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# THE MAPPING CYLINDER AXIOM

# FOR WCHP FIBRATIONS

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**Given a fibre homotopy equivalence $\phi$ over $B$ of fibre spaces $E$ and $E'$ which have the Weak Covering Homotopy Property (WCHP), it is shown that $\phi$ has a generalized mapping cylinder which also possesses the WCHP. The ordinary topological mapping cylinder $M(\phi)$ as a space over $[0, 1] \times B$ does not necessarily possess the WCHP.**

The existence of such a generalized mapping cylinder becomes the fourth of four axioms defining the concept of a "fibration theory."

A simple example of a fibre mapping $\phi$ whose mapping cylinder $M(\phi)$ does not have path lifting properties is given by the fibre map

$$\phi : \{1, 2\} \times B \to \{1, 2\} \times B$$

$$(n, b)\phi = (1, b)$$

over $B$. However, Tulley in the *Duke Mathematical Journal* (1969) has shown that $M(\phi)$ is a fibration if $E$ is compact and $\phi$ is a strong deformation fibre retraction of $E$ onto $E'$.

**2. Structure theories.** A *structure theory* is an assignment of a category $\mathscr{E}(B)$ to each topological space $B$ and of a covariant functor $f^* : \mathscr{E}(C) \to \mathscr{E}(B)$ to each continuous map $f$ of $B$ into $C$, the assignment being such that $id^*$ is the identity functor and $(fg)^*$ is $g^*f^*$ whenever $f$ and $g$ are continuous maps such that $fg$ is defined.

An example of a structure theory is the theory $\mathscr{E}_S$, where each $\mathscr{E}_S(B)$ is the category of "spaces over $B$". An object in $\mathscr{E}_S(B)$ is a continuous map $p$ of a topological space $E$ into $B$. Such an object is denoted by $(E, p)$ or more usually just by $E$. A morphism in the category $\mathscr{E}_S(B)$ from an object $(E, p)$ to an object $(E', p')$ is just a continuous map $\phi : E \to E'$ such that $p'\phi = p$. This is the usual theory of "fibre spaces" without any special path lifting properties. To complete the description of $\mathscr{E}_S$ one needs to construct the functor $g^*$ when given a continuous map $g : B \to C$. Of course, $g^*$ operating on a space $(E, p)$ over $C$ simply gives the usual induced space $(E_g, p_g)$ over $B$ where:

$$E_g = \{(b, e) \in B \times E : g(b) = p(e)\}$$

$$p_g(b, e) = b.$$

If $\phi$ is a morphism in $\mathscr{E}_S(C)$, then $g^*\phi$ is just the appropriate restriction of $id_B \times \phi$.

A more interesting structure theory is the subtheory $\mathscr{E}_W$ of $\mathscr{E}_S$ where the objects in $\mathscr{E}_W(B)$ are those objects of $\mathscr{E}_S(B)$ which have the Weak Covering Homotopy Property (WCHP) [2]. We say that an object $(E, p)$ in $\mathscr{E}_S(B)$ has the WCHP provided that for every topological space $X$ and every homotopy $H$ of $X$ into $B$ such that $H/[0, 1/2] \times X$ is some constant homotopy $[0, 1/2] \times H_0$ and for every map $g : X \to E$ which "lifts" $H_0$ (i.e., $pg = H_0$), there exists an extension $G$ of $\{0\} \times g$,

$$G : I \times X \to E,$$

which "covers" $H$ — i.e. $pG = H$.

The morphisms in $\mathscr{E}_W(B)$ are just those of $\mathscr{E}_S(B)$ which are between objects in $\mathscr{E}_W(B)$. To see that $\mathscr{E}_W$ is a structure theory, it is only necessary to show that if $f$ is a continuous map from $A$ into $B$ and $E$ is an object in $\mathscr{E}_W(B)$, then $f^*E$ is in $\mathscr{E}_W(A)$. This is shown by Proposition 5.14 of Dold [2].

Many of the "fibre-wise" concepts of fibre space theory can be easily adapted to a structure theory $\mathscr{E}$. For example if $E$ is an object (or morphism) in $\mathscr{E}(B)$ and $A$ is a subset of $B$ then by "$E$ restricted to $A$", written $E/A$, one means $j^*E$ where $j$ is the injection of $A$ into $B$. Of course $E$ is called an *extension* of $E/A$; and two extensions of $E/A$ are said to *agree over* $A$.

