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ON FUNCTIONAL EQUATIONS CONNECTED WITH DIRECTED DIVERGENCE, INACCURACY AND GENERALIZED DIRECTED DIVERGENCE

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The measures directed divergence, inaccuracy as well as generalized directed divergence occurring in information theory can be characterized by the symmetry, expansibility, branching, and additivity properties together with some regularity and initial conditions. In this paper some functional equations generalizing those implicit in these characterizations shall be treated.

1. **Introduction.** Let $\Delta_n = \{P = (p_1, p_2, \cdots, p_n) \mid p_i \geqq 0$ and $\sum_{i=1}^n p_i = 1\}$ and $\Delta'_n = \{P = (p_1, p_2, \cdots, p_n) \mid p_i > 0$ and $\sum_{i=1}^n p_i \leqq 1\}$ be the set of all finite complete and incomplete probability distributions respectively. In 1948 C. E. Shannon [16] introduced the following measure of information

$$H_n(P) = -\sum_{i=1}^n p_i \log p_i \,, \tag{1.1}$$

on Δ_n which is now known as Shannon's entropy. This has been generalized to inaccuracy [10]. Inaccuracy and the related quantities directed divergence or information gain [11, 15] and generalized directed divergence [3] are given by

$$H_n(P \| Q) = -\sum_{i=1}^n p_i \log q_i \,, \qquad (P \in \Delta_n,\ Q \in \Delta_n \text{ or } \Delta'_n) \,, \tag{1.2}$$

$$I_n(P \| Q) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \,, \qquad (P \in \Delta_n,\ Q \in \Delta_n \text{ or } \Delta'_n) \,, \tag{1.3}$$

and

$$D_n(P \| Q \mid R) = \sum_{i=1}^n p_i \log \frac{q_i}{r_i} \,, \qquad (P \in \Delta_n,\ Q, R \in \Delta_n \text{ or } \Delta'_n) \tag{1.4}$$

respectively. While characterizing these measures we come across the following functional equations

$$\sum_{i=1}^n \sum_{j=1}^m F(p_i q_j) = \sum_{i=1}^n F(p_i) + \sum_{j=1}^m F(q_j) \,, \quad (P \in \Delta_n,\ Q \in \Delta_m) \,, \tag{1.5}$$

$$\sum_{i=1}^n \sum_{j=1}^m F(p_i q_j, x_i y_j) = \sum_{i=1}^n F(p_i, x_i) + \sum_{j=1}^m F(q_j, y_j) \,, \tag{1.6}$$

$$(P \in \Delta_n,\ Q \in \Delta_m,\ X \in \Delta_n \text{ or } \Delta'_n,\ Y \in \Delta_m \text{ or } \Delta'_m)$$

and

$$\text{(1.7)}\quad \sum_{i=1}^{n}\sum_{j=1}^{m} F(p_iq_j,\, x_iy_j,\, u_iv_j) = \sum_{i=1}^{n} F(p_i,\, x_i,\, u_i) + \sum_{j=1}^{m} F(q_j,\, y_j,\, v_j)\,,$$
$$(P \in \Delta_n,\ Q \in \Delta_m,\ X,\ U \in \Delta_n \text{ or } \Delta'_n,\ Y,\ V \in \Delta_m \text{ or } \Delta'_m)$$

(cf. [2], [4], [5], [6], [7], [8], [9], [13]).

For the motivation to consider (1.6) and (1.7) and the application of this result, refer to the Remark at the end of this paper.

In this paper we consider the functional equation

$$\text{(1.8)}\qquad \sum_{i=1}^{2}\sum_{j=1}^{3} F_{i,j}(p_iq_j,\, x_iy_i) = \sum_{i=1}^{2} G_i(p_i,\, x_i) + \sum_{j=1}^{3} H_j(q_j,\, y_j)\,,$$
$$(P \in \Delta_2,\ Q \in \Delta_3,\ X \in \Delta'_2,\ Y \in \Delta'_3)$$

for unknown functions $F_{i,j}$, G_i, H_j. Then this gives the measurable solutions of (1.6) for all $P \in \Delta_2$, $Q \in \Delta_3$, $X \in \Delta'_2$, $Y \in \Delta'_3$ as a special case. The measurable solution of (1.7) for $P \in \Delta_2$, $Q \in \Delta_3$, X, $U \in \Delta'_2$, Y, $V \in \Delta'_3$ can also be obtained by a reduction to (1.8).

