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EXTENSION OF CONGRUENCES AND HOMOMORPHISMS TO TRANSLATIONAL HULLS

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L. M. Gluskin has shown that if α is an isomorphism of a weakly reductive semigroup S onto a semigroup T, if V is a dense extension of S and T is densely embedded in W then α extends uniquely to an isomorphism of V into W. P. Grillet and M. Petrich have shown that this result can be interpreted in terms of extending α to certain subsemigroups of the translational hull $\Omega(S)$ of S. Here the problem of extending homomorphisms between inverse semigroups is considered. As a preliminary to the main results the problem of extending congruences from S to $\Omega(S)$ is considered and various classes of congruences are shown to be extendable. The main result shows that any homomorphism θ of an inverse semigroup S into an inverse semigroup T such that the ideal, in the semilattice E of idempotents of T, generated by the image of the idempotents of S intersects any principal ideal of E_T in a principal ideal extends naturally to a homomorphism of $\Omega(S)$ into $\Omega(T)$. The extension described is unique with respect to certain natural restrictions.

1. Introduction and extensions of congruences.

We first recall some standard notation (cf. [2] and [5]). For any semigroup S, let

$$\Lambda(S) = \{\lambda \in \mathscr{T}_S : \lambda(xy) = \lambda(x)y, \text{ for all } x, y \in S\},$$
$$P(S) = \{\rho \in \mathscr{T}'_S : (xy)\rho = x(y\rho), \text{ for all } x, y \in S\}$$

where, for any set X, $\mathscr{T}_X(\mathscr{T}'_X)$ denotes the full transformation semigroup on X with functions written on the left (right). Then $\Lambda(S)$ and $P(S)$ are subsemigroups of $\mathscr{T}_S$ and $\mathscr{T}'_S$, respectively. Let

$$\Omega(S) = \{(\lambda, \rho) \in \Lambda(S) \times P(S) : x(\lambda(y)) = ((x)\rho)y, \text{ for all } x, y \in S\}.$$

Then $\Omega(S)$ is a subsemigroup of the direct product $\mathscr{T}_S \times \mathscr{T}'_S$, called the *translational hull of* S. For basic properties of $\Omega(S)$, the reader is referred to [5].

For any $a \in S$, let $\lambda_a \in \Lambda(S)(\rho_a \in P(S))$ be such that

$$\lambda_a(x) = ax, \ (x)\rho_a = xa, \text{ for all } x \in S.$$

Then $(\lambda_a, \rho_a) \in \Omega(S)$ and $\Pi_S : a \to (\lambda_a, \rho_a)$ is a homomorphism of S into $\Omega(S)$. If Π_S is an isomorphism, then S is said to be *weakly reductive*. Thus S is weakly reductive provided that $ax = bx$ and $xa = xb$, for all $x \in S$, implies that $a = b$.

Let S be an ideal of a semigroup V and κ be a congruence on S.

Then κ is *compatible with* V if $(a, b) \in \kappa$, implies that $(va, vb) \in \kappa$ and $(av, bv) \in \kappa$, for all $v \in V$. Let A be a subsemigroup of $\Omega(S)$ containing $\Pi_S(S)$. Then κ is *compatible with* A if $(a, b) \in \kappa$ implies that $(\lambda(a), \lambda(b)) \in \kappa$ and $((a)\rho, (b)\rho) \in \kappa$, for all $(\lambda, \rho) \in A$. If S is weakly reductive then $\Pi_S^{-1} \circ \kappa \circ \Pi_S$ is a congruence on $\Pi_S(S)$, which we denote by κ^π, and κ is compatible with A if and only if κ^π is compatible with A. Now suppose that τ is a congruence on A (alternatively, V) such that $\tau \cap (\Pi_S(S) \times \Pi_S(S)) = \kappa^\pi$ (alternatively, $\tau \cap (S \times S) = \kappa$) then we say that κ *extends to* τ and that τ extends k.

For each $v \in V$, let $\lambda^v(\rho^v)$ denote the mapping $\lambda^v(a) = va((a)\rho^v = av)$, for all $a \in S$, of S into itself. Then $(\lambda^v, \rho^v) \in \Omega(S)$ and $\omega: v \longrightarrow (\lambda^v, \rho^v)$ is a homomorphism of V into $\Omega(S)$. Clearly the restriction of ω to S is just Π_S. The following lemma is straightforward.

LEMMA 1.1. *Let S be an ideal of V and κ a congruence on S. Let S be weakly reductive. Then k extends to a congruence on V if and only if κ is compatible with $\omega(V)$. Moreover, if κ is compatible with $\omega(V)$ then*

$$\kappa^e = \{(u, v) \in V: \textit{for all } a \in S, (ua, va) \in \kappa \textit{ and } (au, av) \in \kappa\}$$

is the maximum congruence on V such that $\kappa^e \cap (S \times S) = \kappa$.

The smallest congruence τ that will extend κ when κ is compatible with $\omega(V)$ is just

$$\tau = \{(u, v) \in V \times V: \textit{either } (u, v) \in S \times S \textit{ and } (u, v) \in \kappa \textit{ or } u = v\} .$$

Consequently, if we are interested in studying those congruences on an ideal S of a semigroup V which extend to congruences on V then we should consider those congruences on S that are compatible with subsemigroups A of $\Omega(S)$ which contain $\Pi_S(S)$ and therefore, in particular, those that are compatible with $\Omega(S)$.

NOTATION. For any semigroup S we write $\mathscr{C}(S)$ for the lattice of congruences on S. For a subsemigroup A of $\Omega(S)$ containing $\Pi_S(S)$ we write $\mathscr{C}_A(S)$ for the set of congruences on S that are compatible with A.

LEMMA 1.2. *Let A be a subsemigroup of $\Omega(S)$ containing $\Pi_S(S)$. Then $\mathscr{C}_A(S)$ is a complete sublattice of the lattice of all congruences on S containing the identity and universal congruences.*

We now consider $\mathscr{C}_A(S)$ for certain classes of semigroups and for the most restrictive case $A = \Omega(S)$.

For any left zero band S (i.e., $xy = x$ for all $x, y \in S$), $\Omega(S) =$

$\{(\lambda, \iota): \lambda$ is a function S, ι the identity on $S\}$ and therefore $\mathscr{C}_{\Omega(S)}(S)$ consists of only the identity and universal congruences.

In general, however, there are many congruences on S which are compatible with $\Omega(S)$ as the following lemmas indicate.

We shall denote by $\mathscr{H}$, Green's relation $\mathscr{H}$ (see [2]).

LEMMA 1.3. *Let S be a semigroup and κ a congruence on S such that $\kappa \subseteqq \mathscr{H}$. Then $\kappa \in \mathscr{C}_{\Omega(S)}(S)$.*

Proof. Let $(a, b) \in \kappa$ and $(\lambda, \rho) \in \Omega(S)$. Then $(a, b) \in \mathscr{H}$ and so there exist elements $x, y \in S$ such that $a = xb$, $b = ay$. Then $b = xby$ and so

$$(1) \qquad \begin{cases} \lambda(a) = \lambda(x)b \\ \lambda(b) = \lambda(x)by \, . \end{cases}$$

Since $(a, b) \in \kappa$ we must have $(b, by) = (ay, by) \in \kappa$ and therefore, by (1), $(\lambda(a), \lambda(b)) \in \kappa$. Similarly $((a)\rho, (b)\rho) \in \kappa$ and so $\kappa \in \mathscr{C}_{\Omega(S)}(S)$.

