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A UNIFIED APPROACH TO BOUNDARY VALUE PROBLEMS ON COMPACT INTERVALS

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Let L be a formal differential operator of order n and consider L as an operator from $C^n([a, b]) \subset L^2([a, b])$ into $L^2([a, b])$. Let $\{\eta_1, \cdots, \eta_l\}$ be a set of linear functionals defined on $C^n([a, b])$ with the property that each $\eta_j T$, $j = 1, \cdots, l$, is continuous, where T is a continuous right inverse of L. Let M be the set of all $f \in C^n([a, b])$ such that $\eta_j(f) = 0$, $1 \leqq j \leqq l$, and N be the set of all $f \in M$ such that $Lf = 0$. It is shown that the inverse of L from $L(M)$, the image of M under L, into $M \cap N^\perp$ is a compact operator and can be represented as an integral operator. In particular, if $l = n$ and $\{\eta_j\}$ is linearly independent, the inverse of L maps $C([a, b])$ onto M and it is compact. The Hilbert-Schmidt expansion theorem is generalized to these inverse operators when L is self-adjoint on M.

The purpose of this paper is to put the homogeneous boundary value problems of ordinary differential equations in an abstract setting, so that the properties which make the whole analysis go through become transparent. We replace the usual boundary condition with a linear functional η defined on $C^n([a, b])$, where n is the order of the differential equation. This has been done by many authors, but their main purpose was to facilitate the notation and, with the exception of Calkin in [1] and Dunford and Schwartz in [2], the topological property of η was never used. In this paper, however, the continuity of ηT, where T is an operator whose construction will be given later, is essential. It is this property which will make the integral representation of the inverse of a differential operator possible. Once the integral representation is obtained, we can generalize the Hilbert-Schmidt expansion theorem to this inverse operator when the differential operator is self-adjoint on M where M is the set of all functions in $C^n([a, b])$ satisfying the boundary conditions. The integral representation of the inverse of a differential operator has been obtained almost always through the use of the Green's function associated with the boundary value problem. But it is easy to find an example for which the Green's function does not exist. Even in such a case, however, we can construct an inverse of a differential operator L from the range $L(M)$ of L on M into $M \cap N^\perp$, where N is the null space of L in M, and this is the operator for which we obtain an integral representation and prove that it is compact.

2. Notations. Throughout the paper L denotes the operator defined by

$$(Lf)(x) = \sum_{k=0}^{n} P_k(x) f^{(n-k)}(x) , \qquad f \in C^n([a, b]) , \tag{1}$$

where each $P_k(x)$ is a continuous function on the closed finite interval $[a, b]$ and $P_0(x) \neq 0$ for any $x \in [a, b]$. We regard L as a mapping from $C^n([a, b]) \subset L^2([a, b])$ into $C([a, b]) \subset L^2([a, b])$, and whenever continuity is mentioned, it is with respect to the norm of $L^2([a, b])$. All the functions are restrictions to $[a, b]$. Thus we simply write $Lf = 0$ meaning $(Lf)(x) = 0$ for all $x \in [a, b]$.

We set

$$S = \{f \in C^n([a, b]) \mid Lf = 0\} . \tag{2}$$

It is a classical result that S is an n-dimensional linear space.

3. Generalized homogeneous boundary value problems. We first of all construct an operator T from $C([a, b])$ into $C^n([a, b])$ having the property $LTf = f$ for all $f \in C([a, b])$. For this purpose we take a basis $\{y_j(x)\}$ of S and consider the system of equations

$$\sum_{k=1}^{n} a_k(x) y_k^{(j-1)}(x) = \delta_{jn} f(x)/P(x) , \qquad 1 \leqq j \leqq n , \tag{3}$$

where $\delta_{jn} = 0$ if $j \neq n$ and $\delta_{nn} = 1$. Since $\{y_j\}$ is a basis for S and $P_0(x) \neq 0$ for any $x \in [a, b]$, the Wronskian $W(x)$ of $\{y_j\}$ does not vanish at any point of $[a, b]$. Hence we can solve the system (3) for $a_k(x)$ and obtain