In solving (1.8) we make use of the following result of C. T. Ng [13]:

THEOREM 1.1. *The measurable solutions of the functional equation*

$$\text{(1.9)}\qquad \sum_{i=1}^{2}\sum_{j=1}^{3} F_{i,j}(p_iq_j) = \sum_{i=1}^{2} G_i(p_i) + \sum_{j=1}^{3} H_j(q_j)\,,$$

for all $P \in \Delta_2$, $Q \in \Delta_3$, *are given by*

$$\text{(1.10)}\quad \begin{cases} H_1(q) = aq\log q + b_1q + c_1,\ H_2(q) = aq\log q + (b_1+d)q + c_4\,, \\ H_3(q) = aq\log q + (b_1+e)q + c_7,\ F_{1,1}(p) = ap\log p + b_2p + c_2\,, \\ F_{1,2}(p) = ap\log p + (b_2+d)p + c_5\,, \\ F_{1,3}(p) = ap\log p + (b_2+e)p + c_8\,, \\ F_{2,1}(p) = ap\log p + b_3p + c_3,\ F_{2,2}(p) = ap\log p + (b_3+d)p + c_6\,, \\ F_{2,3}(p) = ap\log p + (b_3+e)p + c_9,\ G_1(p) = g(p)\,, \\ G_2(p) = -g(1-p) + a[p\log p + (1-p)\log(1-p)] \\ \qquad + (b_3-b_2)p + (b_2-b_1) - c_1 + c_2 + c_3 - c_4 + c_5 + c_6 \\ \qquad - c_7 + c_8 + c_9\,, \end{cases}$$

where a, b_1, b_2, b_3, c_1, c_2, $\cdots$, c_9, d, e *are arbitrary constants and* g *is an arbitrary measurable function.*

2. Measurable solutions of the functional equations (1.6) and (1.8). We first suppose that equation (1.8) is to hold for all $P \in \Delta_2$, $Q \in \Delta_3$, $X \in \Delta'_2$, $Y \in \Delta'_3$, where $F_{i,j}$, G_i, H_j: $[0, 1] \times]0, 1[\to \boldsymbol{R}$ are functions measurable in their first variables.

For arbitrarily fixed x_i, y_j in $]0, 1[$ with $\sum_{i=1}^{2} x_i \leqq 1$, $\sum_{j=1}^{3} y_j \leqq 1$, equation (1.8) is of the form (1.9) in the p_i's and q_j's. Therefore, by Theorem 1.1 there exist 'constants' $a(x_1, x_2, y_1, y_2, y_3)$, $b_i(x_1, x_2, y_1, y_2, y_3)$, $i = 1, 2, 3$, $c_j(x_1, x_2, y_1, y_2, y_3)$, $j = 1, 2, \cdots, 9$, $d(x_1, x_2, y_1, y_2, y_3)$, $e(x_1, x_2, y_1, y_2, y_3)$ and a measurable function $g(\cdot, x_1, x_2, y_1, y_2, y_3)$ such that

$$(2.1)\quad \begin{cases} H_1(q, y_1) = a(x_1, x_2, y_1, y_2, y_3)q \log q + b_1(x_1, x_2, y_1, y_2, y_3)q \\ \qquad + c_1(x_1, x_2, y_1, y_2, y_3)\,, \\ H_2(q, y_2) = a(x_1, x_2, y_1, y_2, y_3)q \log q + (b_1 + d)(x_1, x_2, y_1, y_2, y_3)q \\ \qquad + c_4(x_1, x_2, y_1, y_2, y_3)\,, \\ H_3(q, y_3) = a(x_1, x_2, y_1, y_2, y_3)q \log q + (b_1 + e)(x_1, x_2, y_1, y_2, y_3)q \\ \qquad + c_7(x_1, x_2, y_1, y_2, y_3)\,, \\ F_{1,1}(p, x_1y_1) = a(x_1, x_2, y_1, y_2, y_3)p \log p + b_2(x_1, x_2, y_1, y_2, y_3)p \\ \qquad + c_2(x_1, x_2, y_1, y_2, y_3)\,, \\ F_{1,2}(p, x_1y_2) = a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_2 + d)(x_1, x_2, y_1, y_2, y_3)p \\ \qquad + c_5(x_1, x_2, y_1, y_2, y_3)\,, \\ F_{1,3}(p, x_1y_3) = a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_2 + e)(x_1, x_2, y_1, y_2, y_3)p \\ \qquad + c_8(x_1, x_2, y_1, y_2, y_3)\,, \\ F_{2,1}(p, x_2y_1) = a(x_1, x_2, y_1, y_2, y_3)p \log p + b_3(x_1, x_2, y_1, y_2, y_3)p \\ \qquad + c_3(x_1, x_2, y_1, y_2, y_3)\,, \\ F_{2,2}(p, x_2y_2) = a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_3 + d)(x_1, x_2, y_1, y_2, y_3)p \\ \qquad + c_6(x_1, x_2, y_1, y_2, y_3)\,, \\ F_{2,3}(p, x_2y_3) = a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_3 + e)(x_1, x_2, y_1, y_2, y_3)p \\ \qquad + c_9(x_1, x_2, y_1, y_2, y_3)\,. \end{cases}$$

