluster point of

$(\sigma_n)_{n=1}^{\infty}$ in $M(G)$, and $\sigma$ the measure in $M(G)$ defined by $\sigma(E) = \sigma_\infty(E - x_0)$ for all Borel sets $E$ in $G$. We claim that $\sigma$ has the required properties.

First note that

$$\bigcup_{I \in \mathscr{I}_{n+1}} I \subset \bigcup_{I \in \mathscr{I}_n} I = \operatorname{supp} \sigma_n \subset K_0 - x_0 ,$$

and so we have

$$\text{(13)} \qquad \sigma_\infty \geqq 0 , \quad \sigma_\infty(G) = 1 \quad \text{and} \quad \operatorname{supp} \sigma_\infty \subset \bigcap_{n=1}^{\infty} \Big( \bigcup_{I \in \mathscr{I}_n} I \Big) .$$

Let $p \in N$ be given. It is easily seen from (3), (4), and (11) that

$$\mathscr{I}_{n_p + l_p} = \{L_j^{I_1}\}_j \cup \{L_j^{I_2}\}_j \cup \cdots \cup \{L_j^{I_l}\}_j , \quad \text{where } l = l_p .$$

This, combined with (10) and (13), shows

$$\begin{aligned} &\lambda_k[x + C_{kn_p} + (\operatorname{supp} \sigma_\infty)_1] \\ &\qquad \leqq \sum_{i=1}^{l_p} \lambda_k\Big[ x + C_{kn_p} + \Big( \bigcup_j L_j^{I_i} \Big)_1 \Big] \leqq l_p \cdot (n_p l_p)^{-1} = n_p^{-1} \end{aligned}$$

for all $x \in G$ and all $k = 0, 1, \cdots, n_p$. (Note that $C_{kn} \subset C_{k(n+1)}$ for all $k$ and $n$.) Thus, fixing $x \in G$ and $k \in \{0, 1, 2, \cdots\}$, and letting $p \to \infty$, we have

$$\lambda_k[x + C_k + (\operatorname{supp} \sigma_\infty)_1] = 0 \qquad (x \in G; k = 0, 1, 2, \cdots) .$$

But evidently $\operatorname{supp} \sigma = (\operatorname{supp} \sigma_\infty) + x_0$, and so $(P_1)$ holds.

It remains to show that $\hat{\sigma}$ vanishes at infinity and that $(\operatorname{supp} \sigma) - x_0$ is independent. Although these are proved in [5: Ch. XIII, 151–153 and 155–156], we give their proofs to make the paper self-contained.

Suppose $n_p \leqq n < n_{p+1}$ $(p, n \in N)$, and write $n = n_p + r - 1$ $(r = 1, 2, \cdots, s_p)$. Then we have

$$\text{(14)} \qquad \sum_{K \in \mathscr{K}_n} \sigma_n(K) = \sum_{J \in A_r} \sigma_{n_p}(J) \leqq (\operatorname{Card} A_r) \cdot \max_{J \in A_r} \sigma_{n_p}(J) \leqq M_p^{-1} .$$

Here the equality follows from (4), (6) and (12) while the last inequality follows from (1) and (2). If $\chi \in \hat{F}_n$, then

$$\begin{aligned} | \hat{\sigma}_{n+1}(\chi) - \hat{\sigma}_n(\chi) | &\leqq \sum_{K \in \mathscr{K}_n} \Big| \sum_j b_j^K \widehat{\sigma_n | L_j^K}(\chi) - \widehat{\sigma_n | K}(\chi) \Big| \\ &< \sum_{K \in \mathscr{K}_n} 2^{-n} \sigma_n(K) \leqq 2^{-n} \end{aligned}$$

by (i), (12) and (7). It follows that

$$| \hat{\sigma}_\infty(\chi) - \hat{\sigma}_{n+1}(\chi) | < 2^{-n} \qquad \forall \chi \in \hat{F}_{n+1} ,$$

since $\hat{F}_n \subset \hat{F}_{n+1} \subset \cdots$ by construction. For $\chi \in \hat{G} \backslash \hat{F}_n$, we have

$$\begin{aligned} |\hat{\sigma}_{n+1}(\chi)| &\leqq \sum_{I\in\mathscr{I}_n} |a_I\hat{\lambda}_I(\chi)| + \sum_{K\in\mathscr{K}_n}\sum_j b_j^K\sigma_n(L_j^K) \\ &\leqq \sum_{I\in\mathscr{I}_n} |\widehat{\sigma_n\,|\,I}(\chi)| + \sum_{K\in\mathscr{K}_n}\sigma_n(K) \\ &\leqq 2^{-n}\sum_{I\in\mathscr{I}_n}\sigma_n(I) + M_p^{-1} = 2^{-n} + M_p^{-1} \end{aligned}$$

by (12), (i), (6), (ii) and (14). Hence

$$|\hat{\sigma}_\infty(\chi)| \leqq 2^{-n} + 2^{-n} + M_p^{-1} \qquad \forall\,\chi\in\hat{F}_{n+1}\backslash\hat{F}_n .$$