$$a_j(x) = F_j(x) f(x) , \qquad 1 \leqq j \leqq n , \tag{4}$$

in which each F_j is of the form $Q_j(x)[P_0(x) W(x)]^{-1}$ where $Q_j(x)$ is a polynomial in $y_j^{(k-1)}(x)$, $1 \leqq j \leqq n$, $1 \leqq k \leqq n$. Since $W(x) \neq 0$ for any $x \in [a, b]$, it follows each F_j is continuous on $[a, b]$. We define T by

$$(Tf)(x) = \int_a^x \left[\sum_{j=1}^{n} y_j(x) F_j(\xi) \right] f(\xi) d\xi , \qquad f \in C[a, b] . \tag{5}$$

From (3) and (4) we obtain

$$\sum_{k=1}^{n} y_k^{(j-1)}(x) F_k(x) = \delta_{jn}/P_0(x) , \qquad 1 \leqq j \leqq n . \tag{6}$$

From (5) and (6) we obtain the following properties of T: For each $f \in C([a, b])$,

$$Tf \in C^n([a, b]) \tag{7}$$

$$
\begin{aligned}
D^k(Tf)(x) &= \int_a^x \Big[\sum_{j=1}^{n} y_j^{(k)}(x)F_j(\xi)\Big] f(\xi)d\xi \\
D^n(Tf)(x) &= f(x) + \int_a^x \Big[\sum_{j=1}^{n} y_j^{(n)}(x)F_j(\xi)\Big] f(\xi)d\xi
\end{aligned} \tag{8}
$$

$$
LTf = f\,. \tag{9}
$$

The clue to the generalization to the standard boundary value problems is given by the following very simple observation:

LEMMA 1. *Let T be the operator defined by* (5) *and* $c \in [a, b]$. *Let $\zeta_{k,c}$ be the linear functional defined by $\zeta_{k,c}(f) = f^{(k)}(x)$, $f \in C^n([a, b])$, $0 \leqq k \leqq n-1$. Then each $\zeta_{k,c}T$, $0 \leqq k \leqq n-1$, is continuous, that is, there exists a constant A_k such that*

$$
|\zeta_{k,c}(Tf)| \leqq A_k \|f\|\,, \qquad f \in C([a, b])\,,
$$

where $\| \ \|$ denotes the norm of $L^2([a, b])$.

From the lemma it follows that the linear functional η occurring in the usual boundary value problems all have the property that ηT is continuous.

LEMMA 2. *Let T_0 be an operator having the following properties*:
(i) $LT_0 f = f$ *for all* $f \in C([a, b])$,
(ii) $\|T_0 f\| \leqq A \|f\|$ *for all* $f \in C([a, b])$.

If η is a linear functional such that ηT_0 is continuous, then ηT is also continuous.

Proof. Let $\{z_1, z_2, \cdots, z_n\}$ be an orthonormal basis for S. Then $Tf - T_0 f \in S$ and in fact $Tf - T_0 f = \sum_{j=1}^n (Tf - T_0 f, z_j)z_j$. Hence,

$$
\eta Tf = \eta T_0 f + \sum_{j=1}^{n} (Tf - T_0 f, z_j)\eta(z_j)
$$

from which the assertion follows.

DEFINITION 1. *Let T be the operator defined by* (5). *A linear functional η defined on $C^n([a, b])$ is said to be a boundary functional for L if it has the property that ηT is continuous* (with respect to the norm of $L^2([a, b])$).

DEFINITION 2. A set $\{\eta_1, \cdots, \eta_l\}$ of boundary functionals for L is said to be linearly independent if they are linearly independent as duals on S, that is, if $\sum_{j=1}^l \alpha_j \eta_j(f) = 0$ for all $f \in S$ implies $\alpha_j = 0$, $1 \leqq j \leqq l$, where S is the set defined in (2), or equivalently, if for any basis $\{f_j\}$ of S, the rank of the matrix $[\eta_i(f_j)]$ is l.