LEMMA 1.4. [7] *Let B be an ideal of a semigroup S such that $B^2 = B$. Then $\kappa \in \mathscr{C}_{\Omega(S)}(S)$ where*

$$\kappa = \{(a, b) \in S \times S: \text{ either } \ a, b \in B \text{ or } a = b\} \, .$$

A semigroup S is a *regular* semigroup if $a \in aSa$ for all elements $a \in S$ and a regular semigroup S is an *inverse* semigroup if all idempotents commute. If S is an inverse semigroup then, for each element $a \in S$ there is a unique element x in S, which is usually denoted by a^{-1}, such that $axa = a$ and $xax = x$. For the basic properties of inverse semigroups the reader is referred to [2].

LEMMA 1.5. *Let S be a semigroup and κ be a congruence on S such that S/κ is an inverse semigroup. Then $\kappa \in \mathscr{C}_{\Omega(S)}(S)$.*

Proof. Let $(a, b) \in \kappa$ and $(\lambda, \rho) \in \Omega(S)$. Let $x(y)$ be elements of S such that $(x)\kappa = (\lambda(a))\kappa^{-1}$ and $(y)\kappa = (\lambda(b))\kappa^{-1}$. For any element $u \in S$,

$$(2) \qquad \begin{cases} (u)\kappa(\lambda(a))\kappa = (u\lambda(a))\kappa = ((u)\rho a)\kappa \\ \qquad = ((u)\rho)\kappa(a)\kappa = ((u)\rho)\kappa(b)\kappa \\ \qquad = ((u)\rho b)\kappa = (u\lambda(b))\kappa = (u)\kappa(\lambda(b))\kappa \, . \end{cases}$$

Hence

$$\begin{aligned}(\lambda(a))\kappa &= (\lambda(a))\kappa(x)\kappa(\lambda(a))\kappa\ ,\\ &= (\lambda(a))\kappa(x)\kappa(\lambda(b))\kappa\ , \quad \text{by} \quad (2)\ ,\\ &= (\lambda(a))\kappa(x)\kappa(\lambda(b)y\lambda(b))\kappa\\ &= (\lambda(a))\kappa(x)\kappa(\lambda(b)y)\kappa(\lambda(b))\kappa\\ &= (\lambda(b)y)\kappa(\lambda(a))\kappa(x)\kappa(\lambda(b)y)\kappa(\lambda(b))\kappa\ ,\end{aligned}$$

since S/κ is an inverse semigroup and therefore idempotents commute,

$$\begin{aligned}&= (\lambda(b)y)\kappa(\lambda(b))\kappa(\lambda(a))\kappa^{-1}(\lambda(b)y)\kappa^{-1}(\lambda(b))\kappa\ , \quad \text{by} \quad (2)\ ,\\ &= (\lambda(b))\kappa[(\lambda(b)y)\kappa(\lambda(a))\kappa]^{-1}(\lambda(b))\kappa\ ,\\ &= (\lambda(b))\kappa[(\lambda(b)y)\kappa(\lambda(b))\kappa]^{-1}(\lambda(b))\kappa\ , \quad \text{by} \quad (2)\ ,\\ &= (\lambda(b))\kappa(\lambda(b))\kappa^{-1}(\lambda(b))\kappa = (\lambda(b))\kappa\ .\end{aligned}$$

Thus $(\lambda(a), \lambda(b)) \in \kappa$. Likewise $((a)\rho, (b)\rho) \in \kappa$ and $\kappa \in \mathscr{C}_{\Omega(S)}(S)$.

2. Extensions of homomorphisms. If S is an ideal of a semigroup V then a question closely related to that regarding which congruences extend to V is the question of which homomorphisms extend to V. Naturally, one would not expect to be able to say very much in such a general situation. A context in which the question is more meaningful is indicated in Theorem 2.2.

First, some relevant terminology.

Let S be an ideal of a semigroup V. Then V is a *dense extension* of S if, for any nontrivial congruence σ on V, $\sigma \cap (S \times S)$ is nontrivial. (By a nontrivial relation we mean one which is not equal to the identity relation.) Furthermore, S is said to be a *densely embedded ideal* of V if V is a maximal dense extension; that is, if W is a dense extension of S and $V \subseteq W$ then $V = W$. These concepts can be characterized for weakly reductive semigroups in terms of subsemigroups of $\Omega(S)$ as follows. Let $\omega: v \to (\lambda^v, \rho^v)$ be the homomorphism of V into $\Omega(S)$ introduced in §1.

From [5], we have the following result that illuminates Gluskin's theorem below.

LEMMA 2.1. *Let S be an ideal of the semigroup V and let S be weakly reductive.*

(1) *V is a dense extension of S if and only if ω is an isomorphism of V into $\Omega(S)$.*

(2) *S is a densely embedded ideal of V if and only if ω is an isomorphism of V onto $\Omega(S)$.*

Gluskin [3] ([5], §4, Theorem 1) established the following.

THEOREM 2.2. *Let $\alpha: S \to T$ be an isomorphism between weakly reductive semigroups. Let V be a dense extension of S and let T be* *a densely embedded ideal of W. Then α extends to a unique isomorphism of V into W.*

In other words, if S and T are weakly reductive and $\alpha: S \to T$ is an isomorphism then α will "extend" uniquely to an isomorphism of any subsemigroup of $\Omega(S)$ containing $\Pi_S(S)$ onto a subsemigroup of $\Omega(T)$ containing $\Pi_T(T)$.

We formalize this notion of extension not only for isomorphisms but also homomorphisms as follows.

If A is a subsemigroup of $\Omega(S)$ containing $\Pi_S(S)$ and α is a homomorphism of S into a semigroup T then we say that α *extends* to a homomorphism β of A into $\Omega(T)$ or that β *extends* α if β is a homomorphism of A into $\Omega(T)$ such that

$$\beta(\lambda_a, \rho_a) = (\lambda_{\alpha(a)}, \rho_{\alpha(a)}) , \quad \text{for all} \quad a \in S .$$

LEMMA 2.3. [7] *Let A be a subsemigroup of $\Omega(S)$ containing $\Pi_S(S)$ and let κ be a congruence on S compatible with A. Let $T = S/\kappa$ and α denote the natural homomorphism of S onto T. Then α extends to a homomorphism $\beta: A \to \Omega(T)$. Moreover, if T is weakly reductive then β is unique. If S and T are both weakly reductive then $\beta \circ \beta^{-1} \cap (\Pi_S(S) \times \Pi_S(S)) = \kappa^\pi$ and $\beta \circ \beta^{-1}$ is the largest congruence on A extending κ.*

The homomorphism β is defined in the obvious way by $\beta(\lambda, \rho) = (\lambda', \rho')$ where

$$\lambda'(\alpha(x)) = \alpha(\lambda(x)) \quad \text{and} \quad (\alpha(x))\rho' = \alpha((x)\rho)$$

for all $x \in S$.

We can use this result to make some further observations regarding $\mathscr{C}_{\Omega(S)}(S)$. The reader is referred to [7] for applications of Lemma 2.3 to the translational hull of a semisimple semigroup.