But $\hat{G} = \bigcup_{n=1}^\infty \hat{F}_n$ by (iii) and $\lim_p M_p = \infty$ by construction. Thus the above inequality shows that $\hat{\sigma}_\infty \in C_0(\hat{G})$, or equivalently, that $\hat{\sigma}\in C_0(\hat{G})$.

Finally we prove that $(\operatorname{supp}\sigma) - x_0 = \operatorname{supp}\sigma_\infty$ is independent. Let $x_1, x_2, \cdots, x_t$ be distinct elements of $\operatorname{supp}\sigma_\infty$. It is easy to see that

$$\max_{I\in\mathscr{I}_n} \operatorname{dia}(I) \longrightarrow 0 \quad\text{as}\quad n\longrightarrow\infty .$$

Therefore, there is an $n_0\in N$ such that $x_1, x_2, \cdots, x_t$ belong to distinct sets in $\mathscr{I}_n$ whenever $n\geqq n_0$. Take any $p\in N$ so that $n_p\geqq n_0$ and $M_p > t$, and let

$$A = \{I\in\mathscr{I}_{n_p} : I \text{ contains some } x_n\} .$$

Then $1\leqq \operatorname{Card} A = t < M_p$; hence $A = A_r$ for some $r = 1, 2, \cdots, s_p$. Thus $x_1, x_2, \cdots, x_t$ belong to distinct sets in $\bigcup\{\{L_j^K\}_j : K\in\mathscr{K}_n\}$, where $n = n_p + r - 1$. It follows from (9) that $x_1, x_2, \cdots, x_t$ are $M_p$-independent. Since $p$ can be taken as large as one pleases, we conclude that $x_1, x_2, \cdots, x_t$ are independent.

This establishes Lemma 4.

LEMMA 5. *Let $G$, $\{\lambda_k\}_{k=0}^\infty$ and $\{C_k\}_{k=0}^\infty$ be as in Lemma 4. Let also $\{K_j\}_{j=1}^m$ be finitely many, disjoint, compact subsets of $G$ such that $M_0(K_j)\neq 0$ for all $j = 1, 2, \cdots, m$. Then, for each $n\in N$, there exist $m$ nonzero measures $\mu_j\in M_0^+(K_j)$ such that*

$$(\mathrm{P}_n)\qquad \lambda_k\Big[x + C_k + \Big(\bigcup_{j=1}^m \operatorname{supp}\mu_j\Big)_n\Big] = 0 \qquad (x\in G;\ k = 0, 1, 2, \cdots) .$$

*Proof.* For each $j = 1, 2, \cdots, m$, choose and fix a measure $\tau_j\in M_0^+(K_j)$ with $\|\tau_j\|_M = m^{-1}$ whose support is totally disconnected. In the proof of Lemma 4, replace $\lambda$ and $K_0 - x_0$ by $\lambda' = \sum_{j=1}^m \tau_j$ and $\bigcup_{j=1}^m \operatorname{supp}\tau_j$, respectively; take off condition (9); and let $\sigma_\infty$ be any measure constructed as there with $\sigma_1 = \lambda'$ and $\mathscr{I}_1 = \{\operatorname{supp}\tau_j\}_{j=1}^m$. Then

$$0\neq \mu_j = \sigma_\infty\,|\,K_j \in M_0^+(K_j) \qquad (j = 1, 2, \cdots, m) ,$$

and $\{\mu_j\}_{j=1}^m$ satisfy $(\mathrm{P}_1)$. We repeat the same argument replacing

$\{C_k\}_{k=0}^{\infty}$ and $\{K_j\}_{j=1}^{m}$ by $\{C_k + (\bigcup_{j=1}^{m} \operatorname{supp} \mu_j)_1\}_{k=0}^{\infty}$ and $\{\operatorname{supp} \mu_j\}_{j=1}^{m}$, respectively; and continue this process. At the $n$th step, we will obtain $m$ nonzero measures satisfying the required condition. This completes the proof.