As a direct generalization of the usual homogeneous boundary value problem, we have

THEOREM 1. *Let $\{\eta_1, \cdots, \eta_n\}$ be a set of linearly independent boundary functionals for L and let*

$$M = \{f \in C^n([a, b]) \mid \eta_j(f) = 0\ , \qquad 1 \leqq j \leqq n\}\ . \tag{10}$$

Then the inverse K of L from $C([a, b])$ into M exists, and it is compact. Moreover, there exists a function $K(x, \xi)$ having the following properties:

$$(Kf)(x) = \int_a^b K(x, \xi) f(\xi) d\xi\ , \qquad f \in C([a, b])\ ; \tag{11}$$

for each $x \in [a, b]$

$$K(x, \xi) \in L^2([a, b])\ ; \tag{12}$$

and

$$\int_a^b |\, K(x, \xi)\,|^2 d\xi \leqq B^2 \quad \textit{for all} \quad x \in [a, b] \tag{13}$$

for some constant B.

Proof. Let $\{y_1, \cdots, y_n\}$ be a basis for S. It is straightforward to show that L is one-to-one on M.

Since $\det [\eta_i(y_j)] \neq 0$, given $f \in C([a, b])$, there exist unique C_j, $1 \leqq j \leqq n$, such that

$$\sum_{j=1}^{n} \eta_i(y_j) C_j = -\eta_i(Tf)\ , \qquad 1 \leqq i \leqq n\ . \tag{14}$$

Moreover, each C_j is of the form

$$C_j = \sum_{k=1}^{n} \alpha_{jk} \eta_k(Tf) \tag{15}$$

where α_{kj} are constants which depend only on $\eta_i(y_j)$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n$. From (14) and (15) we obtain

$$\eta_i\left(\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} \eta_k(Tf) y_j + Tf\right) = 0, \qquad 1 \leqq i \leqq n\ . \tag{16}$$

We set

$$(T_0 f)(x) = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} \eta_k(Tf) y_j(x)\ , \qquad f \in C([a, b]) \tag{17}$$

and

$$Kf = T_0 f + Tf\,, \qquad f \in C([a, b])\,. \tag{18}$$

From (16), (17), and (18) we have $\eta_i(Kf) = 0$, $1 \leqq i \leqq n$, $f \in C([a, b])$, so that $Kf \in M$ for all $f \in C([a, b])$. Moreover, from (9), (17), and (18), we see that $L(Kf) = f$ for all $f \in C([a, b])$. The compactness of K follows from the fact that T_0 is a continuous operator with finite dimensional range, so that T_0 (along with T) is also compact. From the hypothesis, for each j, $1 \leqq j \leqq n$, there exists a constant A_j such that

$$|\eta_j(Tf)| \leqq A_j \|f\|\,, \qquad f \in C([a, b])\,.$$

Since $C([a, b])$ is dense in $L^2([a, b])$, we can extend the continuous linear functional $f \to \eta_j(Tf)$ continuously to $L^2([a, b])$ with the same bound A_j. Hence it follows that there exists $G_j \in L^2([a, b])$ such that

$$\eta_j(Tf) = \int_a^b G_j(\xi) f(\xi) d\xi\,, \qquad f \in C([a, b]) \tag{19}$$

and

$$\int_a^b |G_j(\xi)|^2 d\xi \leqq A_j^2\,. \tag{20}$$

Substituting (19) in (17), we obtain

$$(T_0 f)(x) = \int_a^b \left[\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} y_i(x) G_j(\xi)\right] f(\xi) d\xi\,, \qquad f \in C([a, b])\,. \tag{21}$$

We set

$$J_0(x, \xi) = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} y_i(x) G_j(\xi) \tag{22}$$

$$\begin{aligned} J(x, \xi) &= \sum_{j=1}^n y_j(x) F_j(\xi) \quad \text{if} \quad a \leqq \xi \leqq x \\ &= 0 \quad \text{if} \quad \xi > x\,, \end{aligned} \tag{23}$$

$$K(x, \xi) = J_0(x, \xi) + J(x, \xi)\,. \tag{24}$$

Then, from (18), (5), (23), (21), (22), and (24) we obtain

$$(Kf)(x) = \int_a^b K(x, \xi) f(\xi) d\xi\,, \qquad f \in C([a, b])\,.$$

From (20), (22), (23), and (24) and from the fact that y_j and F_j, $1 \leqq j \leqq n$, are continuous functions on the compact set $[a, b]$, it follows that $K(x, \xi)$ has the desired properties (12) and (13).