LEMMA 2.4. *Let κ, τ be congruences on a semigroup S such that $\tau \subseteqq \kappa$. Let $T = S/\tau$. If $\tau \in \mathscr{C}_{\Omega(S)}(S)$ and $\kappa/\tau \in \mathscr{C}_{\Omega(T)}(T)$ then $\kappa \in \mathscr{C}_{\Omega(S)}(S)$.*

Proof. Let $(a, b) \in \kappa$ and $(\lambda, \rho) \in \Omega(S)$. Then $(a\tau, b\tau) \in \kappa/\tau$. Since $\tau \in \mathscr{C}_{\Omega(S)}(S)$ the natural homomorphism $\alpha: S \to T$ extends to a homomorphism $\beta: \Omega(S) \to \Omega(T)$. Let $\beta(\lambda, \rho) = (\lambda', \rho')$. Then

$$((\lambda(a))\tau, (\lambda(b))\tau) = (\lambda'(a\tau), \lambda'(b\tau)) \in \kappa/\tau .$$

Hence

$$(\lambda(a), \lambda(b)) \in \kappa .$$

Similarly $((a)\rho, (b)\rho) \in \kappa$ and $\kappa \in \mathscr{C}_{\Omega(S)}(S)$.

This enables us to say a little more about $\mathscr{C}_{\Omega(S)}(S)$ when S is a regular semigroup.

For any regular semigroup S the relation θ is defined on the lattice of congruences of S as follows [10]

$$(\kappa, \tau) \in \theta \Longleftrightarrow \kappa \cap (E_S \times E_S) = \tau \cap (E_S \times E_S)$$

where, for any semigroup S, E_S denotes the set of idempotents of S. The relation θ is a complete congruence on $\mathscr{C}(S)$ and each θ-class is a complete modular sublattice of $\mathscr{C}(S)$.

PROPOSITION 2.5. *Let S be a regular semigroup. Then $\mathscr{C}_{\Omega(S)}(S)$ is a union of θ-classes.*

Proof. Let θ be any θ-class in $\mathscr{C}(S)$ such that $\theta \cap \mathscr{C}_{\Omega(S)}(S)$ is nonempty. Since $\mathscr{C}_{\Omega(S)}(S)$ and θ are both complete sublattices of $\mathscr{C}(S)$, so is $F = \theta \cap \mathscr{C}_{\Omega(S)}(S)$. Hence F has a smallest member τ, say. Let $T = S/\tau$. Let $\kappa \in \theta$ and $\kappa \geqq \tau$. Since $(\kappa, \tau) \in \theta$, $\kappa/\tau \subseteqq \mathscr{H}$. Hence $\kappa/\tau \in \mathscr{C}_{\Omega(T)}(T)$, by Lemma 1.3. Therefore $\kappa \in \mathscr{C}_{\Omega(S)}(S)$, by Lemma 2.4. Thus $\kappa \in F$.

Now let κ be any element of θ and $\sigma = \kappa \vee \tau$. Then $\kappa \leqq \sigma$ and $\sigma \in F$. Also $\sigma/\kappa \subseteqq \mathscr{H}$.

Let $(\lambda, \rho) \in \Omega(S)$ and $(a, b) \in \kappa$. Then $(a, b) \in \sigma$. Since σ is compatible with $\Omega(S)$, $(\lambda(a), \lambda(b)) \in \sigma$ and $((\lambda(a))\kappa, (\lambda(b))\kappa) \in \sigma/\kappa$. Therefore, $((\lambda(a))\kappa, (\lambda(b))\kappa) \in \mathscr{H}$ and so there is an element $c \in S$ such that $(c)\kappa(\lambda(a))\kappa = (\lambda(b))\kappa$. Hence,

$$\begin{cases} (\lambda(b))\kappa = (c\lambda(a))\kappa = ((c)\rho a)\kappa \\ \qquad = ((c)\rho)\kappa(a)(\kappa) = ((c)\rho)\kappa(b)\kappa = (c\lambda(b))\kappa = (c)\kappa(\lambda(b))\kappa\ . \end{cases}$$

Consequently, left translation by $(c)\kappa$ induces the identity mapping on the right ideal generated by $(\lambda(b))\kappa$. In particular, $(c)\kappa(\lambda(a))\kappa = (\lambda(a))\kappa$. Therefore $(\lambda(a))\kappa = (\lambda(b))\kappa$. Similarly, $((a)\rho, (b)\rho) \in \kappa$ and so $\kappa \in \mathscr{C}_{\Omega(S)}(S)$. Thus $\theta \subseteqq \mathscr{C}_{\Omega(S)}(S)$ and the result follows.

We shall call a semigroup S *fundamental* if there are no nontrivial congruences on S contained in $\mathscr{H}$. We conclude this section with a result which is a consequence of some of our observations regarding the extension of congruences and of some independent interest.

PROPOSITION 2.6. *Let S be a regular semigroup. Then S is fundamental if and only if $\Omega(S)$ is fundamental.*

Proof. For the purposes of this proposition, we identify S with $\Pi_S(S)$. First suppose that S is not fundamental and let ρ be a nontrivial congruence on S contained in $\mathscr{H}$. Then $\rho \in \mathscr{C}_{\Omega(S)}(S)$, by Lemma 1.3. Let σ be the minimum extension of ρ to a congruence on $\Omega(S)$ described in Lemma 1.1. (Here $S = \Pi_S(S)$ and $V = \Omega(S)$.) Clearly $\sigma \subseteqq \mathscr{H}$, Green's relation on $\Omega(S)$, and σ is nontrivial. Therefore $\Omega(S)$ is not fundamental.

Conversely, suppose that σ is a nontrivial congruence on $\Omega(S)$, such that $\sigma \subseteqq \mathscr{H}$. By Lemma 2.1, $\rho = \sigma \cap (S \times S)$ is a nontrivial congruence on S. Since S is a regular ideal of $\Omega(S)$, the restriction of $\mathscr{H}$ on $\Omega(S)$ to S is just Green's relation $\mathscr{H}$ on S. Hence ρ is a nontrivial congruence on S and $\rho \subseteqq \mathscr{H}$. Therefore S is not fundamental.

3. Extensions of homomorphisms between inverse semigroups.

From Lemmas 1.5 and 2.3, we see that, if $\alpha: S \rightarrow T$ is an epimorphism between inverse semigroups then α extends uniquely to a homomorphism $\beta: \Omega(S) \rightarrow \Omega(T)$. In general [7], β is not an epimorphism.

Although it seems to be difficult, in the general case, to say much about extending semigroup monomorphisms one might expect to be able to say something when the semigroups concerned are inverse semigroups. To do so, however, it is not surprising that one needs to know more about the translational hull of an inverse semigroup.

The first result in this direction appears to be the observation due to Ponizovski.

LEMMA 3.1. (Ponizovski [6]). *If S is an inverse semigroup then $\Omega(S)$ is also an inverse semigroup.*

Then J. Ault [1] considered those inverse semigroups S for which $\Pi_S(S) = \Omega(S) \backslash$(Unit group of $\Omega(S)$) and orthogonal sums of such semigroups. Any inverse semigroup S for which E_S is a chain is such a semigroup.

For an arbitrary inverse semigroup S, J. Ault also characterized those $\lambda \in \Lambda(S)$ for which there is a $\rho \in P(S)$ such that (λ, ρ) belongs to the unit group $\Sigma(S)$ of $\Omega(S)$. Since the mapping $\Pi_\Lambda: (\lambda, \rho) \rightarrow \lambda$ is an isomorphism of $\Omega(S)$ into $\Lambda(S)$ it would seem natural to attempt to characterize $\Sigma(S)$ (and, indeed, $\Omega(S)$) by characterizing $\Pi_\Lambda(\Sigma(S))$, as J. Ault did (and $\Pi_\Lambda(\Omega(S))$, as described below).