*Proof of Theorem* 1 *for metrizable groups.* Suppose that $G$ is a given metrizable group. Let $\{\lambda_k\}_{k=0}^{\infty}$, $\{C_k\}_{k=0}^{\infty}$, $x_0$, and $K_0$ be as in Lemma 4. Let also $\hat{G} = \bigcup_{n=1}^{\infty} \hat{E}_n$ be as in the proof of Lemma 4. We construct a sequence $(\mathscr{I}_n)_{n=1}^{\infty}$ of finite collections of disjoint compact subsets of $K_0$, a sequence $(\sigma_n)_{n=1}^{\infty}$ of probability measures in $M_0(K_0)$, and a consequence $(\hat{F}_n)_{n=1}^{\infty}$ of compact subsets of $\hat{G}$. They satisfy the following four conditions. Every $\sigma_n$ has the form

$$\text{(i)} \qquad \sigma_n = \sum_{I \in \mathscr{I}_n} a_I \mu_I ,$$

where each $a_I$ is a real positive number, $\mu_I$ a probability measure in $M_0(I)$ with $\operatorname{supp} \mu_I = I$, and

$$\|\sigma_n\|_M = \sum_{I \in \mathscr{I}_n} a_I = 1 .$$

$$\text{(ii)} \qquad \sup \{|\hat{\mu}_I(\chi)| : \chi \in \hat{G} \backslash \hat{F}_n\} < 2^{-n} a_I \qquad \forall I \in \mathscr{I}_n .$$

$$\text{(iii)} \qquad \hat{E}_n \subset \hat{F}_n .$$

$$\text{(iv)} \qquad \lambda_k[x + C_k + (K_1)_1 + (K_2)_2 + \cdots + (K_n)_n] = 0 \quad (x \in G;\ k = 0, 1, 2, \cdots) ,$$

where $K_1 = \bigcup \{I : I \in \mathscr{I}_1\}$ and $K_n = \bigcup \{I \in \mathscr{I}_n : I \notin \mathscr{I}_{n-1}\}$ for $n = 2, 3, \cdots$.

For $n = 1$, we apply Lemma 4 to obtain a probability measure $\sigma_1 \in M_0(K_0)$ such that $(\operatorname{supp} \sigma_1) - x_0$ is independent and

$$\lambda_k[x + C_k + (\operatorname{supp} \sigma_1)_1] = 0 \qquad (x \in G;\ k = 0, 1, 2, \cdots) .$$

Set $\mathscr{I}_1 = \{I = \operatorname{supp} \sigma_1\}$, $\mu_I = \sigma_1$, $a_I = 1$, and take any compact subset $\hat{F}_1$ of $\hat{G}$ satisfying (ii) and (iii) for $n = 1$.

Suppose that $(\mathscr{I}_i)_{i=1}^{n}$, $(\sigma_i)_{i=1}^{n}$ and $(\hat{F}_i)_{i=1}^{n}$ have been constructed for some $n \in \boldsymbol{N}$. Choose and fix any $I_n \in \mathscr{I}_n$ with

$$(1) \qquad a_{I_n} = \sup \{a_I : I \in \mathscr{I}_n\} .$$

Applying Lemmas 3 and 5, we can find a (finite) collection $\{b_{nj}\}_j$ of real numbers, a collection $\{L_{nj}\}_j$ of disjoint compact subsets of $I_n$, and a collection $\{\mu_{nj}\}_j$ of probability measures in $M_0(I_n)$ with $\operatorname{supp} \mu_{nj} = L_{nj}$ which satisfy the following four conditions.

$$(2) \qquad 0 < b_{nj} < n^{-1} \cdot \min_{I \in \mathscr{I}_n} a_I \qquad \forall j .$$

(3) $$\sum_j b_{nj} = a_{I_n}$$

(4) $$\left|\sum_j b_{nj}\hat{\mu}_{nj}(\chi) - a_{I_n}\hat{\mu}_{I_n}(\chi)\right| < 2^{-n} \qquad \forall \chi \in \hat{F}_n .$$

(5) $$\lambda_k\left[x + C_k + (K_1)_1 + \cdots + (K_n)_n + \left(\bigcup_j L_{nj}\right)_{n+1}\right] = 0 \qquad \forall x \in G$$

for all $k = 0, 1, 2, \cdots$. Put $\mathscr{I}_{n+1} = (\mathscr{I}_n \backslash \{I_n\}) \cup \{L_{nj}\}_j$, and $a_I = b_{nj}$, $\mu_I = \mu_{nj}$ for $I = L_{nj}$ $\forall j$. Define $\sigma_{n+1}$ by the right-hand side of (i) with $n$ replaced by $n+1$. Finally, we take any compact subset $\hat{F}_{n+1}$ of $\hat{G}$, with $\hat{F}_{n+1} \supset \hat{E}_{n+1} \cup \hat{F}_n$, so that (ii) holds with $n$ replaced by $n+1$.