$$(2.2)\quad \begin{cases} G_1(p, x_1) = g(p, x_1, x_2, y_1, y_2, y_3)\,, \\ G_2(p, x_2) = -g(1 - p, x_1, x_2, y_1, y_2, y_3) + a(x_1, x_2, y_1, y_2, y_3)[p \log p \\ \qquad + (1 - p) \log (1 - p)] + (b_3 - b_2)(x_1, x_2, y_1, y_2, y_3)p \\ \qquad + (b_2 - b_1 - c_1 + c_2 + c_3 - c_4 + c_5 + c_6 - c_7 + c_8 \\ \qquad + c_9)(x_1, x_2, y_1, y_2, y_3)\,. \end{cases}$$

From (2.1) we get

$$(2.3)\qquad a(x_1, x_2, y_1, y_2, y_3) \equiv \text{constant} = a$$

and

$$(2.4)\quad \begin{cases} b_1(x, x_2, y_1, y_2, y_3) \equiv \text{a function of } y_1 \text{ only} = b_1(y_1)\,, \\ b_1(y_1) + d(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } y_2 = \theta_1(y_2)\,, \\ b_1(y_1) + e(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } y_3 = \phi_1(y_3)\,, \\ b_2(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_1y_1 = b_2(x_1y_1)\,, \end{cases}$$

$$(2.4)\quad \begin{cases} b_2(x_1y_1) + d(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_1y_2 = \theta_2(x_1y_2)\,, \\ b_2(x_1y_1) + e(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_1y_3 = \phi_2(x_1y_3)\,, \\ b_3(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_2y_1 = b_3(x_2y_1)\,, \\ b_3(x_2y_1) + d(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_2y_2 = \theta_3(x_2y_2)\,, \\ b_3(x_2y_1) + e(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_2y_3 = \phi_3(x_2y_3)\,, \end{cases}$$

where x_i, y_j are in $]0, 1[$ with $\sum_{i=1}^{2} x_i \leqq 1$ and $\sum_{j=1}^{3} y_j \leqq 1$.
Similarly

$$(2.5)\quad \begin{cases} c_1(x_1, x_2, y_1, y_2, y_3) = c_1(y_1)\,, \\ c_2(x_1, x_2, y_1, y_2, y_3) = c_2(x_1y_1)\,, \\ c_3(x_1, x_2, y_1, y_2, y_3) = c_3(x_2y_1)\,, \\ c_4(x_1, x_2, y_1, y_2, y_3) = c_4(y_2)\,, \\ c_5(x_1, x_2, y_1, y_2, y_3) = c_5(x_1y_2)\,, \\ c_6(x_1, x_2, y_1, y_2, y_3) = c_6(x_2y_2)\,, \\ c_7(x_1, x_2, y_1, y_2, y_3) = c_7(y_3)\,, \\ c_8(x_1, x_2, y_1, y_2, y_3) = c_8(x_1y_3)\,, \\ c_9(x_1, x_2, y_1, y_2, y_3) = c_9(x_2y_3)\,, \end{cases}$$

where x_i, y_j are in $]0, 1[$ with $\sum_{i=1}^{2} x_i \leqq 1$ and $\sum_{j=1}^{3} y_j \leqq 1$.