Throughout the remainder of this paper, for any inverse semigroup S and any $\lambda \in \Lambda(S)$, θ_λ will denote the mapping of E_S into E_S defined by

$$\theta_\lambda(e) = \lambda(e)\lambda(e)^{-1}\,, \quad \textit{for all} \quad e \in E_S\,.$$

In her characterization of $\Sigma(S)$, J. Ault introduced these mappings θ_λ and observed that if $(\lambda, \rho) \in \Sigma(S)$ then θ_λ is an automorphism of E_S. These mappings are also valuable in discussing $\Pi_A(\Omega(S))$.

Before stating the main result, we need one or two preliminary observations.

For any semigroup A and any $a \in A$ the mapping $a \to \lambda_a$ is a homomorphism of A into $\Lambda(A)$. If A is left reductive ($ax = bx$, for all $x \in A$, implies $a = b$), for example, if A is an inverse semigroup, then this mapping is an isomorphism. Let $\Gamma(A) = \{\lambda_a : a \in A\}$.

For any mapping α, $\Delta(\alpha)$ and $\nabla(\alpha)$ denote the *domain* and *range* of α, respectively.

If A is a semilattice then B is a *P-ideal* of A if $B \cap I_x$ is a principal ideal of A for all principal ideals I_x. Then $\Omega(A)$ and $\Lambda(A)$ are just (isomorphic to) the semilattice of all P-ideals of A [1]. The element $(\lambda, \rho) \in \Omega(A)$ or $\lambda \in \Lambda(A)$ corresponds to the P-ideal $\nabla(\lambda)$ and any P-ideal B corresponds to the element $(\lambda, \rho) \in \Omega(A)$ or $\lambda \in \Lambda(A)$ where

$$\lambda(e) = (e)\rho = f \text{ where } Ae \cap B = Af .$$

THEOREM 3.2. [9] *For an inverse semigroup S, $\Pi_A(\Omega(S))$ can variously be described as:*

(1) *the idealizer of $\Gamma(S)$ in $\Lambda(S)$;*

(2) *the unique maximal inverse subsemigroup of $\Lambda(S)$ containing $\Gamma(S)$;*

(3) *the unique maximal inverse subsemigroup of $\Lambda(S)$ with the idealizer of $\Gamma(E_S)$ in $\Lambda(S)$ as its set of idempotents;*

(4) *the unique maximal inverse subsemigroup of $\Lambda(S)$ with $\Lambda(E_S)$ as its set of idempotents;*

(5) *the set of all $\lambda \in \Lambda(S)$ such that*

(a) *$\nabla(\theta_\lambda)$ is a P-ideal,*

and

(b) *θ_λ is a homomorphism;*

(6) *the set of all $\lambda \in \Lambda(S)$ such that*

(a) *$\nabla(\theta_\lambda)$ is a P-ideal,*

and

(b) *$\Delta_\lambda = \{\lambda(e)^{-1}\lambda(e) : e \in E_S\}$ is a P-ideal,*

and

(c) *the restriction of θ_λ is an isomorphism of Δ_λ onto $\nabla(\theta_\lambda)$.*

Armed with these results we can now return to the problem of extending homomorphisms. The reader is referred to [8] for a discussion of some examples that indicate some limits to one's expectations.

We shall need some observations regarding idempotents in $\Omega(S)$.

LEMMA 3.3. *Let S be an inverse semigroup and let (λ, ρ), (l, r) be idempotents of $\Omega(S)$. Then*

(1) $\lambda^2 = \lambda$, $\rho^2 = \rho$;

(2) $\lambda(E_S) \subseteqq E_S$, $(E_S)\rho \subseteqq E_S$;

(3) $\Delta_\lambda = \lambda(E_S)$ *is a P-ideal of* E_S;

(4) *for any* $e \in E_S$,

$$E_S e \cap \Delta_\lambda = E_S \lambda(e) ;$$

(5) $(\lambda, \rho) \leqq (l, r)$ *if and only if* $\Delta_\lambda \subseteqq \Delta_l$.

Conversely, if P is any P-ideal of E_S, then $(\lambda, \rho) \in E_{\Omega(S)}$ where, for any $a \in S$,

$$\lambda(a) = ea \text{ where } E_S aa^{-1} \cap P = E_S e ,$$
$$(a)\rho = af \text{ where } E_S a^{-1}a \cap P = E_S f .$$

Moreover, $\Delta_\lambda = P$.

Proof. Observations (1) and (2) follow from [1] while (3) and (4) follow from [9], Lemma 2.5. The assertion (5) follows from (3) and the fact that elements of $\Omega(S)$ are completely determined by their actions on E_S.

COROLLARY 3.4. *Let S be an inverse semigroup and (λ, ρ) be an idempotent of $\Omega(S)$. Then*

$$(3)\qquad \begin{cases} (\lambda, \rho) = \bigvee \{(\lambda_e, \rho_e) : e \in E_S \text{ and } (\lambda_e, \rho_e) \leqq (\lambda, \rho)\} , \\ \qquad\quad = \bigvee \{(\lambda_e, \rho_e) : e \in \Delta_\lambda\} , \end{cases}$$

where $\vee A$, for a subset A of $E_{\Omega(S)}$, denotes the least upper bound of A in $E_{\Omega(S)}$ which, of course, need not exist for all A.

Proof. That

$$\{(\lambda_e, \rho_e) : e \in E_S \quad \text{and} \quad (\lambda_e, \rho_e) \leqq (\lambda, \rho)\} = \{(\lambda_e, \rho_e) : e \in \Delta_\lambda\}$$

is immediate from Lemma 3.3, (5) and observation that, for any $e \in E_{S'}$

$$\Delta_{\lambda_e} = E_S e .$$

Hence, $(\lambda, \rho) \geqq (\lambda_e, \rho_e)$, for all $e \in \Delta_\lambda$. On the other hand, since Δ_λ is an ideal of E_S,

$$\Delta_\lambda = \bigcup \{E_S e : e \in \Delta_\lambda\} = \bigcup \{\Delta_{\lambda_e} : e \in \Delta_\lambda\}$$

and if (l, r) is any idempotent of $\Omega(S)$ such that $(l, r) \geqq (\lambda_e, \rho_e)$, for all $e \in \Delta_\lambda$, we must have

$$\Delta_l \supset \bigcup \{\Delta_{\lambda_e} : e \in \Delta_\lambda\}$$

and, therefore, $(l, r) \geqq (\lambda, \rho)$. Thus (3) holds.

Finally,

LEMMA 3.5. ([9], Lemma 2.5). *Let S be an inverse semigroup. Let $(\lambda, \rho) \in \Omega(S)$ and (κ, σ) be the inverse of (λ, ρ) in $\Omega(S)$. Then*

(1) $\lambda(e)^{-1}\lambda(e) = \kappa\lambda(e)$, *for all* $e \in E_S$;

(2) $Ee \cap \Delta_\lambda = E\kappa\lambda(e) = E\lambda(e)^{-1}\lambda(e)$, *for all* $e \in E_S$.

We can now establish the main extension theorem for monomorphisms.