We next consider the case in which the number of boundary functionals is not necessarily n and may not be linearly independent.

THEOREM 2. *Let* $\{\eta_j\}_1^l$ *be a finite collection of boundary functionals for* L. *Let*

$$\begin{aligned} M_0 &= \{f \in C^n([a, b]) \mid \eta_j(f) = 0 , \quad 1 \leqq j \leqq l\} \\ N &= M_0 \cap S = \{f \in M_0 \mid Lf = 0\} \\ M_1 &= M_0 \cap N^\perp = \{f \in M_0 \mid (f, g) = 0 \text{ for all } g \in N\} \\ L(M_0) &= \{f \mid f = Lu \text{ for some } u \in M_0\}. \end{aligned}$$

Then there exists an operator K_1 *from* $L(M_0)$ *into* M_1 *such that*

(i) $LK_1 f = f$ *for all* $f \in L(M_0)$,

(ii) K_1 *is compact on* $L(M_0)$,

(iii) *there exists a function* $K_1(x, \xi)$ *such that* $(K_1 f)(x) = \int_a^b K_1(x, \xi) f(\xi) d\xi$, $f \in L(M_0)$; *for each* $x \in [a, b]$, $K_1(x, \xi) \in L^2([a, b])$, *and* $\int_a^b |K_1(x, \xi)|^2 d\xi \leqq B_1^2$ *for all* $x \in [a, b]$ *for some constant* B_1.

Proof. Let $\dim N = p$. Let $\{g_j\}$ be an orthonormal basis for N and let

$$\eta_{l+j}(f) = (f, g_j) , \quad f \in L^2([a, b]) , \quad 1 \leqq j \leqq p .$$

Then clearly η_{l+j}, $1 \leqq j \leqq p$, are boundary functionals for L. Moreover,

$$M_1 = \{f \in C^n([a, b]) \mid \eta_j(f) = 0 , \quad 1 \leqq j \leqq l + p\} .$$

Since $M_0 = M_1 \oplus N$ and N is the null space of L on M_0, L is one-to-one on M_1, so that if a solution in M_1 of the equation $Ly = f$ does exist, it is unique. Let $\{y_i\}$ be a basis for S. Then if $f \in L(M_0) = L(M_1)$, there exist unique C_j, $1 \leqq j \leqq n$, such that

$$\sum_{j=1}^{n} C_j y_j(x) + (Tf)(x)$$

is the solution of $Ly = f$ in M_1, that is,

$$L\Big(\sum_{j=1}^{n} C_j y_j + Tf\Big) = f \tag{25}$$

and

$$\eta_i\Big(\sum_{j=1}^{n} C_j y_j + Tf\Big) = 0 , \quad 1 \leqq i \leqq l + p . \tag{26}$$

From (26) it is seen that each C_j is of the form

$$C_j = \sum_{k=1}^{l+p} \alpha_{jk} \eta_k(Tf)$$

where $\alpha_{jk} \in C$ depend only on $\eta_j(y_k)$, $1 \leqq j \leqq l + p$, $1 \leqq k \leqq n$. Let

$$(K_1f)(x) = \sum_{i=1}^{n}\sum_{j=1}^{l+p} \alpha_{ij}\eta_j(Tf)y_i(x) + (Tf)(x)\,, \quad f \in L(M_0)\,. \tag{27}$$

Then from (25), (26), and (27) we have $K_1 f \in M_1$ and $L(K_1 f) = f$ for all $f \in L(M_0)$. The proof that K_1 is compact and that it can be represented as an integral operator satisfying (i)–(iii) is the same as in the proof of Theorem 1.