Again given $E$, if $C$ is another topological space then the *product* $C \times E$ can be defined to be $pr^*E$ where $pr$ is the projection of $C \times B$ onto $B$. Notice that $C \times E$ is in $\mathscr{E}(C \times B)$ and not in $\mathscr{E}(B)$. Now let $I$ denote the unit interval of the real line. A *homotopy* (formerly fibre homotopy) $\psi$ over $B$ is a morphism in $\mathscr{E}(I \times B)$. The homotopy $\psi$ is said to be from $j_0^*\psi$ to $j_1^*\psi$ where $j_i$ is the injection map of $B$ onto $\{i\} \times B$ within $I \times B$. Another way of saying that a homotopy $\psi$ is from $\phi_0$ to $\phi_1$ is to say that $\psi$ restricted to $\{i\} \times B$ is $\{i\} \times \phi_i$, for $i = 0, 1$.

Once the concept of homotopy has been introduced, one may immediately speak of morphisms that are *homotopy equivalences* and of one morphism being the *homotopy inverse* of another and of one object being *homotopy equivalent* to another.

In [3] Tulley presents a concept similar to that of homotopy equivalence of two objects $E$ and $E'$ in $\mathscr{E}(B)$. A *connection* from $E$ to $E'$ is an object $H$ in $\mathscr{E}(I \times B)$ such that $j_0{}^*H$ is $E$ and $j_1{}^*H$ is $E'$.

**3. Fibration theories.** Structure theories in general do not have enough properties to be very interesting. However if the four axioms below are added, a theory is obtained in which many of the usual fibre space results can be obtained. Note that a *numerable* covering is one which is refined by the supports of some locally finite partition of unity.

DEFINITION 1. A structure theory $\mathscr{E}$ is said to be a *fibration theory* if it enjoys the following four axioms.

AXIOM I. Let $\{U_i\}$ be a numerable open covering of a space $B$ and $\{E_i\}$ a system of objects (or morphisms) over each $U_i$ such that each $E_i$ agrees with each $E_j$ over $U_i \cap U_j$. Then there is a unique common extension of the $E_i$ to $B$.

AXIOM II. Let $\{U_i\}$ be an open numerable covering of a space $B$ and let $\phi$ be a morphism in $\mathscr{E}(B)$ such that each restriction $\phi/U_i$ is a homotopy equivalence. Then $\phi$ is a homotopy equivalence.

AXIOM III. If there is a connection from $E$ to $E'$, where $E$ and $E'$ are two objects in $\mathscr{E}(B)$, then $E$ is homotopic to $E'$.

AXIOM IV. (Mapping Cylinder Axiom) Let $\phi$ be a homotopy equivalence in $\mathscr{E}(B)$ from $E$ to $E'$. Then there is an object $M(\phi)$ in $\mathscr{E}(I \times B)$ which serves as a mapping cylinder for $\phi$. By this it is meant that $M(\phi)$ is a connection from $E$ to $E'$ and that there is a *characterizing* homotopy equivalence $\psi_M$ from $M(\phi)$ to $I \times E'$ which has the boundary relations $\psi_M/\{0\} \times B = \{0\} \times \phi$ and $\psi_M/\{1\} \times B = id$.

THEOREM 2. *The theory $\mathscr{E}_W$ of WCHP fibre spaces is a fibration theory.*

The proof of this theorem occupies the rest of the paper. Axioms I, II, and III are from the work of Dold [**2**; respectively Propositions 5.12, 3.3, and 6.6]. Axiom III is also equivalent to a well known property of fibrations. Namely if $f$ and $g$ are two homotopic maps of a space $A$ into a space $B$ and of $E$ is a WCHP fibration over $B$, then $f^*E$ and $g^*E$ are fiber homotopy equivalent (merely homotopy equivalent in the language of $\mathscr{E}_W$). To apply this to the situation of Axiom III note that the injection maps $j_0$ and $j_1$ of $B$

into $I \times B$ are homotopic. Hence if $H$ is a connection from $E$ to $E'$, then by definition $j_0{}^*H$ is $E$ and $j_1{}^*H$ is $E'$ while by the above property $j_0{}^*H$ is homotopy equivalent to $j_1{}^*H$. Conversely one can show:

PROPOSITION 3. *Let $\mathscr{E}$ be a fibration theory and $f$ and $g$ homotopic maps of a space $A$ into a space $B$. If $E$ is an object in $\mathscr{E}$ $(B)$ then $f^*E$ and $g^*E$ are homotopic.*

*Proof.* By assumption there is a homotopy $h : I \times A \to B$ which is from $f$ to $g$. Define $H$ to be $h^*E$. Clearly then $H$ is a connection from $j_0{}^*H = j_0{}^*h^*E = (hj_0)^*E = f^*E$ to $j_1{}^*H = j_1{}^*h^*E = (hj_1)^*E = g^*E$. Hence by Axiom III, $f^*E$ is homotopic to $g^*E$.

Axiom IV is important for two reasons. First of all it clearly provides a converse for Axiom III.