THEOREM 3.6. *Let S and T be inverse semigroups and $\theta: S \to T$ be a monomorphism such that the ideal*

$$\langle \theta(E_S) \rangle = \{f \in E_T : f \leqq \theta(e), \text{ for some } e \in E_S\}$$

generated by $\theta(E_S)$ in E_T is a P-ideal. Then θ extends to a monomorphism φ of $\Omega(S)$ into $\Omega(T)$ such that, for any $(\lambda, \rho) \in E_{\Omega(S)}$

$$(4) \qquad \begin{aligned} \varphi(\lambda, \rho) &= \bigvee \{(\lambda_{\theta(e)}, \rho_{\theta(e)}): e \in E_S \text{ and } (\lambda_e, \rho_e) \leqq (\lambda, \rho)\} \\ &= \bigvee \{(\lambda_{\theta(e)}, \rho_{\theta(e)}): e \in \Delta_\lambda\} . \end{aligned}$$

Moreover, φ is the unique extension of θ such that

$$(5) \qquad \varphi(\iota, \iota) = \bigvee \{(\lambda_{\theta(e)}, \rho_{\theta(e)}): e \in E_S\} ,$$

where ι denotes the identity left and right translation, and therefore is the unique extension of θ satisfying (4). *Thus if ψ is an extension of θ such that $\psi(\iota, \iota) = \varphi(\iota, \iota)$ then $\psi = \varphi$.*

Proof. Let $\lambda \in \Lambda(S)$. Define a mapping λ' as follows. Let $a \in T$ and $aa^{-1} = e$. Let

$$E_T e \cap \langle \theta(E_S) \rangle = E_T e'$$

and $e'' \in E_S$ be such that $e' \leqq \theta(e'')$. Then define

$$\lambda'(a) = \theta(\lambda(e''))a .$$

Suppose that $e'', f'' \in E_S$ are such that $e' \leqq \theta(e''), \theta(f'')$. Then $e' \leqq \theta(e''f'')$. Also

$$h = \theta(\lambda(e''))^{-1}\theta(\lambda(e''))aa^{-1} \in E_T e' .$$

Hence, $h \leqq e' \leqq \theta(e''f'')$ and $h\theta(e''f'') = h$. Therefore,

$$\begin{cases} \theta(\lambda(e''))a = \theta(\lambda(e''))ha = \theta(\lambda(e''))\theta(e''f'')ha \\ \qquad = \theta(\lambda(e'')\theta(e''f'')\theta(\lambda(e''))^{-1}\theta(\lambda(e''))aa^{-1}a \\ \qquad = \theta(\lambda(e''))\theta(e''f'')a = \theta(\lambda(e'')e''f'')a = \theta(\lambda(e''f''))a . \end{cases}$$

Similarly,

$$\theta(\lambda(f''))a = \theta(\lambda(e''f''))a$$

and λ' is well defined.

To see that λ' is in $\Lambda(T)$ let $a, b \in T$, $e = aa^{-1}$, $f = abb^{-1}a^{-1}$. Let $e' \in E_T$, $e'' \in E_S$ be such that

$$E_T e \cap \langle \theta(E_S) \rangle = E_T e' \quad \text{and} \quad e' \leqq \theta(e'') .$$

Let f' and f'' be defined similarly. Then

$$\lambda'(a)b = \theta(\lambda(e''))ab ,$$
$$\lambda'(ab) = \theta(\lambda(f''))ab .$$

Since $f \leqq e$, $f' \leqq e'$ and so we may choose f'' so that $f'' \leqq e''$. (If $f'' \not\leqq e''$ take $e''f''$ as a new f''.) Clearly $f' = e'f$.

Let $x = \theta(\lambda(e''))^{-1}\theta(\lambda(e''))f$. Since

$$\theta(\lambda(e'')) = \theta(\lambda(e'')e'') = \theta(\lambda(e''))\theta(e'')$$

we have that

$$x = x\theta(e'')f = x\theta(e'')ef = xe'f = xf' = xf'\theta(f'') = x\theta(f'') .$$

Hence

$$\begin{aligned} \lambda'(a)b &= \theta(\lambda(e''))ab = \theta(\lambda(e''))xab \\ &= \theta(\lambda(e''))x\theta(f'')ab = \theta(\lambda(e''))\theta(f'')ab = \theta(\lambda(e'')f'')ab \\ &= \theta(\lambda(f''))ab = \lambda'(ab) . \end{aligned}$$

Therefore $\lambda' \in \Lambda(T)$.

Now let $\lambda, l \in \Lambda(S)$ and λ', l' be defined as above. Let $a \in T$, $e = aa^{-1}$ and e', e'' be defined as before. Then

$$\begin{aligned} \lambda'l'(a) &= \lambda'(\theta(l(e''))a) \\ &= \lambda'(\theta(l(e'')))a , \end{aligned}$$

and it is not difficult to see that $\lambda'(\theta(l(e''))) = \theta(\lambda l(e''))$. Thus $\lambda'l'(a) = (\lambda l)'$ (a) and the mapping $\lambda \to \lambda'$ is a monomorphism of $\Lambda(S)$ into $\Lambda(T)$.

We define a mapping $\rho \to \rho'$ of $P(S) \to P(T)$ similarly. Let $a \in T$, $a^{-1}a = e$,

$$E_T e \cap \langle \theta(E_S) \rangle = E_T e'$$

and $e'' \in E_S$ be such that $e' \leqq \theta(e'')$. Define

$$(a)\rho' = a(\theta(e'')\rho) .$$

As for the left translations, $\rho' \in P(T)$ and $\rho \to \rho'$ is a monomorphism of $P(S) \to P(T)$.

Suppose now that $(\lambda, \rho) \in \Omega(S)$. We wish to show that $(\lambda', \rho') \in \Omega(T)$. It remains to show that λ' and ρ' are linked. Let $a, b \in T$, $e = a^{-1}a$, $f = bb^{-1}$ and let e', e'', f', f'' be defined as before. Then

$$x = e\theta(\lambda(f''))\theta(\lambda(f''))^{-1} \in E_T e \cap \langle \theta(E_S) \rangle$$

and so $x \leqq e' \leqq \theta(e'')$. Similarly, if $y = \theta((e'')\rho)^{-1}\theta((e'')\rho)f$ then $y \leqq \theta(f'')$. Hence

$$\begin{aligned} a\lambda'(b) &= a\theta(\lambda(f''))b = ax\theta(\lambda(f''))b \\ &= ax\theta(e'')\theta(\lambda(f''))b = a\theta(e''\lambda(f''))b = a\theta((e'')\rho f'')b \\ &= a\theta((e'')\rho)\theta(f'')b = a\theta((e'')\rho)y\theta(f'')b = a\theta((e'')\rho)yb \\ &= (a)\rho' b \ . \end{aligned}$$

Thus $(\lambda', \rho') \in \Omega(T)$ and clearly $\varphi: (\lambda, \rho) \to (\lambda', \rho')$ is a monomorphism.

We now show that φ extends θ. Let $a \in T$, $e = aa^{-1}$ and e', e'' be as before. Let $x \in S$. Then

$$\begin{aligned} (\lambda_x)'(a) &= \theta(\lambda_x(e''))a = \theta(xe'')a = \theta(x)\theta(e'')a \\ &= \theta(x)a \ , \quad \text{since} \quad \theta(x)^{-1}\theta(x)e \leqq \theta(e'') \ . \end{aligned}$$

Thus $(\lambda_x)' = \lambda_{\theta(x)}$. Similarly $(\rho_x)' = \rho_{\theta(x)}$ and the monomorphism $\varphi: (\lambda, \rho) \to (\lambda', \rho')$ extends θ.