4. Generalized Hilbert-Schmidt theorems. We get back to Theorem 1. We need to extend the operator K to $L^2([a, b])$ and establish some elementary facts before we can state the next theorem.

Since each $\eta_j T$ is a linear functional continuous on the dense set $C([a, b])$, it has a unique continuous extension to $L^2([a, b])$. Hence T_0 has a unique continuous extension $\hat{T}_0$ to $L^2([a, b])$. We observe that

$$\hat{T}_0 f \in C^n([a, b]) \quad \text{for all} \quad f \in L^2([a, b])\,. \tag{28}$$

The integral defining T makes sense even if $f \in L^2([a, b])$, and denoting this extended operator by $\hat{T}$, it is easily seen from (5) that

$$\hat{T} f \in C([a, b]) \quad \text{for all} \quad f \in L^2([a, b])\,. \tag{29}$$

Let

$$\hat{K} = \hat{T}_0 + \hat{T}\,. \tag{30}$$

Then $\hat{K}$ is the continuous extension of K and $\hat{K}$ is compact on $L^2([a, b])$. From (28) and (29), we have

$$\hat{K} f \in C([a, b]) \quad \text{for all} \quad f \in L^2([a, b])\,. \tag{31}$$

With these definitions and notations, we have the following lemma.

LEMMA 3. *Let λ be a nonzero eigenvalue of $\hat{K}$ and φ be the corresponding eigenfunction. Then $\varphi \in C^n([a, b])$, and in fact $\varphi \in M$, where M is defined by* (10).

Proof. By definition, $\hat{K}\varphi = \lambda\varphi$ and $\varphi \in L^2([a, b])$. Hence from (31) we have $\hat{K}_\varphi \in C([a, b])$. Since $\lambda \neq 0$, it follows $\varphi \in C([a, b])$. But this means that $\hat{K}_\varphi = K_\varphi \in M$, so that $\varphi \in M$.

From this lemma it follows that every eigenfunction of $\hat{K}$ corresponding to a nonzero eigenvalue is also an eigenfunction of L.

We now prove the Hilbert-Schmidt theorem for the inverse operator of Theorem 1 when L is self-adjoint on M, that is, when $(Lf, g) = (f, Lg)$ for all $f, g \in M$. (For the conditions as to what differential operators can be self-adjoint, see Naimark [3].) Before we state the theorem, however, we observe that if L is self-adjoint on M, K is self-adjoint on $C([a, b])$ and so $\hat{K}$ is self-adjoint on $L^2([a, b])$. Moreover, since the range of $\hat{K}$ contains an infinite dimensional subspace M, it follows that $\hat{K}$ has infinitely many eigenvalues. With these and the elementary properties of self-adjoint compact operators on a Hilbert space in mind, we state the theorem.

THEOREM 3. *Let L have the additional property that it is self-adjoint on M. Let $\{\varphi_j\}$ be the complete system of orthonormal eigenfunctions of $\hat{K}$ corresponding to nonzero eigenvalues, and λ_j be the eigenvalue of $\hat{K}$ corresponding to φ_j. Then for each $f \in L^2([a, b])$,*

$$\hat{K}f = \sum_{j=1}^{\infty} (\hat{K}f, \varphi_j)\varphi_j = \sum_{j=1}^{\infty} \lambda_j (f, \varphi_j)\varphi_j \tag{32}$$

and the series converges to $\hat{K}f$ uniformly on $[a, b]$.

Proof. The fact that the series converges to $\hat{K}f$ in $L^2([a, b])$ follows from the theory of self-adjoint compact operators on a Hilbert space.