The simultaneous equations (2.4) are equivalent to

$$(2.6)\quad \begin{cases} d(x_1, x_2, y_1, y_2, y_3) = \theta_1(y_2) - b_1(y_1) = \theta_2(x_1y_2) - b_2(x_1y_1) \\ \qquad = \theta_3(x_2y_2) - b_3(x_2y_2)\,, \\ e(x_1, x_2, y_1, y_2, y_3) = \phi_1(y_3) - b_1(y_1) = \phi_2(x_1y_3) - b_2(x_1y_1) \\ \qquad = \phi_3(x_2y_3) - b_3(x_2y_1)\,, \end{cases}$$

where x_i, y_j are in $]0, 1[$ with $x_1 + x_2 \leqq 1$, $y_1 + y_2 + y_3 \leqq 1$.

We shall give the general solutions of equation (2.6) through the following lemma.

LEMMA 2.1. *The general solutions of the functional equation*

$$(2.7)\quad f(rs) - g(rt) = h(s) - k(t)\,,$$

for all $r, s, t \in]0, 1[$ *with* $s + t \leqq 1$, *are given by*

$$(2.8)\quad \begin{cases} f(x) = \psi(x) + A\,, \\ g(x) = \psi(x) + A + C\,, \\ h(x) = \psi(x) + B\,, \\ k(x) = \psi(x) + B + C\,, \end{cases}$$

for all $x \in]0, 1[$, *where* A, B, C *are constants and* $\psi\colon]0, \infty[\to \boldsymbol{R}$ (reals) *is a solution of the Cauchy equation,*

$$\psi(rs) = \psi(r) + \psi(s) . \tag{2.9}$$

Proof. We rewrite equation (2.7) as

$$f(rs) - h(s) = g(rt) - k(t) , \tag{2.10}$$

for all $r, s, t \in]0, 1[$ with $s + t \leqq 1$. Thus $f(rs) - h(s)$ is a function of r only, say

$$f(rs) - h(s) = l(r) , \tag{2.11}$$

for all $r, s \in]0, 1[$. Thus by [11, p. 59] there exists $\psi:]0, \infty[\to \boldsymbol{R}$ satisfying

$$\psi(rs) = \psi(r) + \psi(s) , \tag{2.9}$$

for all $r, s \in]0, \infty[$ such that it represents f, h, and l through the equations

$$\begin{cases} f(x) = \psi(x) + A , \\ h(x) = \psi(x) + B , \\ l(x) = \psi(x) + A - B , \end{cases} \tag{2.12}$$

for all $x \in]0, 1[$, where A and B are arbitrary constants. Similarly g and k are given by

$$\begin{cases} g(x) = \psi(x) + A + C , \\ k(x) = \psi(x) + B + C , \end{cases} \tag{2.13}$$

for all $x \in]0, 1[$ and where C is an arbitrary constant. This completes the proof of Lemma 2.1.

Thus the general solution of the equations (2.6) is given by

$$\begin{cases} b_i(x) = \psi(x) + A_i , & i = 1, 2, 3 \\ \theta_i(x) = \psi(x) + A_i + B , & i = 1, 2, 3 \\ \phi_i(x) = \psi(x) + A_i + C , & i = 1, 2, 3 \end{cases} \tag{2.14}$$

for all $x \in]0, 1[$, where A_i, B, C are constants and ψ is a solution of the Cauchy equation (2.9).

Now we shall determine the function g and the 'constants' c_i's in equation (2.2). We prepare our result by the following lemma.

LEMMA 2.2. *Let $k_i:]0, 1[\to \boldsymbol{R}$, $i = 1, 2, 3$ be functions satisfying the functional equation*

$$k_1(r) + k_2(rs) + k_3(rt) = T(s, t) \tag{2.15}$$

for all $r, s, t \in]0, 1[$ *with* $s + t \leqq 1$. *Then, and only then, there exist functions* ψ, ϕ: $]0, \infty[\to \boldsymbol{R}$ *which are solutions of* (2.9) *and constants* A, B, C *such that*

$$(2.16)\qquad \begin{cases} k_1(x) = -\psi(x) - \phi(x) + C\,, \\ k_2(x) = \psi(x) + A\,, \\ k_3(x) = \phi(x) + B\,. \end{cases}$$