To see that (4) holds, let (λ, ρ) be an idempotent of $\Omega(S)$. Then (λ', ρ') is an idempotent of $\Omega(T)$. Let $e \in \Delta_{\lambda'}$. Then $e = \lambda'(e)$. Let e', e'' be as in the definition of $\lambda'(e)$. Then

$$e = \lambda'(e) = \theta(\lambda(e''))e \ .$$

Therefore, $e \leqq \theta(\lambda(e''))$ where $f'' = \lambda(e'') \in \Delta_\lambda$. Thus $e \in \Delta_{\lambda_{f'}}$ where $f' = \theta(f'')$. Hence

$$(6) \qquad \Delta_{\lambda'} \subseteqq \bigcup \{\Delta_{\lambda_{f'}}: f' = \theta(f'') \quad \text{and} \quad f'' \in \Delta_\lambda\} \ .$$

Since the converse inclusion clearly holds, we have equality in (6) and therefore

$$(\lambda', \rho') = \bigvee \{(\lambda_{f'}, \rho_{f'}): f' = \theta(f'') \quad \text{and} \quad f'' \in \Delta_\lambda\} \ .$$

Hence (4) holds. In particular we note from this that

$$(7) \qquad \Delta_{\iota'} = \langle \theta(E_S) \rangle$$

and

$$(8) \qquad (\iota', \iota') = \bigvee \{(\lambda_{\theta(e)}, \rho_{\theta(e)}): e \in E_S\} \ .$$

Let ψ be any other extension of θ that satisfies (5). For any $(\lambda, \rho) \in \Omega(S)$, let $\psi(\lambda, \rho) = (\lambda'', \rho'')$. From (5) and (8) it follows that

(9) $$(\iota'', \iota'') = (\iota', \iota') .$$

Let (λ, ρ) be any element of $\Omega(S)$ and let (κ, σ) be the inverse of (λ, ρ) in $\Omega(S)$. Then it follows, from Lemma 3.5, that

$$\Delta_\lambda = \Delta_{\kappa\lambda} \quad \text{and} \quad \Delta_{\lambda''} = \Delta_{\kappa''\lambda''} .$$

Let $a \in T$ and $aa^{-1} = e$. Let e', e'' be as before. Then

$$\lambda''(a) = \lambda''(e)a = \lambda''(\lambda''(e)^{-1}\lambda''(e))a = \lambda''(g)a ,$$

where $g = \lambda''(e)^{-1}\lambda''(e) \in \Delta_{\lambda''} = \Delta_{\kappa''\lambda''} \subset \Delta_{\iota''}$. Hence, from (7) and (9), $g \in \Delta_{\lambda_{f'}}$, for some $f' = \theta(f'')$ with $f'' \in E_S$. Since $g \leqq e$ we therefore have $g \in E_T e \cap \langle \theta(E_S) \rangle$ and so $g \leqq e' \leqq \theta(e'')$. Hence $g = g\theta(e'')$. Also

$$\begin{aligned}(\lambda'', \rho'')(\lambda_{\theta(e'')}, \rho_{\theta(e'')}) &= \psi(\lambda, \rho)\psi(\lambda_{e''}, \rho_{e''}) = \psi((\lambda, \rho)(\lambda_{e''}, \rho_{e''})) \\ &= \psi(\lambda_{\lambda(e'')}, \rho_{\lambda(e'')}) = (\lambda_{\theta\lambda(e'')}, \rho_{\theta\lambda(e'')}) .\end{aligned}$$

Therefore, since T is weakly reductive,

$$\lambda''\lambda_{\theta(e'')} = \lambda_{\theta\lambda(e'')} .$$

Also, from Lemma 3.5,

$$E_T e \cap \Delta_{\lambda''} = E_T g$$

while, for $x = \lambda''(\theta(e''))$,

$$x^{-1}xe \in E_T e \cap \Delta_{\lambda''} .$$

Hence

(10) $$x^{-1}xeg = x^{-1}xe .$$

Therefore,

$$\begin{aligned}\lambda''(a) &= \lambda''(g)a = \lambda''(\theta(e'')g)a = xga = xx^{-1}xgea \\ &= xa , \quad \text{by} \quad (10) = \lambda''(\theta(e''))a = \lambda''\lambda_{\theta(e'')}(a) \\ &= \lambda_{\theta\lambda(e'')}(a) = (\theta(\lambda(e'')))a = \lambda'(a) .\end{aligned}$$

Thus $\lambda'' = \lambda'$, $\psi(\lambda, \rho) = \varphi(\lambda, \rho)$ and $\psi = \varphi$.

Combining Lemmas 1.5 and 2.3 with Theorem 3.6 we have the following result.

COROLLARY 3.7. *Let $\theta: S \rightarrow T$ be a homomorphism of the inverse semigroup S into the inverse semigroup T such that $\langle \theta(E_S) \rangle$ is a P-ideal in E_T. Then θ extends to a homomorphism $\varphi: \Omega(S) \rightarrow \Omega(T)$ which is unique with respect to the condition*

$$\varphi(\iota, \iota) = \bigvee \{(\lambda_{\theta(e)}, \rho_{\theta(e)}): e \in E_S\}$$

where ι denotes the identity left and right translation of S.

Proof. Let $\theta = \theta_1\theta_2$ where θ_2 is the natural epimorphism of S onto $S/\theta \circ \theta^{-1}$ and θ_1 is the embedding of $S/\theta \circ \theta^{-1}$ into T such that $\theta = \theta_1\theta_2$. Let φ be the composite of the natural extensions of θ_1 and θ_2 described in Lemma 2.3 and Theorem 3.6. Then φ extends θ.

The uniqueness part does not follow immediately from the statement of Theorem 3.6 but the proof of uniqueness in Theorem 3.6 will carry over almost verbatim.

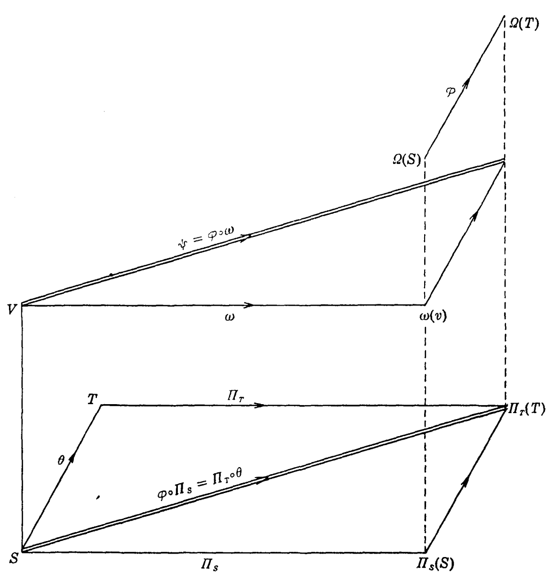
So far we have been concerned with extending a homomorphism $\theta: S \rightarrow T$ between inverse semigronps to a homomorphism $\varphi: \Omega(S) \rightarrow \Omega(T)$. Having done that we can easily generalize the domain of our extension if we also liberalize our definition of an extension. Let S be an ideal of a semigroup V and $\theta: S \rightarrow T$ be a homomorphism. Then we shall say that a homomorphism $\psi: V \rightarrow \Omega(T)$ *extends* θ if

$$\psi(a) = (\lambda_{\theta(a)}, \rho_{\theta(a)}) \, , \quad \text{for all} \quad a \in S \, .$$

Then we have the following result.