This completes the induction. Let $\sigma$ be a weak-* cluster point of $(\sigma_n)_{n=1}^{\infty}$ in $M(G)$. Then it is easy to prove that $\sigma$ has all the required properties (see [5: Ch. XIII, 151–153]). This establishes Theorem 1 for metrizable groups.

To prove the general case, we need one more lemma.

LEMMA 6. *Let* $\{\lambda_k\}_{k=0}^{\infty}$ *and* $\{C_k\}_{k=0}^{\infty}$ *be as in Theorem* 1. *Then, given a* $\sigma$*-compact subset* $\hat{F}$ *of* $\hat{G}$, *we can find a* $\sigma$*-compact, noncompact, open subgroup* $\Gamma$ *of* $\hat{G}$ *so that* $\hat{F} \subset \Gamma$ *and*

(i) $$\lambda_k[x + C_k + H_\Gamma] = 0 \qquad (x \in G; k = 0, 1, 2, \cdots) ,$$

*where* $H_\Gamma$ *denotes the annihilator of* $\Gamma$.

*Proof.* Let $C_{kn}$ be as in the proof of Lemma 4, and let $\mathscr{F}$ be the family of all $\sigma$-compact, noncompact, open subgroups of $\hat{G}$ which contain $\hat{F}$. Since every $C_{kn}$ is compact, we have

(1) $$C_{kn} = \bigcap \{C_{kn} + H_\Gamma : \Gamma \in \mathscr{F}\} \qquad (k = 0, 1, 2, \cdots; n = 1, 2, \cdots) .$$

Applying Lemma 1, we can find neighborhoods $V_{kn}$ of 0 so that

(2) $$\lambda_k[x + C_{kn} + V_{kn}] < n^{-1} \qquad (k = 0, 1, 2, \cdots; n = 1, 2, \cdots : x \in G) .$$

By (1), there exist subgroups $\Gamma_{kn}$ in $\mathscr{F}$ such that

(3) $$C_{kn} + H_{kn} \subset C_{kn} + V_{kn} \qquad (k = 0, 1, 2, \cdots; n = 1, 2, \cdots) ,$$

where $H_{kn}$ is the annihilator of $\Gamma_{kn}$. Let $\Gamma$ be any subgroup in $\mathscr{F}$ which contains all $\Gamma_{kn}$. Then, it follows from (2) and (3) that (i) holds. This completes the proof.

*Proof of Theorem* 1 *for general groups*. Let $G$ be an arbitrary nondiscrete LCA group, and let $\{\lambda_k\}_{k=0}^{\infty}$ and $\{C_k\}_{k=0}^{\infty}$ be as in Theorem 1. For $\hat{F} = \{\chi \in \hat{G} : \hat{\lambda}_0(\chi) \neq 0\}$, take a $\Gamma \subset G$ as in Lemma 6. Setting

$H = H_\Gamma$, we denote by $\pi$ and $m_H$ the natural mapping of $G$ onto $G_0 = G/H$ and the Haar measure of $H$ with $m_H(H) = 1$, respectively. For each $\mu \in M(G)$, define a measure $\mu' \in M(G_0)$ by setting

$$(1) \qquad \int_{G_0} f d\mu' = \int_G f \circ \pi d\mu \qquad \forall f \in C_0(G_0) .$$

Identifying $\Gamma$ with $\hat{G}_0$ in the usual way, we see $\hat{\mu}' = \hat{\mu} \mid \Gamma$ for all $\mu \in M(G)$, so that $0 \neq \lambda_0' \in M_0(G_0)$. On the other hand, we have

$$(2) \qquad \lambda_k'[x' + C_k'] = 0 \qquad (x' \in G_0;\ k = 0, 1, 2, \cdots)$$

by (i) in Lemma 6 and (1), where $C_k' = \pi(C_k)$. Therefore $\{\lambda_k'\}_{k=0}^\infty \subset M_c^+(G)$, and we can apply our result for metrizable groups to find a nonzero measure $\sigma' \in M_0^+(\operatorname{supp} \lambda_0')$ with compact support such that

$$(3) \quad \lambda_k'[x' + C_k' + G_p(\operatorname{supp} \sigma')] = 0 \qquad (x' \in G_0;\ k = 0, 1, 2, \cdots) .$$