Proof. As the right side of (2.15) is independent of r, we have

$$(2.17)\qquad k_1(r) + k_2(rs) + k_3(rt) = k_1(r') + k_2(r's) + k_3(r't)\,,$$

for all $r, r', s, t \in]0, 1[$ with $s + t \leqq 1$. For arbitrary $s, s' \in]0, 1[$ we can choose $t \in]0, 1[$ such that $s + t, s' + t \leqq 1$ and thus from (2.17) we get

$$(2.18)\qquad k_2(rs) - k_2(r's) = k_2(rs') - k_2(r's')\,,$$

for all $r, r', s, s' \in]0, 1[$. We can now fix r' and s' arbitrarily and then equation (2.18) reduces to

$$(2.19)\qquad k_2(rs) = l_1(r) + l_2(s)\,,$$

for all $r, s \in]0, 1[$, (for some functions l_i), which is an equation similar to (2.11). Thus there exists a function ψ: $]0, \infty[\to \boldsymbol{R}$ satisfying (2.9) such that

$$k_2(x) = \psi(x) + A\,,$$

for all $x \in]0, 1[$, where A is a constant. Similarly there exists ϕ: $]0, \infty[\to \boldsymbol{R}$ satisfying (2.9) such that

$$k_3(x) = \phi(x) + B\,,$$

for all $x \in]0, 1[$. If we replace k_2, k_3 by ψ, ϕ respectively in equation (2.17) while fixing r' we get k_1 as is in (2.16). This proves our lemma.

From equation (2.2), we see that g is a function of p and x_1 only, say

$$(2.20)\qquad g(p, x_1, x_2, y_1, y_2, y_3) = g(p, x_1)\,.$$

Now, from equation (2.2), we see that $-c_1(y_1) + c_2(x_1y_1) + c_3(x_2y_1)$ is independent of y_1 and therefore by Lemma 2.2 we have

$$(2.21)\qquad \begin{cases} c_1(x) = \psi_1(x) + \phi_1(x) + D_1\,, \\ c_2(x) = \psi_1(x) + E_1\,, \\ c_3(x) = \phi_1(x) + F_1\,, \end{cases}$$

for all $x \in]0, 1[$, where ψ_1 and ϕ_1 are solutions of the equation (2.9) and D_1, E_1, F_1 are arbitrary constants. Similarly we have

$$(2.22)\qquad \begin{cases} c_4(x) = \psi_2(x) + \phi_2(x) + D_2\,, \\ c_5(x) = \psi_2(x) + E_2\,, \\ c_6(x) = \varphi_2(x) + F_2\,, \\ c_7(x) = \psi_3(x) + \phi_3(x) + D_3\,, \\ c_8(x) = \psi_3(x) + E_3\,, \\ c_9(x) = \phi_3(x) + F_3\,, \end{cases}$$

where ψ_2, ϕ_2, ψ_3, ϕ_3 are solutions of (2.9) again. If we replace the c_i's in the second equation of (2.2) by equations (2.20), (2.21), and (2.22) we see that $-g(1-p, x_1) - \psi(x_1)p + \psi(x_1) + \psi_1(x_1) + \psi_2(x_1) + \psi_3(x_1)$ is independent of x_1, say

$$(2.23)\qquad \begin{aligned} g(1-p, x_1) = g(1-p) - \psi(x_1)p + \psi(x_1) \\ + \psi_1(x_1) + \psi_2(x_1) + \psi_3(x_1) \end{aligned}$$

for all $p \in [0, 1]$ and $x_1 \in]0, 1[$, where $g\colon [0, 1] \to \boldsymbol{R}$ is an arbitrary measurable function.

Combining equations (2.1), (2.2), (2.3), (2.4), (2.5), (2.14), (2.21), (2.22), and (2.23) we are ready to conclude the following theorem.

THEOREM 2.1. *Let* $F_{i\,j}$, G_i, $H_j\colon [0, 1] \times]0, 1[\to \boldsymbol{R}$ $(i = 1, 2,\ j = 1, 2, 3)$ *be functions which are measurable in their first variables. Then these functions satisfy the functional equation* (1.8) *if and only if there exist* ψ, ψ_i, ϕ_i: $]0, \infty[\to \boldsymbol{R}$ *all satisfy the Cauchy equation* (2.9) *such that*