PROPOSITION 4. *Let $\mathscr{E}$ be a fibration theory. If two objects $E$ and $E'$ in $\mathscr{E}$ $(B)$ are homotopic then there is a connection from $E$ to $E'$.*

See Theorem 12 of Tulley [3] for a version of this proposition in the theory of Covering Homotopy Property fiber spaces. Axiom IV is also important for obstruction theory.

**4. Mapping cylinder axiom.** There is little difficulty in verifying Axiom IV for the theory $\mathscr{E}_S$. The difficulty arises in constructing mapping cylinders that possess the WCHP. In fact in the theory $\mathscr{E}_S$ we can construct a mapping cylinder for any morphism $\phi$ from $E$ to $E'$. Of course the characteristic morphism will not be a homotopy equivalence if the morphism $\phi$ is not. The mapping cylinder $M(\phi)$ is defined to be the quotient space of $I \times E \cup \{1\} \times E'$ under the relation $(1, e) \equiv (1, \phi(e))$. The equivalence class of a point $x$ in this quotient is denoted by $\{x\}$. The fiber projection map $p(\phi)$ to be used with $M(\phi)$ is defined by

$$p(\phi)\{t,x\} = \begin{cases} (t,p(x)) & \text{for } (t,x) \text{ in } I \times E \\ (1,p'(x)) & \text{for } (t,x) \text{ in } \{1\} \times E' \end{cases} .$$

The fact that $M(\phi)$ is a quotient space insures that $p(\phi)$ is well defined and continuous. The characteristic morphism $\psi_M$ is defined by:

$$\psi_M\{t,x\} = \begin{cases} (t,\varphi(x)) & \text{for } (t,x) \text{ in } I \times E \\ (t,x) & \text{for } (t,x) \text{ in } \{1\} \times E' \end{cases} .$$

Again this is well defined and continuous because $M(\phi)$ was a quotient space. It is straightforward to verify that $\psi_M$ is a morphism and that $M(\phi)$ and $\psi_M$ satisfy the conditions of Axiom IV.

The purpose of this paper is to establish Axiom IV for the subtheory $\mathscr{E}_W$. Suppose then that $\phi$ is a morphism between two objects which have the WCHP. There are two special cases in which the particular mapping cylinder $M(\phi)$ constructed above will also have the WCHP. In both cases this will be a result of showing that $\psi_M$ is a homotopy equivalence. For $I \times E'$ is induced from $E'$ and so has the WCHP and if $\psi_M$ is a homotopy equivalence, then the following version of Proposition 5.2 of [2] shows that $M(\phi)$ is also in $\mathscr{E}_W(I \times B)$.

PROPOSITION 5. *Let $(E, p)$ and $(E', p')$ be two objects in $\mathscr{E}_S(B)$ which are homotopic* (i.e., fiber homotopy equivalent). *If either of them has the WCHP, then the other does.*

**5. Special cases of the mapping cylinder axiom.** The two special cases when $M(\phi)$ has the WCHP are when $\phi$ is a strong deformation retract or when $\phi$ is a fibration in the theory $\mathscr{E}_S$. These terms are defined precisely below.

DEFINITION 6. Let $\phi$ be a morphism in $\mathscr{E}_S(B)$ from an object $E$ to an object $E'$. Then $\phi$ is a *strong deformation retract* (sdr) in $\mathscr{E}_S$ iff there is a morphism $\rho$ from $E'$ to $E$ such that $\rho\phi = id$ and a homotopy $\theta$ from the identity of $E'$ to $\phi\rho$ such that $\theta(I \times \phi) = I \times \phi$. The last condition insures that the homotopy $\theta$ is "constant" on the image space $\phi(E)$.

DEFINITION 7. Let $\pi$ be a morphism in $\mathscr{E}_S(B)$ from $E$ to $E'$. Then $\pi$ is an $\mathscr{E}_S$-fibration iff it possesses the following homotopy lifting property:

Suppose that $\phi$ is a morphism of an object $X$ in $\mathscr{E}_S(B)$ to $E$ which "lifts" one end of a homotopy $\rho : I \times X \to I \times E'$. That is to say that $\{0\} \times \pi\phi = \theta/\{0\} \times B$. Then $\{0\} \times \phi$ can be extended to a homotopy $\psi : I \times X \to I \times E$ which lifts $\theta$ i.e. $(I \times \pi)\psi = \theta$. If $\pi$ from $E$ to $E'$ is an $\mathscr{E}_S$-fibration, then $E'$ is called the *base* object and $E$ the *total* object.