Using the Cauchy's inequality, we obtain

$$\left| \sum_{j=1}^{m} (f, \varphi_j)\lambda_j\varphi_j(x) \right| \leqq \left[\sum_{j=1}^{m} |(f, \varphi_j)|^2 \right]^{1/2} \left[\sum_{j=1}^{m} \lambda_j^2 |\varphi_j(x)|^2 \right]^{1/2} .$$

Since $\lambda_j\varphi_j(x) = \int_a^b K(x, \xi)\varphi_j(\xi)d\xi$, fixing $x \in [a, b]$, recalling (12) and (13), and applying the Bessel's inequality to the function $K(x, \xi)$, we obtain

$$\sum_{j=1}^{\infty} \lambda_j^2 |\varphi_j(x)|^2 \leqq \int_a^b |K(x, \xi)|^2 d\xi \leqq B^2 .$$

Hence,

$$\left| \sum_{j=l}^{m} (f, \varphi_j)\lambda_j\varphi_j(x) \right| \leqq B\left[\sum_{j=l}^{m} |(f, \varphi_j)|^2 \right]^{1/2}$$

for all $x \in [a, b]$, from which we see that the sequence of the partial sums $S_l(x)$ of the series in (32) is uniformly Cauchy on $[a, b]$. Each φ_j is continuous by Lemma 3, so that $S_l \in C([a, b])$ for each l. Hence the limit of the partial sums $S_l(x)$ is also continuous on $[a, b]$. But $\{S_l\}$ converges to $\hat{K}f$ in $L^2([a, b])$ and $\hat{K}f \in C([a, b])$ by (31). Hence the function to which $\{S_l(x)\}$ converges uniformly must be $\hat{K}f$, which completes the proof.

As an immediate consequence of Theorem 1, Theorem 3, and Lemma 3, we have the following theorem:

THEOREM 4. *Let* $\{\eta_1, \cdots, \eta_n\}$ *be a linearly independent set of boundary functionals for* L *and let* L *be self-djoint on* $M = \{f \in C^n([a, b]) \mid \eta_j(f) = 0,\ 1 \leqq j \leqq n\}$. *Let* $\{\varphi_j\}$ *be the complete system of orthonormal eigenfunctions of* L *and* μ_j *be the eigenvalue corresponding to* φ_j. *Then for every* $f \in C([a, b])$ *the solution of the generalized boundary value problem*

$$Ly = f$$
$$\eta_j(y) = 0\ , \qquad 1 \leqq j \leqq n$$

exists and can be represented by the series

$$\sum_{j=1}^{\infty} \mu_j^{-1}(f, \varphi_j)\varphi_j$$

the series converging to the solution uniformly on $[a, b]$.

In order to facilitate the statements of the following two theorems, we state conditions and definitions used in the theorems.

(i) $\{\eta_1, \cdots, \eta_l\}$, $1 \leqq l \leqq n$, *is* a set of boundary functionals for L, in which $\{\eta_1, \cdots, \eta_r\}$ is linearly independent in the sense of Definition 2.

(ii) $M_0 = \{f \in C^n([a, b]) \mid \eta_j(f) = 0,\ 1 \leqq j \leqq l\}$.

(iii) L is self-adjoint on M_0.

(iv) $N = M_0 \cap S$; $\dim N = p = n - r$; $q = l - r$.

(v) $\{\psi_1, \cdots, \psi_p\}$ is an orthogonal basis for N.

(vi) $\zeta_j(f) = (f, \psi_j)$, $f \in L^2([a, b])$, $1 \leqq j \leqq p$. Then each ζ_j is a boundary functional for L and the set $\{\eta_1, \cdots, \eta_r, \zeta_1, \cdots, \zeta_p\}$ is linearly independent in the sense of Definition 2.

(vii) $M_1 = \left\{f \in C^n([a, b]) \middle| \begin{array}{l} \eta_j(f) = 0,\ 1 \leqq j \leqq r \\ \zeta_k(f) = 0,\ 1 \leqq k \leqq p \end{array}\right\}$. Then by Theorem 1, there exists a compact operator K such that $LKf = f$ and $Kf \in M_1$ for all $f \in C([a, b])$.

(viii) $\{\varphi_j\}$ is the complete system of (orthonormal) eigenfunctions of L in $M_0 \cap M_1$ and μ_j is the eigenvalue corresponding to φ_j, that is, $\varphi_j \in M_0 \cap M_1$ and $L\varphi_j = \mu_j\varphi_j$.