$$(2.24)\qquad \begin{cases} H_1(q, y) = aq\log q + [\psi(y) + A_1]q + \psi_1(y) + \phi_1(y) + D_1\,, \\ H_2(q, y) = aq\log q + [\psi(y) + A_1 + B]q + \psi_2(y) + \phi_2(y) + D_2\,, \\ H_3(q, y) = aq\log q + [\psi(y) + A_1 + c]q + \psi_3(y) + \phi_3(y) + D_3\,, \\ F_{1,1}(p, y) = ap\log p + [\psi(y) + A_2]p + \psi_1(y) + E_1\,, \\ F_{1,2}(p, y) = ap\log p + [\psi(y) + A_2 + B]p + \psi_2(y) + E_2\,, \\ F_{1,3}(p, y) = ap\log p + [\psi(y) + A_2 + c]p + \psi_3(y) + E_3\,, \\ F_{2,1}(p, y) = ap\log p + [\psi(y) + A_3]p + \phi_1(y) + F_1\,, \\ F_{2,2}(p, y) = ap\log p + [\psi(y) + A_3 + B]p + \phi_2(y) + F_2\,, \\ F_{2,3}(p, y) = ap\log p + [\psi(y) + A_3 + c]p + \phi_3(y) + F_3\,, \\ G_1(p, x) = g(p) + \psi(x)p + \psi_1(x) + \psi_2(x) + \psi_3(x)\,, \\ G_2(p, x) = -g(1-p) + a[p\log p + (1-p)\log(1-p)] \\ \qquad + [\psi(x) + A_3 - A_2]p + \phi_1(x) + \phi_2(x) + \phi_3(x) + A_2 \\ \qquad - A_1 - D_1 - D_2 - D_3 + E_1 + E_2 + E_3 + F_1 + F_2 + F_3\,, \end{cases}$$

for all $p, q \in [0, 1]$, $x, y \in]0, 1[$, *where* a, A_i, B, c, D_i, E_i, F_i, $i = 1, 2, 3$, *are all constants, and* g *is an arbitrary measurable function.*

THEOREM 2.2. *If* $F\colon [0, 1] \times]0, 1[\to \boldsymbol{R}$ *is measurable in its first variable, then it satisfies the functional equation* (1.6) *for all* $P \in \Delta_2$, $Q \in \Delta_3$, $X \in \Delta_2'$, $Y \in \Delta_3'$ *if and only if* F *is of the form*

$$F(p, x) = ap \log p + [\psi(x) + A]p \,, \tag{2.25}$$

for all $p \in [0, 1]$, $x \in]0, 1[$, *where* ψ *is a solution of the Cauchy equation* (2.9) *and* a, A *are constants.*

3. On the measurable solutions of the functional equation (1.7). Let $F\colon [0, 1] \times]0, 1[\times]0, 1[\to \boldsymbol{R}$ be measurable in its first variable and satisfy the equation (1.7) for all $P \in \Delta_2$, $Q \in \Delta_3$, $X, U \in \Delta_2'$, $Y, V \in \Delta_3'$.

For each fixed u_i, v_j equation (1.7) reduces to the form (1.8). Thus by Theorem 2.1 there exist in particular ψ, ψ_1, ψ_2, ϕ_1, ϕ_2 satisfying the Cauchy equation (2.9) in their first variables and A_1, A_2, A_3, a, B, D_1, D_2, E_1, E_2, F_1 such that

$$\tag{3.1} \left\{ \begin{aligned} F(q, y, v_1) &= a(u_1, u_2, v_1, v_2, v_3) q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + A_1(u_1, u_2, v_1, v_2, v_3)]q + (\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + D_1(u_1, u_2, v_1, v_2, v_3) \,, \\ F(q, y, v_2) &= a(u_1, u_2, v_1, v_2, v_3) q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + (A_1 + B)(u_1, u_2, v_1, v_2, v_3)]q + (\psi_2 + \phi_2) \\ &\quad\quad (y, u_1, u_2, v_1, v_2, v_3) + D_2(u_1, u_2, v_1, v_2, v_3) \,, \\ F(q, y, u_1 v_1) &= a(u_1, u_2, v_1, v_2, v_3) q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + A_2(u_1, u_2, v_1, v_2, v_3)]q + \psi_1(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + E_1(u_1, u_2, v_1, v_2, v_3) \,, \\ F(q, y, u_1 v_2) &= a(u_1, u_2, v_1, v_2, v_3) q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + (A_2 + B)(u_1, u_2, v_1, v_2, v_3)]q \\ &\quad + \psi_2(y, u_1, u_2, v_1, v_2, v_3) + E_2(u_1, u_2, v_1, v_2, v_3) \,, \\ F(q, y, u_2 v_1) &= a(u_1, u_2, v_1, v_2, v_3) q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + A_3(u_1, u_2, v_1, v_2, v_3)]q + \phi_1(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + F_1(u_1, u_2, v_1, v_2, v_3) \,. \end{aligned} \right.$$