Now define a measure $\sigma \in M(G)$ by setting

$$(4) \qquad \int_G f d\sigma = \int_{G_0} \left\{ \int_H f(x + t) dm_H(t) \right\} d\sigma'(x') \qquad \forall f \in C_0(G) .$$

As is easily seen, we then have

$$(5) \qquad \operatorname{supp} \sigma = \pi^{-1}[\operatorname{supp} \sigma'] \quad \text{and} \quad \operatorname{supp} \lambda_0 = \pi^{-1}[\operatorname{supp} \lambda_0']$$

(note that $\sigma * m_H = \sigma$ and $\lambda_0 * m_H = \lambda_0$). It is also easy to check that $0 \neq \sigma \in M_0^+(G)$, that $\operatorname{supp} \sigma$ is a compact subset of $\operatorname{supp} \lambda_0$, and that

$$\lambda_k[x + C_k + G_p(\operatorname{supp} \sigma)] = \lambda_k'[x' + C_k' + G_p(\operatorname{supp} \sigma')] = 0$$

for all $x \in G$ and all $k = 0, 1, 2, \cdots$.

This establishes Theorem 1.

*Proof of Corollary.* Let $Y$ be as in the present Corollary. Setting $C_k = \{0\}$ for all $k$ and applying Theorem 1 to $\{|\lambda_k|\}_{k=0}^\infty$, we obtain a nonzero measure $\sigma \in M_0^+(\operatorname{supp} \lambda_0)$ with compact support such that

$$(1) \qquad |\tau| [x + G_p(\operatorname{supp} \sigma)] = 0 \qquad (x \in G)$$

holds for all $\tau \in \bigcup_{k=0}^\infty L^1(\lambda_k)$. But then we have

$$\begin{aligned} |\nu * \tau| [x + G_p(\operatorname{supp} \sigma)] &\leqq (|\nu| * |\tau|)[x + G_p(\operatorname{supp} \sigma)] \\ &\leqq \int_G |\tau| [x - y + G_p(\operatorname{supp} \sigma)] d |\nu| (y) = 0 \end{aligned}$$

for all $x \in G$ whenever $\nu \in M(G)$ and $\tau \in \bigcup_{k=0}^\infty L^1(\lambda_k)$. Since the ideal $Y$ is generated by $\bigcup_{k=0}^\infty L^1(\lambda_k)$, this implies that (1) holds for all $\tau \in Y$.

The last statement in the Corollary is now trivial and the proof is complete.

To prove Theorem 2, we need some notation. Let $\mathscr{T}$ be a nonempty family of (locally) Borel measurable subgroups of $G$ such that for any countable subfamily $\mathscr{T}_0$ of $\mathscr{T}$ there exists a subgroup $H \in \mathscr{T}$ which contains all $L \in \mathscr{T}_0$. Define

$$I(\mathscr{T}) = \{\mu \in M(G): |\mu|(x + H) = 0 \ \forall x \in G \text{ and } \forall H \in \mathscr{T}\}$$

and

$$\mathscr{R}(\mathscr{T}) = \{\nu \in M(G): |\nu|(G \backslash (D + H)) = 0 \text{ for some countable } D \subset G \text{ and some } H \in \mathscr{T}\}.$$

Then it is easy to prove the following (cf. [1]):

(a) $I(\mathscr{T})$ is a closed ideal in $M(G)$ such that $I(\mathscr{T})^* = I(\mathscr{T})$.

(b) $\mathscr{R}(\mathscr{T})$ is a closed subalgebra of $M(G)$ such that $\mathscr{R}(\mathscr{T})^* = \mathscr{R}(\mathscr{T})$.

(c) $M(G) = I(\mathscr{T}) + \mathscr{R}(\mathscr{T})$ and $I(\mathscr{T}) \cap \mathscr{R}(\mathscr{T}) = \{0\}$.

We denote by $\Phi_{\mathscr{T}}$ the projection of $M(G)$ onto $\mathscr{R}(\mathscr{T})$ which is induced by the direct sum decomposition $M(G) = I(\mathscr{T}) + \mathscr{R}(\mathscr{T})$. Note that $\Phi_{\mathscr{T}}$ is a *-homomorphism of $M(G)$ onto $\mathscr{R}(\mathscr{T})$.