THEOREM 5. *For every* $f \in C([a, b])$ *we have*

$$Kf = \sum_{j=1}^{\infty} \mu_j^{-1}(f, \varphi_j)\varphi_j + \sum_{j=1}^{q} (f, \psi_j)K\psi_j$$

and the series (together with the second sum) *converges to* Kf *uniformly on* $[a, b]$.

Proof. Since L is self-adjoint on $M_0 \cap M_1$, K is self-adjoint on $L(M_0 \cap M_1)$, the image of $M_0 \cap M_1$ under L. We want to show that

$L(M_0 \cap M_1)$ is invariant under K. We first show that $Kg \in M_0 \cap M_1$ for every $g \in C([a, b]) \cap N^{\perp}$. To this end we show that every $f \in C([a, b])$ can be expressed in the form

$$g + \sum_{j=1}^{q} (f, \psi_j)\psi_j$$

with $g \in L(M_0 \cap M_1)$.

Let $e_k = K\psi_k$, $1 \leqq k \leqq q$. We claim that the determinant of the $q \times q$ matrix $[\eta_{r+i}(e_j)]$ is not zero. Suppose the contrary. Then there exist $\alpha_j \in C$, not all zero, such that

$$\sum_{j=1}^{q} \eta_{r+i}(e_j)\alpha_j = 0 , \qquad 1 \leqq i \leqq q ,$$

or

$$\eta_{r+i}\left(\sum_{j=1}^{q} \alpha_j e_j\right) = 0, \qquad 1 \leqq i \leqq q .$$

Since $\sum_{j=1}^{q} \alpha_j e_j \in M_1$, it follows $\eta_k(\sum_{j=1}^{q} \alpha_j e_j) = 0$, $1 \leqq k \leqq l$, that is, $\sum_{j=1}^{q} \alpha_j e_j \in M_0$. Hence, $(L(\sum_{j=1}^{q} \alpha_j e_j), \psi) = 0$ for all $\psi \in N$ since L is self-adjoint on M_0 and $N \subset M_0$. Since $L(\sum_{i=1}^{q} \alpha_i e_i) = \sum_{j=1}^{q} \alpha_j \psi_j \in N$, it follows that $\sum_{j=1}^{q} \alpha_j \psi_j = 0$, which implies $\alpha_j = 0$, $1 \leqq j \leqq q$, contradicting the choice of α_j's. Hence, $\det [\eta_{r+i}(e_j)] \neq 0$. Let $f \in C([a, b])$. Then $Kf \in M_1$, and since $\det [\eta_{r+i}(e_j)] \neq 0$, there exists a unique set $\{\beta_j\}_1^q$, $\beta_j \in C$, such that

$$\sum_{j=1}^{q} \eta_{r+i}(e_j)\beta_j = \eta_{r+i}(Kf) , \qquad 1 \leqq i \leqq q ,$$

or

$$\eta_{r+i}\left(Kf - \sum_{j=1}^{q} \beta_j e_j\right) = 0 , \qquad 1 \leqq i \leqq q ,$$

from which we have

$$Kf - \sum_{j=1}^{q} \beta_j e_j \in M_0 \cap M_1 .$$

Let $g = f - \sum_{j=1}^{q} \beta_j \psi_j$. Then $Kg \in M_0 \cap M_1$ and we have

$$f = g + \sum_{j=1}^{q} \beta_j \psi_j .$$

Since $g = L(Kg) \in L(M_0 \cap M_1)$ and L is self-adjoint on M_0, we have $(g, \psi) = 0$ for all $\psi \in N$. Hence $(f, \psi_k) = (g, \psi_k) + \sum_{j=1}^{q} \beta_j(\psi_j, \psi_k) = \beta_k$, $1 \leqq k \leqq q$. Thus, for every $f \in C([a, b])$

$$f = g + \sum_{j=1}^{q} (f, \psi_j)\psi_j \tag{33}$$

with $Kg \in M_0 \cap M_1$. It follows immediately from (33) that

$$Kf \in M_0 \cap M_1 \quad \text{for all} \quad f \in C([a, b]) \cap N^- . \tag{34}$$

To prove the invariance of $L(M_0 \cap M_1)$ under K, let $g \in L(M_0 \cap M_1)$. Then $Kg \in M_0 \cap M_1$, and since $M_1 \subset N^-$, we have $Kg \in N^\perp$. Moreover, $Kg \in C([a, b])$. Hence from (34), $K(Kg) \in M_0 \cap M_1$ or $Kg \in L(M_0 \cap M_1)$. Thus $L(M_0 \cap M_1)$ is invariant under K.