$$a(u_1, u_2, v_1, v_2, v_3) \equiv a \text{ constant} = a \,. \tag{3.2}$$

Hence it follows that

$$\tag{3.3} \begin{aligned} &\psi(y, u_1, u_2, v_1, v_2, v_3) + A_1(u_1, u_2, v_1, v_2, v_3) \\ &\quad \equiv a \text{ function of } y \text{ and } v_1 \text{ only} = \theta(y, v_1) \,, \end{aligned}$$

$$\psi(y, u_1, u_2, v_1, v_2, v_3) + A_1(u_1, u_2, v_1, v_2, v_3) + B(u_1, u_2, v_1, v_2, v_3) = \theta(y, v_2) , \tag{3.4}$$

$$\psi(y, u_1, u_2, v_1, v_2, v_3) + A_2(u_1, u_2, v_1, v_2, v_3) = \theta(y, u_1 v_1) , \tag{3.5}$$

$$\begin{aligned} &\psi(y, u_1, u_2, v_1, v_2, v_3) + A_2(u_1, u_2, v_1, v_2, v_3) + B(u_1, u_2, v_1, v_2, v_3) \\ &\quad = \theta(y, u_1 v_2) . \end{aligned} \tag{3.6}$$

From equations (3.3) to (3.6) we have

$$\theta(y, v_2) - \theta(y, v_1) = \theta(y, u_1 v_2) - \theta(y, u_1 v_1) \tag{3.7}$$

and

$$A_2(u_1, u_2, v_1, v_2, v_3) - A_1(u_1, u_2, v_1, v_2, v_3) = \theta(y, u_1 v_1) - \theta(y, v_1) . \tag{3.8}$$

For (3.7), by Lemma 2.1 there exists, for each fixed y, a function $\theta_1(\cdot, y)$ satisfying the Cauchy equation (2.9) and a constant $\theta_2(y)$ such that, we have

$$\theta(y, v) = \theta_1(v, y) + \theta_2(y) . \tag{3.9}$$

Now equations (3.8) and (3.9) yield

$$\theta_1(v, y) \equiv \text{a function of } v \text{ alone} = \theta_1(v) . \tag{3.10}$$

Thus we can rewrite the first equation of (3.1) as

$$\begin{aligned} F(q, y, v_1) &= aq \log q + [\theta_1(v_1) + \theta_2(y)]q \\ &\quad + (\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3) + D_1(u_1, u_2, v_1, v_2, v_3) . \end{aligned} \tag{3.11}$$

From (3.11) we see that $(\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3) + D_1(u_1, u_2, v_1, v_2, v_3)$ depends on y and v_1 only. Since ψ_1, ϕ_1 satisfy the Cauchy equation (2.9), $(\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3)$ and $D_1(u_1, u_2, v_1, v_2, v_3)$ depend on (y, v_1) and v_1 only respectively. Thus we can write (3.11) in the form

$$\begin{aligned} F(q, y, v) &= aq \log q + [\theta_1(v) + \theta_2(y)]q \\ &\quad + \alpha_1(y, v) + \alpha_2(v) , \end{aligned} \tag{3.12}$$

where θ_1 and α_1 $(\cdot, v)$ satisfy the Cauchy equation (2.9).

From the first, third, and fifth equations of (3.1) and (3.12) we have

$$\alpha_1(y, v_1) = \alpha_1(y, u_1 v_1) + \alpha_1(y, u_2 v_1) ,$$

for all $u_1, u_2, v_1 \in]0, 1[$ with $u_1 + u_2 \leqq 1$. Hence α_1 is independent of the second variable and we may write the equation (3.12) as

$$F(q, y, v) = aq \log q + [\theta_1(v) + \theta_2(y)]q + \alpha_1(y) + \alpha_2(v) , \tag{3.13}$$

for all $q \in [0, 1]$, $y, v \in]0, 1[$ where θ_1 and α_1 are solutions of the Cauchy equation (2.9). If we interchange the roles of the second and the third arguments of F in the above procedure we see that θ_2, α_2 are also solutions of the Cauchy equation (2.9).