*Proof of Theorem* 2. Let $\{\lambda_k\}_{k=0}^{\infty}$ and $Y$ be as in Corollary; without loss of generality, we may assume that $\lambda_k \geqq 0$ for all $k$. By Lemma 6, there is a $\sigma$-compact, noncompact, open subgroup $\Gamma$ of $\hat{G}$ such that

$$(1) \qquad \lambda_k(x + H) = 0 \qquad (x \in G;\ k = 0, 1, 2, \cdots),$$

where $H$ is the annihilator of $\Gamma$. Let $G_0 = G/H$, and let $\mu \to \mu'$ be the mapping of $M(G)$ onto $M(G_0)$ defined in the proof of Theorem 1 for general groups. Note that $\mu \to \mu'$ is a *-homomorphism. Since (1) implies $\lambda'_k \in M_c(G)$ for all $k$, Theorem 1 assures that there exists a nonzero measure $\sigma' \in M_0^+(G_0)$ with compact support such that $K' = \operatorname{supp} \sigma'$ is independent and

$$(2) \qquad \lambda'_k[x' + G_p(K')] = 0 \qquad (x' \in G_0;\ k = 0, 1, 2, \cdots).$$

Let $\omega_1$ be the first countable ordinal and let $W = \{1, 2, \cdots\}$ be the well-ordered set consisting of all ordinals smaller than $\omega_1$. We now construct a family $\{L'_\alpha: \alpha \in W\}$ of disjoint compact subsets of $K'$ such that

$$(3) \qquad M_0(L'_\alpha) \neq \{0\} \quad \text{and} \quad \sigma'(L'_\alpha) = 0$$

for all $\alpha \in W$. First, by Theorem 1, there exists a compact subset $L'_1$ of $K'$ having property (3). Let $\beta \in W$, $\beta \geqq 2$, and suppose that $L'_\alpha$ has been constructed for all $\alpha \in W$ with $\alpha < \beta$. Then $E'_\beta = \bigcup \{L'_\alpha: \alpha < \beta\}$ is $\sigma$-compact, and by (3), has $\sigma'$-measure zero. Therefore, there exists a compact subset $F'_\beta$ of $K' \backslash E'_\beta$ having positive $\sigma'$-

measure. Applying Theorem 1 again, we can find a compact subset $L'_\beta$ of $F'_\beta$ so that (3) holds for $\alpha = \beta$. By transfinite induction, we obtain a family $\{L'_\alpha : \alpha \in W\}$ of disjoint compact subsets of $K'$ satisfying (3).

Let $\mathscr{P}(W)$ be the family of all nonempty subsets of $W$; hence Card $\mathscr{P}(W) = 2^\omega$, where $\omega$ denotes the smallest uncountable cardinal. For each $A \in \mathscr{P}(W)$ and $\chi \in \Gamma$, we construct a complex homomorphism $\Psi_{A\chi}$ of $M(G)$ as follows. Let $\mathscr{T} = \mathscr{T}_A$ be the family of subgroups of $G$ each of which is generated by $\bigcup \{L'_\alpha : \alpha \in B\}$ for some countable subset $B$ of $A$. We define $\Psi_{A\chi}$ by setting

$$(4) \qquad \Psi_{A\chi}(\mu) = \widehat{\Phi_{\mathscr{T}}((\chi\mu)')}(1) \qquad (\mu \in M(G)) .$$

It is easy to see that $\Psi_{A\chi}$ is a symmetric complex homomorphism of $M(G)$. Also $\Psi_{A\chi} \neq 0$ because

$$(5) \qquad \Psi_{A\chi}(\delta_x) = \chi(x) \neq 0 \qquad (x \in G) ,$$

where $\delta_x$ denotes the unit mass at $x$.

Fixing an $A \in \mathscr{P}(W)$ and $\chi \in \Gamma$, we now prove

$$(6) \quad \bigcup \{L^1(\mu): \mu \in Y\} \cup M_a(G) \subset \operatorname{Ker} \Psi_{A\chi} \quad \text{but} \quad M_0(G) \not\subset \operatorname{Ker} \Psi_{A\chi} .$$

First note that $\nu \in M^+(G)$ and $\mu \in L^1(\nu)$ imply $\mu' \in L^1(\nu')$. In fact, if $w$ is a bounded Borel function on $G$, we have

$$\left| \int_{G_0} f d(w\nu)' \right| = \left| \int_G (f \circ \pi) w d\nu \right| \leqq \int_G (|f| \circ \pi) |w| d\nu$$
$$\leqq \| w \|_\infty \int_G (|f| \circ \pi) d\nu = \| w \|_\infty \cdot \| f \|_{L^1(\nu')}$$

for all $f \in C_0(G_0)$, so that $(w\nu)' \in L^1(\nu')$. Since the mapping $\tau \to \tau'$ is norm-decreasing, we see

$$\inf \{\| \mu' - \tau' \|_M : \tau' \in L^1(\nu')\} \leqq \| \mu' - (w\nu)' \|_M \leqq \| \mu - w\nu \|_M .$$