PROPOSITION 8. *Let $\phi$ be a sdr in $\mathscr{E}_S$. Then the characteristic morphism $\psi_M$ of the mapping cylinder of $\phi$ is a homotopy equivalence.*

*Proof.* Let $\rho$ be the retraction for $\phi$ and $\theta$ the homotopy from $id$ to $\phi\rho$ which is "constant" on $\phi(E)$. Namely let:

$$\rho : E' \to E$$

$$\rho\phi = id$$

$$\theta/\{0\} \times B = id$$

$$\theta(I \times \phi) = I \times \phi.$$

$$\theta : I \times E' \to I \times E'$$

$$\theta/\{1\} \times B = \{1\} \times \phi\rho$$

One needs a homotopy inverse for $\psi_M$. This is defined by setting

$$\psi^-(t, e') = \{t, \rho(e')\}$$

$$\psi^- : I \times E' \to M(\phi).$$

To show that $\psi^-$ is a homotopy inverse of the morphism $\psi_M$, one must show that $\psi_M\psi^-$ and $\psi^-\psi_M$ are homotopic to the appropriate identities. It is clear from the definition that $\psi_M\psi^-$ is $I \times \phi\rho$ because $\psi^-$ is essentially $I \times \rho$. Let $pr_B$ denote the natural projection of $I \times B$ onto $B$. It is clear that the morphism $(I \times pr_B)^*\theta$ is a homotopy from the identity to $I \times \phi\rho$.

All that remains now is the construction of a homotopy from the identity of $M(\phi)$ to the morphism $\psi^-\psi_M$. First let us note that the composition $\psi^-\psi_M$ is given by the rules:

$$\psi^-\psi_M\{t, e\} = \{t, e\}$$

$$\psi^-\psi_M\{1, e'\} = \{1, \rho e'\} = \{1, \phi\rho e'\}.$$

Since $\theta$ is a homotopy from the identity of $E'$ to $\phi\rho$, the above formulas suggest that we define a homotopy

$$\gamma : I \times M(\phi) \to I \times M(\phi),$$

whose second coordinate $\gamma_t$ is specified by the rules:

$$\gamma_t\{s, e\} = \{s, e\}$$

$$\gamma_t\{1, e'\} = \{1, \theta_t e'\}$$

$$\gamma_t : M(\phi) \to M(\phi).$$

By $\theta_t$ we mean the second coordinate of the homotopy $\theta$ :

$$\theta(t, e') = (t, \theta_t e').$$

The only possible ambiguity in the definition of $\gamma_t$ comes from the fact that $\{1, e\}$ is $\{1, \phi e\}$. According to the above rules, the image of the latter is $\{1, \theta_t \phi e\}$. But by assumption $\theta(I \times \phi)$ is just $(I \times \phi)$ and hence $\theta_t \phi(e)$ is just $\phi(e)$. This means that the image is $\{1, e\} = \{1, \phi e\}$ in either case — so there is no ambiguity. This argument shows that $\gamma$ is well defined, while from the definitions, it can be seen that $\gamma$ is a morphism and thus a homotopy from the identity of $M(\phi)$ to $\psi^{-}\psi_M$.

More mechanisms are required to prove that the characteristic morphism $\psi_M$ is a homotopy equivalence when $\phi$ is a $\mathscr{E}_S$-fibration.

DEFINITION 9. Let $\pi$ and $\pi'$ be two $\mathscr{E}_S$-fibrations with the same base object $E_b$,

$$\pi : E \to E_b \qquad \pi' : E' \to E_b.$$

A *fiber morphism* from $\pi$ to $\pi'$ is a morphism $\phi$ from $E$ to $E'$ such that $\pi'\phi$ is $\pi$. Similarily if $\phi$ and $\phi'$ are two fibre morphisms from $\pi$ to $\pi'$, then a *$\mathscr{E}_S$-fibre homotopy* from $\phi$ to $\phi'$ is a homotopy:

$$\theta : I \times E \to I \times E'$$

from $\phi$ to $\phi'$ such that $(I \times \pi')\theta = (I \times \pi)$. Continuing as above, we can define all of the usual fibre concepts in the theory $\mathscr{E}_S$ rather than in the category of topological spaces. In particular we have the concept of a *$\mathscr{E}_S$-fibre homotopy equivalence* and of a *$\mathscr{E}_S$-fibre homotopy inverse.*

The above notation was introduced in order to state a generalization of Theorem 6.1 of Dold [2]. The proof is essentially that which appears in his paper. One only needs to be careful to perform all constructions in the theory $\mathscr{E}_S$ rather than in the category of topological spaces.

PROPOSITION 10. *Let $\pi$ and $\pi'$ be two $\mathscr{E}_S$-fibrations with the same base object:*

$$\pi : E \to E_b \qquad \textit{and} \qquad \pi' : E' \to E_b.$$

*If $\phi$ is a $\mathscr{E}_S$ fibre morphism from $\pi$ to $\pi'$ which is a homotopy equivalence in $\mathscr{E}_S$, then it is also a $\mathscr{E}_S$ fibre homotopy equivalence.*

With the above tool, we can demonstrate the analog of Proposition 8 for $\mathscr{E}_S$ fibrations.