Substituting (3.13) into (1.7), taking into account that θ_i, α_i are solutions of the Cauchy equation (2.9) we get $\alpha_i \equiv 0$. Thus we have proved the following theorem.

THEOREM 3.1. *Let* $F: [0, 1] \times]0, 1[\times]0, 1[\to \boldsymbol{R}$ *be measurable in its first variable. Then* F *satisfies the functional equation* (1.7) *if and only if* F *has the form*

$$F(q, y, v) = aq \log q + [\theta_1(v) + \theta_2(y)]q \,, \tag{3.14}$$

where $\theta_1, \theta_2:]0, \infty[\to \boldsymbol{R}$ *satisfy the Cauchy equation* (2.9).

COROLLARY 3.1. *Let* $F: ([0, 1] \times]0, 1[\times]0, 1[) \cup \{(0, 0, [0, 1[)\} \cup \{(1, 1,]0, 1])\} \cup \{(0, [0, 1[, 0)\} \cup \{(1,]0, 1], 1)\} \to \boldsymbol{R}$ *be measurable in its first variable. Then it satisfies the equation* (1.7) *if and only if* F *has the form given by* (3.14) *on* $[0, 1] \times]0, 1[\times]0, 1[$ *and on the boundary* $F(0, 0, \cdot) \equiv 0$, $F(1, 1, \cdot) = \theta_1(\cdot)$, $F(0, \cdot, 0) \equiv 0$ and $F(1, \cdot, 1) = \theta_2(\cdot)$.

REMARK. The measures H_n, I_n, D_n in (1.2), (1.3), (1.4) possess in particular properties: (a) *Symmetry*: H_n, I_n, D_n are symmetric in the pairs (p_i, q_i), (p_i, q_i), (p_i, q_i, r_i) respectively, (b) *Expansibility*: If $P = (p_1, p_2, \cdots, p_n)$, $Q = (q_1, q_2, \cdots, q_n)$, $R = (r_1, r_2, \cdots, r_n)$ and $P' = (p_1, p_2, \cdots, p_n, 0)$, $Q' = (q_1, q_2, \cdots, q_n, 0)$, $R' = (r_1, r_2, \cdots, r_n, 0)$, then $H_n(P \| Q) = H_{n+1}(P' \| Q')$, $I_n(P \| Q) = I_{n+1}(P' \| Q')$ and $D_n(P \| Q | R) = D_{n+1}(P' \| Q' | R')$, (c) *Branching*: If $P = (p_1, p_2, \cdots, p_n)$, $Q = (q_1, q_2, \cdots, q_n)$, $R = (r_1, r_2, \cdots, r_n)$ and $P' = (p_1 + p_2, p_3, \cdots, p_n)$, $Q' = (q_1 + q_2, q_3, \cdots, q_n)$ and $R' = (r_1 + r_2, r_3, \cdots, r_n)$, then $H_n(P \| Q) - H_{n-1}(P' \| Q')$, $I_n(P \| Q) - I_{n-1}(P' \| Q')$ and $D_n(P \| Q | R) - D_{n-1}(P' \| Q' | R)$ depend on (p_1, p_2, q_1, q_2), (p_1, p_2, q_1, q_2) and $(p_1, p_2, q_1, q_2, r_1, r_2)$ respectively. It is shown by C. T. Ng [14] that these three properties are equivalent to the representability of H_n, I_n, D_n in the form $H_n(P \| Q) = \sum_{i=1}^n f(p_i, q_i)$, $I_n(P \| Q) = \sum_{i=1}^n g(p_i, q_i)$ and $D_n(P \| Q | R) = \sum_{i=1}^n h(p_i, q_i, r_i)$ where f, g, h are any function satisfying $f(0, 0) = g(0, 0) = h(0, 0, 0) = 0$. From these representations, the additivity property of these measures motivates the study of the functional equations (1.6) and (1.7).

The Theorems 2.2 and 3.1 lead to a characterization of directed divergence and inaccuracy and of generalized directed divergence respectively. These three measures are determined by (a) Symmetry, (b) Expansibility, (c) Branching, (d) Additivity, and (e) Regularity conditions such as Lebesgue measurability and appropriate initial conditions.

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