Since $w$ was arbitrary and $\mu \in L^1(\nu)$, this implies $\mu' \in L^1(\nu')$. Suppose now that $\nu \in M(G)$ and $\lambda \in L^1(\lambda_k)$ for some $k$. Then, the above observation and (2) show

$$| (\nu * \lambda)' | (x' + T') = | \nu' * \lambda' | (x' + T') \leqq (| \nu' | * | \lambda' |)(x' + T')$$
$$\leqq (| \nu' | * | \lambda' |)[x' + G_p(K')] = 0$$

for all $x' \in G_0$ and all $T' \in \mathscr{T} = \mathscr{T}_A$. Since the linear span of the sets $M(G) * L^1(\lambda_k)$, $k = 0, 1, 2, \cdots$, is dense in $Y$, it follows that

$$| \tau' | (x' + T') = 0 \qquad (x' \in G_0;\ T' \in \mathscr{T})$$

holds for all $\tau \in Y$, and so for all $\tau \in \bigcup \{L^1(\mu): \mu \in Y\}$. Therefore we have

$$\bigcup \{L^1(\mu): \mu \in Y\} \subset \operatorname{Ker} \Psi_{A\chi} .$$

Note now that $\lambda_0' \neq 0$, and so $G_p(\operatorname{supp} \sigma')$ has no interior point by (2); hence the Haar measure of $G_p(\operatorname{supp} \sigma')$ is zero. Since $M_a(G)' = M_a(G_0)$, it follows that $M_a(G) \subset \operatorname{Ker} \Psi_{A\chi}$. To prove that $M_0(G) \not\subset \operatorname{Ker} \Psi_{A\chi}$, take any $\alpha \in A$. Then $L_\alpha' \subset G_p(L_\alpha') \in \mathscr{T}$, and so $\Phi_{\mathscr{T}}[M_0(L_\alpha')] = M_0(L_\alpha') \neq \{0\}$. This establishes (6) because $M_0(G)' = M_0(G_0)$.

Finally, take any $A, B \in \mathscr{P}(W)$ and any $\chi, \gamma \in \Gamma$. If $\chi \neq \gamma$, (5) implies that $\Psi_{A\chi} \neq \Psi_{B\gamma}$. If $A \neq B$ (say $A \not\supset B$), take any $\beta \in B \backslash A$; we claim

$$(7) \qquad M_c(L_\beta') \subset \operatorname{Ker} \Phi_{\mathscr{T}} \qquad \text{where } \mathscr{T} = \mathscr{T}_A .$$

In fact, let $T'$ be an arbitrary subgroup in $\mathscr{T}$; there exists a countable subset $A_0$ of $A$ such that $T' = G_p(\bigcup \{L_\alpha': \alpha \in A_0\})$. Since $K'$ is independent and since $L_\beta'$ and $\bigcup \{L_\alpha': \alpha \in A_0\}$ are disjoint subsets of $K'$, it follows that $L_\beta' \cap (x' + T')$ contains at most one point for each $x' \in G_0$. In particular, if $\mu' \in M_c(L_\beta')$, then $|\mu'|(x' + T') = 0$ for all $x' \in G_0$. Since $T' \in \mathscr{T}$ was arbitrary, we see that (7) holds. On the other hand, we have $\Phi_{\mathscr{U}}(M(L_\beta')) = M(L_\beta')$ for $\mathscr{U} = \mathscr{T}_B$. Thus $\Psi_{A\chi} \neq \Psi_{B\gamma}$, as is easily seen. This clearly establishes Theorem 2.

REMARKS. (i) If $G$ is a metrizable $I$-group, then the element $x_0$ in Theorem 1 (and Corollary) can be chosen $x_0 = 0$. In fact, take any nonzero $\lambda_0 \in M_0^+(G)$, and assume that $E_{qy} = \{x \in G: qx = y\}$ has positive $\lambda_0$-measure for some $q \in N$ and some $y \in G$. Let $\mu_0$ be the restriction of $\lambda_0$ to $E_{qy}$, so that $0 \neq \mu_0 = M_0^+(G)$. It is trivial that $E_{qy}$ is a coset of some closed subgroup $H$ of $G$ which is of bounded order. If $\Gamma$ is the annihilator of $H$, we see $|\hat{\mu}_0| = \text{const} \neq 0$ on $\Gamma$. Since $\hat{\mu}_0$ vanishes at infinity, it follows that $\Gamma$ is compact, or, equivalently, that $H$ is an open subgroup of $G$. This is a contradiction because $G$ is an $I$-group while $H$ is of bounded order. Thus, our assertion follows from the last paragraph of the proof of Lemma 2 and the proof of Theorem 1 for metrizable groups.