PROPOSITION 11. *Let $\phi$ be a homotopy equivalence in $\mathscr{E}_S(B)$ which is also a $\mathscr{E}_S$ fibration. Then the characteristic morphism $\psi_M$ of the mapping cylinder $M(\phi)$ is a homotopy equivalence.*

*Proof.* Suppose that $\phi$ is a morphism from $E$ to $E'$ and consider the following commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\ \phi\ } & E' \\ \phi\big\downarrow & & \big\downarrow id \\ E' & \xrightarrow{\ id\ } & E' \end{array}$$

(Diagram 1)

From the diagram, it can be seen that $\phi$ is a fiber morphism from itself to the identity of $E'$ — which is also a $\mathscr{E}_S$ fibration. Applying Proposition 10, we see that $\phi$ must be a $\mathscr{E}_S$ fibre homotopy equivalence in addition to being a homotopy equivalence. Consequently, it has a $\mathscr{E}_S$ fibre homotopy inverse $\rho$:

$$\begin{array}{ccc} E' & \xrightarrow{\ \rho\ } & E \\ id\big\downarrow & & \big\downarrow \phi \\ E' & \xrightarrow{\ id\ } & E' \end{array}$$

(Diagram 2)

From the fact that it must be a $\mathscr{E}_S$ fibre morphism, we see that $\rho$ is an actual right hand inverse of $\phi$. On the other hand, we know that there must be a $\mathscr{E}_S$ fibre homotopy:

$$\theta : I \times E \to I \times E$$

from the identity of $E$ to the composition $\rho\phi$. In other words, we have:

$$(I \times \phi)\theta = I \times \phi$$

$$\theta/\{0\} \times B = \{0\} \times id = id$$

$$\theta/\{1\} \times B = \{1\} \times \rho\phi.$$

The proposed homotopy inverse of $\psi_M$ is defined just as in Proposition 8:

$$\psi^{-}(t, e') = \{t, \rho(e')\}$$

$$\psi^{-}: I \times E' \rightarrow M(\phi).$$

This time however, the composition $\psi_M \psi^{-}$ is the identity. This is because $\phi\rho$ is the identity. What is necessary is a homotopy from the identity of $M(\phi)$ to the composition $\psi^{-}\psi_M$. First, we compute the latter:

$$\psi^{-}\psi_M\{t, e\} = \psi^{-}(t, \phi e) = \{t, \rho\phi e\}.$$

$$\psi^{-}\psi_M\{1, e'\} = \psi^{-}(1, e') = \{1, \rho e'\}$$

$$= \{1, \phi\rho e'\} = \{1, e'\}.$$

This suggests that we construct the desired homotopy:

$$\gamma : I \times M(\phi) \rightarrow I \times M(\phi),$$

by defining its second coordinate $\gamma_t$ in terms of the second coordinate $\theta_t$ of the homotopy $\theta$:

$$\gamma_t\{s, e\} = \{s, \theta_t e\}$$

$$\gamma_t\{1, e'\} = \{1, e'\}.$$

To make sure that these rules are consistent with each other in the case $\{1, e\} = \{1, \phi e\}$, consider that according to the first rule, the image should be:

$$\{1, \theta_t e\} = \{1, \phi\theta_t e\},$$

which because $(I \times \phi)\theta$ is just $I \times \phi$ — becomes $\{1, \phi e\}$. The latter is the image of $\{1, \phi e\} = \{1, e\}$ under the second rule.

It is now clear that $\gamma$ is a morphism and a homotopy from the identity of $M(\phi)$ to $\psi^{-}\psi_M$, and hence the characteristic morphism $\psi_M$ is indeed a homotopy equivalence.

**6. Factoring arbitrary morphisms.** Axiom IV has now been shown in two special situations. In one of these, one deals only with strong deformation retracts (in $\mathscr{E}_S$) and in the other, one deals only with $\mathscr{E}_S$ fibrations. These two cases, it turns out, will suffice to demonstrate the axiom in general. The reason is that every morphism in $\mathscr{E}_S$ factors into a sdr followed by a $\mathscr{E}_S$-fibration. The construction involved is simply a variant of the one which turns any topological map into a Hurewicz fibration (see [**1**]). What one must do is alter the classical construction so that it applies to fibre maps as well (i.e., to morphisms in $\mathscr{E}_S$).