COROLLARY 3.8. *Let $\theta: S \rightarrow T$ be a homomorphism between inverse semigroups such that $\langle \theta(E_S) \rangle$ is a P-ideal in E_T. Let S be an ideal of the semigroup V. Then θ extends to a homomorphism $\psi: V \rightarrow \Omega(T)$.*

Proof. Let $\omega: V \rightarrow \Omega(S)$ be the homomorphism introduced in §1. Then the restriction of ω to S is just Π_S. Let $\varphi: \Omega(S) \rightarrow \Omega(T)$ be the extension of θ described in Corollary 3.7. Then $\varphi \circ \omega: V \rightarrow \Omega(T)$ and



$$\varphi \circ \omega(a) = \varphi \circ \Pi_S(a) = \varphi(\lambda_a, \rho_a) = (\lambda_{\theta(a)}, \rho_{\varphi(a)}) .$$

Thus $\psi = \varphi \circ \omega$ is an extension of θ. The situation is displayed in the above diagram.

An interesting feature of the natural extension φ of θ in Theorem 3.6 is that it preserves convexity. Recall that a subset A of a partially ordered set X is said to be *convex* if $a \leqq x \leqq b$, $a, b \in A$, $x \in X$ implies that $x \in A$.

PROPOSITION 3.9. *If, in addition to the hypothesis of Theorem 3.6, $\theta(E_S)$ is convex in E_T then $\varphi(E_{\Omega(S)})$ is convex in $E_{\Omega(T)}$ and φ is the unique extension of θ such that the image of $E_{\Omega(S)}$ is convex in $E_{\Omega(T)}$.*

Proof. We wish to show that $\varphi(E_{\Omega(S)})$ is convex. Let (λ, ρ), $(\mu, \nu) \in E_{\Omega(S)}$ and $(l, r) \in E_{\Omega(T)}$ be such that

$$(\lambda', \rho') \leqq (l, r) \leqq (\mu', \nu') .$$

Then, from Lemma 3.3, we must clearly have

$$\Delta_{\lambda'} \subseteqq \Delta_l \subseteqq \Delta_{\mu'}$$

and therefore

$$\Delta_{\lambda'} \cap \theta(E_S) \subseteqq \Delta_l \cap \theta(E_S) \subseteqq \Delta_{\mu'} \cap \theta(E_S) . \tag{11}$$

Let the three expressions in (11) be labelled A_1, A_2, and A_3, respectively. Note that

$$\Delta_{\lambda'} = \{e \in E_T : e \leqq \theta(f'') \quad \text{for some } f'' \in \Delta_\lambda\}$$

and similarly for $\Delta_{\mu'}$. Let $e'' \in E_S$. Then

$$\begin{aligned} \theta(E_S)\theta(e'') \cap A_2 &\subseteqq E_T\theta(e'') \cap \Delta_l \\ &= E_T f \ , \quad \text{for some } f \in E_T . \end{aligned}$$

Since $\Delta_l \subseteqq \Delta_{\mu'}$ we must have $f \leqq \theta(f'')$, for some $f'' \in E_S$. On the other hand,

$$\Delta_\mu \cap E_S e'' = E_S g'' ,$$

for some $g'' \in E_S$, and so

$$\theta(E_S)\theta(e'') \cap A_2 \supseteqq \theta(E_S)\theta(e'') \cap A_1 \supseteqq \theta(E_S)\theta(g'') .$$

Hence,

$$\theta(g'') \leqq f \leqq \theta(f'')$$

and, since $\theta(E_S)$ is convex, $f \in \theta(E_S)$. Since we also have $f \in \Delta_l$, it follows that $f \in A_2$. Hence

$$\theta(E_S)\theta(e'') \cap A_2 = \theta(E_S)f$$

and A_2 is a P-ideal of $\theta(E_S)$. Let $A = \theta^{-1}(A_2)$. Then A is a P-ideal of E_S. Let (ε, δ) be the element of $E_{\Omega(S)}$, as in Lemma 3.3, with $\Delta_\varepsilon = A$. Then $\varphi(\varepsilon, \delta) = (\varepsilon', \delta')$ so that to show that $\varphi(\varepsilon, \delta) = (l, r)$ it only remains to be shown that $\Delta_{\varepsilon'} = \Delta_l$. But clearly

$$\Delta_{\varepsilon'} = \langle \theta(\Delta_\varepsilon) \rangle = \langle \theta(A) \rangle = \langle A_2 \rangle \subseteqq \Delta_l .$$

Let $e \in \Delta_l$. Since $\Delta_l \subseteqq \Delta_{\mu'}$, we have that $e \leqq \theta(e'')$, for some $e'' \in E_S$. Let $f'' \in E_S$ be such that

$$A_2 \cap \theta(E_S)\theta(e'') = \theta(E_S)\theta(f'') ,$$

and let $g \in E_T$ be such that

$$\Delta_l \cap E_T\theta(e'') = E_T g .$$

Then $e \leqq g$ while

$$\theta(f'') \leqq g \leqq \theta(e'') .$$

Therefore $g \in \theta(E_S)$, since $\theta(E_S)$ is convex, and

$$g \in \Delta_l \cap \theta(E_S) = A_2 .$$

Thus $e \in \langle A_2 \rangle$, $\Delta_l = \Delta_{\varepsilon'}$, $l = \varepsilon'$ and therefore $(l, r) = (\varepsilon', \delta')$.

Finally, to establish the uniqueness of φ, let ψ be any other extension such that $\psi(E_{\Omega(S)})$ is convex. From Theorem 3.6,

$$\varphi(\iota, \iota) = \bigvee \{\lambda_{\theta(e)}, \rho_{\theta(e)}): e \in E_S\}$$

and clearly, since $(\iota, \iota) \geqq (\lambda_e, \rho_e)$, for all $e \in E_S$,

$$\psi(\iota, \iota) \geqq (\lambda_{\theta(e)}, \rho_{\theta(e)}) , \quad \text{for all } e \in E_S .$$

Hence

$$\psi(\iota, \iota) \geqq \varphi(\iota, \iota) .$$

Since $\psi(E_{\Omega(S)})$ is convex in $E_{\Omega(T)}$, there is some element $(\lambda, \rho) \in E_{\Omega(S)}$ with $\psi(\lambda, \rho) = \varphi(\iota, \iota)$. Then

$$\psi(\lambda, \rho) \geqq (\lambda_{\theta(e)}, \rho_{\theta(e)}) = \psi(\lambda_e, \rho_e) ,$$

for all $e \in E_S$. Hence

$$(\lambda, \rho) \geqq (\lambda_e, \rho_e) ,$$

for all $e \in E_S$. But there is only one idempotent in $\Omega(S)$ with this property, namely (ι, ι). Hence $\psi(\iota, \iota) = \varphi(\iota, \iota)$ and, by Theorem 3.6,

$\varphi = \psi$.

In general, φ may not be the only homomorphism of $\Omega(S)$ into $\Omega(T)$ that extends θ, even if $\theta(E_S)$ is convex in E_T. An example to illustrate this will be found in [8].

The knowledge that certain homomorphisms between inverse semigroups extend to their translational hulls can be useful in determining the nature of the translational hull of individual inverse semigroups. By relating an inverse semigroup S via homomorphisms to other inverse semigroups for which the translational hulls are known one can again insight into $\Omega(S)$.