(ii) Let $\hat{G}^-$ denote the closure of $\hat{G}$ in the maximal ideal space $\Delta_G$ of $M(G)$, and let $Y$ be as in Theorem 2. Then, for some $\tau \in M_0^+(G)$, the set $E_\tau$ of all symmetric $\Theta \in \Delta_G$ such that

$$\bigcup \{L^1(\mu): \mu \in Y\} \cup M_a(G) \subset \Theta \quad \text{and} \quad \hat{\tau}(\Theta) = 1$$

has cardinal number $\geqq 2^\omega$, where $\hat{\tau}$ denotes the Gelfand transform of $\tau$. Note that $E_\tau$ is a closed subset of $\Delta_G$ disjoint from $\hat{G}^-$. To see this, redefine $\mathscr{P}(W)$ in the proof of Theorem 2 to be the family of all subsets of $W$ containing $1 \in W$, and fix any probability measure $\tau \in M_0^+(G)$ such that $\tau' \in M(L_1')$. Then we have

$$\Psi_{A1}(\tau) = \hat{\tau}'(1) = \hat{\tau}(1) = 1 \qquad (A \in \mathscr{P}(W)) .$$

(iii) Let $Y$ be as in Theorem 2. Then there exist a measure $\tau \in M_0^+(G)$, a nondiscrete LCA group $G_0$, and an independent compact subset $K'$ thereof, with $M_0(K') \neq \{0\}$, having the following property: the set of all asymmetric $\Theta \in \Delta_G$ such that

$$\bigcup \{L^1(\mu): \mu \in Y\} \cup M_a(G) \subset \Theta \quad \text{and} \quad \hat{\tau}(\Theta) = 1$$

has cardinal number $\geqq \operatorname{Card} M_c(K')^*$, where $M_c(K')^*$ denotes the conjugate space of $M_c(K')$. This can be proved using the proof of Theorem 2 and a theorem of Hewitt and Kakutani [2]. We omit the details.

(iv) Some analogs to our results hold for non-abelian groups. For example, we have the following: Let $G$ be a nondiscrete locally compact group, $\{\lambda_k\}_{k=0}^\infty \subset M_c^+(G)$, $\lambda_0 \neq 0$, and let $\{C_k\}_{k=0}^\infty$ be a countable family of $\sigma$-compact subsets of $G$ such that

$$\lambda_k(xC_k) = 0 \qquad (x \in G; k = 0, 1, 2, \cdots) .$$

Then there exists a nonzero measure $\sigma \in M_c^+(\operatorname{supp} \lambda_0)$ with compact support such that

$$\lambda_k[xG_p(\operatorname{supp} \sigma)C_k] = 0 \qquad (x \in G; k = 0, 1, 2, \cdots) .$$

## REFERENCES

1. E. Hewitt, *The asymmetry of certain algebras of Fourier-Stieltjes transforms*, Michigan Math. J., **5** (1955), 149-158.
2. E. Hewitt and S. Kakutani, *A class of multiplicative linear functionals on the measure algebra of a locally compact abelian group*, Illinois J. Math., **4** (1960), 553-574.
3. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. 1, Springer-Verlag, Heidelberg, 1963.
4. E. Hewitt and K. R. Stromberg, *A remark on Fourier-Stieltjes transforms*, An. Acad. Brasil. Ci., **34** (1962), 175-180.
5. L.-Å. Lindahl and F. Poulsen, *Thin Sets in Harmonic Analysis*, Marcel-Dekker, Inc., New York, 1971.
6. L. Pigno and S. Saeki, *Fourier-Stieltjes transforms which vanish at infinity*, (submitted).
7. W. Rudin, *Fourier-Stieltjes transforms of measures on independent sets*, Bull. Amer. Math. Soc., **66** (1960), 199-202.
8. ———, *Fourier Analysis on Groups*, Interscience, New York, 1962.
9. N. Th. Varopoulos, *Sets of multiplicity in locally compact abelian groups*, Ann. Inst. Fourier (Grenoble), **16** (1966), 123-158.

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