To begin with then, let $\phi$ be a morphism in $\mathscr{E}_S(B)$:

$$\phi : E \to E',$$

where $(E, p)$ and $(E', p')$ are two objects in $\mathscr{E}_S(B)$. We define the "path space" (fibrewise) of the object $(E', p')$ to be the following subspace of the path space of $E'$:

$$P(E', p') = \{f \in (E')^I : p'f \text{ is constant}\},$$

where, of course, one uses the compact open topology in $(E')^I$. This space $P(E', p')$ is the space of *fibre-wise paths* — i.e., paths which remain within a single fibre. One can construct an object in $\mathscr{E}_S(B)$ from it by:

$$E(\phi) = \{(e, f) \in E \times P(E', p') : \phi e = f(0)\}$$

$$p(\phi)(e, f) = p(e) = p'f(t)$$

$$p(\phi) : E(\phi) \to B.$$

Denote this object just by $E(\phi)$. It will be the total object of a $\mathscr{E}_S$ fibration $\pi(\phi)$ in $\mathscr{E}_S(B)$. Define:

$$\pi(\phi)(e, f) = f(1)$$

$$\pi(\phi) : E(\phi) \to E'.$$

The proof that the morphism $\pi(\phi)$ has the Covering Homotopy Property in the theory $\mathscr{E}_S$ is almost exactly the same as the classical proof [**1**]. The crucial fact is that whenever two paths in $P(E', p')$ can be added together, the resulting path is still in $P(E', p')$ — i.e., the addition of fibre-wise paths is a fibre-wise path. This is the only real modification of the original proof.

As in the classical situation, the $\mathscr{E}_S$ fibration $\pi(\phi)$ will be a factor of $\phi$. The other factor is constructed as follows:

$$j(\phi) : E \to E(\phi)$$

$$j(\phi)(e) = (e, c(\phi e)),$$

where $c(\phi e)$ is the constant path at the point $\phi e$ in $E'$. It is clear that this is a fibre-wise path, and hence it is in $P(E', p')$. The image $j(\phi)(e)$ is thus clearly in $E(\phi)$. To see that $j(\phi)$ is a morphism, compute:

$$p(\phi)j(\phi)(e) = p(\phi)(e, c(\phi e)) = p(e).$$

Now $j(\phi)$ is a sdr — but again the proof is not presented because it is essentially the same as the classical one [**1**].

The morphism $\phi$ can now be factored into the composition of $j(\phi)$ and $\pi(\phi)$:

$$\pi(\phi)j(\phi)(e) = \pi(\phi)(e, c(\phi e))$$

$$= c(\phi e)(0) = \phi(e).$$

Thus as desired, $\phi$ can be factored into a sdr and a $\mathscr{E}_S$-fibration in $\mathscr{E}_S(B)$:

$$\phi = \pi(\phi)j(\phi).$$

**7. Alternate proof of Axiom II.** An important observation is the fact that the construction:

$$E \xrightarrow{j(\phi)} E(\phi) \xrightarrow{\pi(\phi)} E'$$

is natural with respect to restriction. Namely, if $A$ is a subspace of $B$, then we have:

$$E(\phi)/A = E(\phi/A)$$

$$j(\phi)/A = j(\phi) \qquad \text{and} \qquad \pi(\phi)/A = \pi(\phi/A).$$

With this in mind, one can give another proof (see [2] Theorem 3.3) of Axiom II for $\mathscr{E}_W$ without too much difficulty.

PROPOSITION 12. *Let $\phi$ be a morphism in $\mathscr{E}_W(B)$. Suppose that $\{U_i\}$ is an open numerable covering of $B$ such that each restriction $\phi/U_i$ is a homotopy equivalence. Then $\phi$ is also a homotopy equivalence.*

*Proof.* First notice that any strong deformation retract is certainly a homotopy equivalence. It is then clear from the factorization:

$$\phi = \pi(\phi)j(\phi),$$

that $\phi$ is a homotopy equivalence exactly when $\pi(\phi)$ is one. Recalling that the factorization is natural with respect to restriction, consider:

$$\phi/U_i = \pi(\phi/U_i)j(\phi/U_i).$$

It is given that each $\phi/U_i$ is a homotopy equivalence. Thus each $\pi(\phi/U_i)$ is also one. If one could show that $\pi(\phi)$ is a homotopy equivalence, the proof would be done.

The latter will be demonstrated with the aid of the mapping cylinders $M(\pi(\phi/U_i))$. Proposition 11 tells us that these possess the WCHP. But on the other hand by naturality each $M(\pi(\phi/U_i))$ is the restriction of $M(\pi(\phi))$ to $I \times U_i$. Clearly $\{I \times U_i\}$ is a numerable open covering of $I \times B$, and hence Axiom I asserts that $M(\pi(\phi))$ has the WCHP — i.e., is in $\mathscr{E}_W(I \times B)$. The fact that $\pi(\phi)$ (and hence $\phi$) is a homotopy equivalence follows from the next theorem — which is a kind of converse to Propositions 8 and 11.