We let:

K_0 be the restriction of K to $L(M_0 \cap M_1)$,

H_0 be the closure of $L(M_0 \cap M_1)$ in $L^2([a, b])$,

$\hat{K}_0$ be the continuous extension of K_0 to H_0.

Then $\hat{K}_0$ is a compact self-adjoint operator on the Hilbert space H_0. We next show that if $\varphi \in H_0$ and $\lambda \neq 0$ such that $\hat{K}_0\varphi = \lambda\varphi_0$, then $\varphi \in L(M_0 \cap M_1)$.

Since K is representable as an integral operator by Theorem 1, $\hat{K}_0$ is representable as an integral operator and consequently just as in the case of Theorem 1 we can show that

$$\hat{K}_0 f \in C([a, b]) \quad \text{for all} \quad f \in H_0 ,$$

so that $\varphi = \lambda^{-1}K_0\varphi \in C([a, b])$. Hence by (33) we can write $\varphi = g + \sum_{j=1}^{q} (\varphi, \psi_j)\psi_j$ with $g \in L(M_0 \cap M_1)$. But $(f, \psi) = 0$ for all $f \in L(M_0 \cap M_1)$ and for all $\psi \in N$. It follows from this that $(f, \psi) = 0$ for all $f \in H_0$ and for all $\psi \in N$. Hence $(\varphi, \psi_j) = 0$, $1 \leqq j \leqq q$, so that $\varphi = g \in L(M_0 \cap M_1)$. Hence $K\varphi = K_0\varphi = \hat{K}_0\varphi = \lambda\varphi$ and so $L\varphi = \lambda^{-1}\varphi$. It follows from this that $\{\varphi_j\}$ is also the complete system of (orthonormal) eigenfunctions of K_0 corresponding to the nonzero eigenvalues of K_0. Hence by the same argument as in the proof of Theorem 3, for every $f \in H_0$

$$\hat{K}_0 f = \sum_{j=1}^{\infty} (\hat{K}_0 f, \varphi_j)\varphi_j = \sum_{j=1}^{\infty} \mu_j^{-1}(f, \varphi_j)\varphi_j$$

the series converging to $\hat{K}_0 f$ uniformly on $[a, b]$. Now let $f \in C([a, b])$ and

$$g = f - \sum_{j=1}^{q} (f, \psi_j)\psi_j .$$

Then from (33) $g \in L(M_0 \cap M_1)$, so that

$$\begin{aligned} Kf &= K_0 g + \sum_{j=1}^{q} (f, \psi_j)K\psi_j \\ &= \sum_{j=1}^{\infty} \mu_j^{-1}(g, \varphi_j)\varphi_j + \sum_{j=1}^{q} (f, \psi_j)K\psi_j \\ &= \sum_{j=1}^{\infty} \mu_j^{-1}(f, \varphi_j)\varphi_j + \sum_{j=1}^{q} (f, \psi_j)K\psi_j \end{aligned}$$

since $(\psi_k, \varphi_j) = 0$ for all k and j, and the series (together with the second sum) converges to Kf uniformly on $[a, b]$.

THEOREM 6. *Under the conditions* (i)–(viii), *for every* $f \in C([a, b]) \cap N^{\perp}$, *the generalized homogeneous boundary value problem*

$$Ly = f$$
$$\eta_j(y) = 0\,, \qquad 1 \leqq j \leqq l$$

has a unique solution in M_0 *and the solution can be represented by the series*

$$\sum_{j=1}^{\infty} \mu_j^{-1}(f, \varphi_j)\varphi_j$$

the series converging to the solution uniformly on $[a, b]$.

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