For instance, the Howie-Munn representation of an inverse semigroup S is described as follows [4]. Let E be the semilattice of idempotents of S. Let T_E denote the inverse semigroup of all isomorphisms of principal ideals of E onto principal ideals and let U_E denote the inverse semigroup of all isomorphisms of P-ideals of E onto P-ideals of E. Note that T_E is a subsemigroup of U_E. The mapping $\theta: a \rightarrow \theta_a$ where

$$(1) \quad \Delta(\theta_a) = Ea^{-1}a ;$$

$$(2) \quad \theta_a(e) = aea^{-1} , \quad \text{for all} \quad e \in \Delta(\theta_a)$$

is a homomorphism of S into T_E such that $\theta \circ \theta^{-1}$ is the maximum idempotent separating congruence on S.

Since θ maps E onto the idempotents of T_E, θ extends to a homomorphism $\varphi: \Omega(S) \rightarrow \Omega(T_E)$. This extension is considered in more detail in [8] where it is shown that $\Omega(T_E) = U_E$ and $\phi \circ \phi^{-1}$ is the maximum idempotent separating congruence on $\Omega(S)$.

5. Composition of extensions.

Let us call a homomorphism $\alpha: S \rightarrow T$ between inverse semigroups a *P-homomorphism* if $\langle \alpha(E_S) \rangle$ is a P-ideal in E_T.

In this final section we show that the extension of a composite of P-homomorphisms is the composite of the extensions. Implicit in this statement, of course, is the claim that the composite of two P-homomorphisms is a P-homomorphism. We tackle this first.

LEMMA 5.1. *If $\alpha: S \rightarrow T$ and $\beta: T \rightarrow U$ are P-homorphisms, then so is $\beta \circ \alpha$.*

Proof. Let $e \in E_U$. Then there exists an element $e' \in E_U$ such that

$$E_U e \cap \langle \beta(E_T) \rangle = E_U e' . \tag{12}$$

Let $f \in E_T$ be such that $e' \leqq \beta(f)$. Then there exists an element $f' \in E_T$ such that

$$E_T f \cap \langle \alpha(E_S) \rangle = E_T f' . \tag{13}$$

Let $g \in E_S$ be such that $f' \leqq \alpha(g)$. Clearly

$$e\beta\alpha(g) \in E_U e \cap \langle \beta\alpha(E_S) \rangle .$$

Let x be any element of $E_U e \cap \langle \beta\alpha(E_S) \rangle$. Ther

$$x \leqq e , \tag{14}$$

and, for some $h \in E_S$,

$$x \leqq \beta\alpha(h) \tag{15}$$

and

$$x \in E_U e \cap \langle \beta\alpha(E_S) \rangle \subseteqq E_U e \cap \langle \beta(E_T) \rangle .$$

Hence, by (12),

$$x \leqq e' \leqq \beta(f) . \tag{16}$$

From (15) and (16),

$$x \leqq \beta(f)\beta\alpha(h) = \beta(f\alpha(h)) \tag{17}$$

where

$$f\alpha(h) \in E_T f \cap \langle \alpha(E_S) \rangle .$$

Hence, by (13),

$$f\alpha(h) \leqq f' \leqq \alpha(g) .$$

Therefore, from (17),

$$x \leqq \beta\alpha(g) ,$$

and, from (14),

$$x \leqq e\beta\alpha(g) .$$

Thus

$$E_U e \cap \langle \beta\alpha(E_S) \rangle = E_U e\beta\alpha(g) ,$$

and $\beta \circ \alpha$ is a P-homomorphism.

NOTATION. For any P-homomorphism $\alpha: S \to T$ let $\Omega(\alpha)$ denote the extension of α to a homomorphism of $\Omega(S) \to \Omega(T)$ described in Corollary 3.7.

Let $\mathscr{C}_1$ denote the category of inverse semigroups and P-homomorphisms. Note that a homomorphism $\alpha: S \to T$ between inverse semigroups where S has an identity is necessarily a P-homomorphism. Hence $\Omega(\alpha) \in \mathscr{C}_1$ for all $\alpha \in \mathscr{C}_1$.

THEOREM 5.2. *$\Omega: \mathscr{C}_1 \to \mathscr{C}_1$ defined on objects and morphisms by*

$$\Omega: S \to \Omega(S)$$
$$\Omega: \{\alpha: S \longrightarrow T\} \longrightarrow \{\Omega(\alpha): \Omega(S) \longrightarrow \Omega(T)\}$$

is a covariant functor. Moreover, for any morphism $\alpha: S \to T$ in $\mathscr{C}_1$, the diagram

$$\begin{array}{ccc} S & \xrightarrow{\Pi_S} & \Omega(S) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\Omega(\alpha)} \\ T & \xrightarrow{\Pi_T} & \Omega(T) \end{array}$$

commutes and therefore $\{\Pi_S: S \in \text{objects of } \mathscr{C}_1\}$ is a natural transformation from the identity functor: $\mathscr{C}_1 \to \mathscr{C}_1$ to Ω.

Proof. To show that Ω is indeed a covariant functor it remains to be shown that, for any P-homomorphisms $\alpha: S \to T$, $\beta: T \to U$, it is the case that $\Omega(\beta \circ \alpha) = \Omega(\beta) \circ \Omega(\alpha)$. Since it is clear that $\Omega(\beta) \circ \Omega(\alpha)$ is an extension of $\beta \circ \alpha$, by Corollary 3.7, all that is required is to show that $\Omega(\beta \circ \alpha)(\iota, \iota) = \Omega(\beta) \circ \Omega(\alpha)(\iota, \iota)$.

Let $\Omega(\alpha)(\iota, \iota) = (\kappa, \rho)$, $\Omega(\beta)(\kappa, \rho) = (\lambda, \sigma)$ and $\Omega(\beta \circ \alpha)(\iota, \iota) = (\mu, \tau)$. Then κ, λ, μ are idempotents. Let $e \in \Delta_\lambda$. Then, for $e' \in E_U$ and $f \in E_T$ such that

$$E_U e \cap \langle \beta(E_T) \rangle = E_U e' \quad \text{and} \quad e' \leqq \beta(f)$$

we have

$$e = \lambda(e) = \beta(\kappa(f))e$$

and, for $f' \in E_T$, $g \in E_S$ such that

$$E_T f \cap \langle \alpha(E_S) \rangle = E_T f' \quad \text{and} \quad f' \leqq \alpha(g)$$

we have

$$\kappa(f) = \alpha(\iota(g))f = \alpha(g)f \; .$$

Thus

$$e = \beta(\kappa(f))e = \beta\alpha(g)\beta(f)e$$

and

$$\mu(e) = \mu(\beta\alpha(g))\beta(f)e = \beta\alpha(\iota(g))\beta(f)e = \beta\alpha(g)\beta(f)e = e \; .$$

Thus $e \in \Delta_\mu$. Conversely, let $e \in \Delta_\mu$, e', e'' be such that

$$E_U e \cap \langle \beta\alpha(E_S) \rangle = E_U e',\ e' \leqq \beta\alpha(e'') .$$

Then

$$e = \mu(e) = \beta\alpha(\iota(e''))e = \beta\alpha(e'')e .$$

Hence

$$\lambda(e) = \lambda(\beta\alpha(e''))e = \beta(\kappa(\alpha(e'')))e = \beta\alpha(\iota(e''))e = \beta\alpha(e'')e = e$$

and $e \in \Delta_\lambda$. Thus $\Delta_\lambda = \Delta_\mu$, $\lambda = \mu$, $(\lambda, \sigma) = (\mu, \tau)$ and $\Omega(\beta \circ \alpha) = \Omega(\beta) \circ \Omega(\alpha)$.

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