PROPOSITION 13. *Let $\gamma$ be a morphism in $\mathscr{E}_W(B)$:*

$$\gamma : E \rightarrow E'.$$

*If $M(\gamma)$ is in $\mathscr{E}_W(I \times B)$, then the characteristic morphism $\psi_M$ is a homotopy equivalence — and so $\gamma$ must also be one.*

*Proof.* Consider that the natural map:

$$f : I \times B \rightarrow \{1\} \times B$$

$$f(t, b) = (1, b)$$

is homotopic to the identity of $I \times B$, and hence by an application of Proposition 3 for $\mathscr{E}_W$, we know that $M(\gamma)$ is homotopic to $f^*M(\gamma) = I \times E'$. Let $\psi$ be a homotopy equivalency from $1 \times E'$ to $M(\gamma)$ with homotopy inverse $\psi^-$:

$$\psi : I \times E' \to M(\gamma)$$

$$\psi^- : M(\gamma) \to I \times E'.$$

then $\psi_M \psi$ is a morphism from $I \times E'$ to $I \times E'$ such that

$$\psi_M \psi / \{1\} \times B = id\,(\psi/\{1\} \times B).$$

But since $\psi$ is a homotopy equivalence, the restriction $\psi/\{1\} \times B$ is also one. Applying Lemma 14 below to $\psi_M \psi$, it must be a homotopy equivalence, and since $\psi_M$ is homotopic to $\psi_M \psi \psi^-$, then $\psi_M$ must also be a homotopy equivalence. Then the fact that $\gamma$ is a homotopy equivalence follows from the observation that $\psi_M / \{0\} \times B = \{0\} \times \gamma$.

LEMMA 14. *Let $\psi$ be a homotopy from $\phi$ to $\phi'$. If $\phi$ is homotopy equivalence, then so is $\psi$ and hence by induction so is $\phi'$.*

*Proof.* Let $f : I \times I \to I$ be defined by $f(s, t) = st$, and define $\theta = (f \times id_B)^*\psi$. Thus $\theta$ is from $I \times I \times E$ to $I \times I \times E'$. Then by computation it can be seen that:

$$\theta/\{0\} \times I \times B = \{0\} \times I \times \phi$$

$$\theta/\{1\} \times I \times B = \{1\} \times \psi,$$

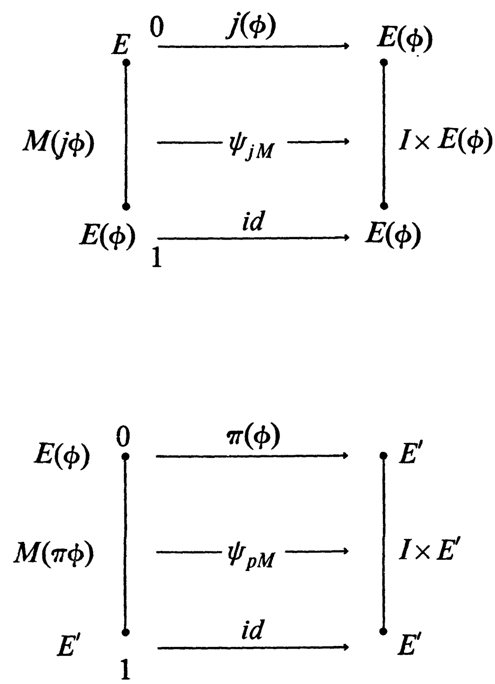
so that $\theta$ is a homotopy from $\psi$ to $I \times \phi$. The latter is a homotopy equivalence and so $\psi$ is one also.

**8. General case of the mapping cylinder axiom.** One can now demonstrate Axiom IV for the theory $\mathscr{E}_W$. Let then $\phi$ be a homotopy equivalence in the category $\mathscr{E}_W(B)$ from $E$ to $E'$. Again use the factorization $\phi = \pi(\phi)\, j(\phi)$. But $j(\phi)$ is a homotopy equivalence — so $\pi(\phi)$ must also be one. Propositions 8 and 11 then assert that the characteristic morphisms:

$$\psi_{jM} : M(j(\phi)) \to I \times E(\phi)$$

$$\psi_{pM} : M(\pi(\phi)) \to I \times E',$$

are homotopy equivalences. But the objects $E'$ and $E(\phi)$ are now in $\mathscr{E}_W(B)$ and so by Proposition 5 the mapping cylinders $M(j(\phi))$ and $M(\pi(\phi))$ have the WCHP (are in $\mathscr{E}_W(I \times B)$). These mapping cylinders can be placed end to end to provide a mapping cylinder $M$ for $\phi$. Essentially one identifies the zero end of $\psi_{pM}$ with the unit end of $\psi_{jM}$. The boundary relations of $\psi_{jM}$ and $\psi_{pM}$ are illustrated below.




(Diagram 3)

Since the characteristic morphisms must be patched together as well, one sees that the first of these must be modified so that its value at one will coincide with the value of the second at zero. Call the modified homotopy equivalence $\psi_1$:

$$\psi_1 : M(j(\phi)) \to I \times E'.$$

It is given by the formula $\psi_1 = (I \times \pi(\phi))\psi_{jM}$.

To overlap these homotopy equivalences a little in the middle in order to perform our construction over an open covering of $I \times B$, one uses the "preparation" maps:

$$f_1 : [0, 2/3) \to I$$

$$f_1(t) = \min(1, 3t)$$

$$f_2 : (1/3, 1] \to I$$

$$f_2(t) = \max(1, 3t - 2).$$

Applying these maps, one obtains the following homotopy equivalences and objects in the weak theory $\mathscr{E}_W$:

$$f_1{}^*\psi_1 : f_1{}^*M(j(\phi)) \to [0, 2/3) \times E'$$

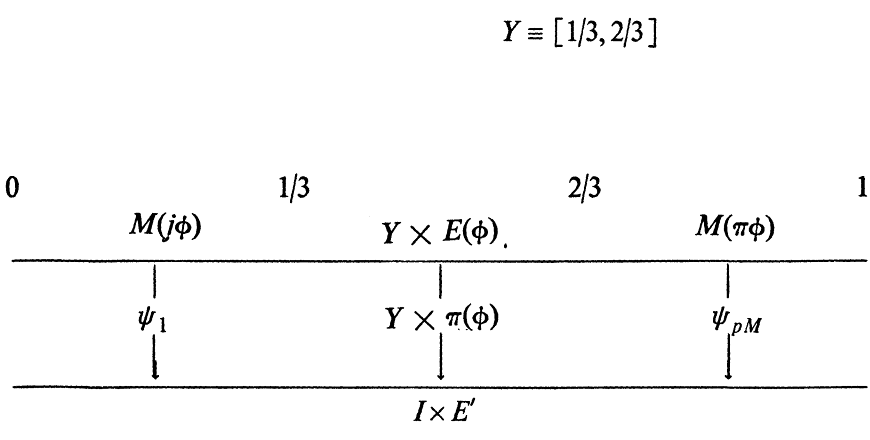
$$f_2{}^*\psi_{pM} : f_2{}^*M(\pi(\phi)) \to (1/3, 1] \times E'.$$

These homotopy equivalences fit together over the intersection $X = (1/3, 2/3) \times B$. To see this note that:

$$\psi_1/\{1\} \times B = \{1\} \times \pi(\phi)$$

$$\psi_{pM}/\{0\} \times B = \{0\} \times \pi(\phi),$$

while $f_1$ maps the interval $(1/3, 2/3)$ to one and $f_2$ maps it to zero. Applying Axiom I we obtain a unique morphism $\psi_M$ which extends both $f_1{}^*\psi_1$ and $f_2{}^*\psi_{pM}$. This is illustrated below:




(Diagram 4)

A uniqueness argument shows that the range of $\psi_M$ must be the object $I \times E'$, while Axiom II insures that $\psi_M$ is a homotopy equivalence. If the domain of $\psi_M$ is denoted by $M$, then this will be the desired mapping cylinder of $\phi$. To see this, one must demonstrate that $M$ and $\psi_M$ have the correct boundary relations. Notice that $f_1$ maps zero to zero while $f_2$ maps one to one. In view of the formulas:

$$\psi_1/\{0\} \times B = \{0\} \times \pi(\phi)\hat{j}(\phi) = \{0\} \times \phi$$

$$\psi_{pM}/\{1\} \times B = \{1\} \times id,$$

one sees that $\psi_M$ has the proper relations:

$$\psi_M/\{0\} \times B = \{0\} \times \phi$$

$$\psi_M/\{1\} \times B = \{1\} \times id = id,$$

and hence the domain $M$ of $\psi_M$ has the corresponding relations:

$$M/\{0\} \times B = \{0\} \times E$$

$$M/\{1\} \times B = \{1\} \times E'.$$

This finishes the proof of Axiom IV for the theory $\mathscr{E}_W$.

**9.** Remarks. The four axioms of a fibration theory are enough to permit a type of obstruction theory. This has been carried out in the author's dissertation, "Fiber Spaces and the Higher Homotopy Cocycle Relations," University of Notre Dame 1965. The most important axiom in this respect is the mapping cylinder axiom, which allows the extension of an object from the boundary of a cell to the entire cell, provided that the object over the boundary is homotopy equivalent to a trivial object.

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