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STRONGLY REGULAR GRAPHS AND GROUP DIVISIBLE DESIGNS

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The counting techniques of the author's earlier work on strongly regular graphs are used to prove the converse of a result of R. C. Bose and S. S. Shrikhande on geometric and pseudo-geometric graphs $(q^2 + 1, q + 1, 1)$.

0. Introduction. In the present paper, we use the counting techniques of the author's earlier work [5] to prove the converse of a result of R.C. Bose and S.S. Shrikhande [3] on geometric and pseudo-geometric graphs $(q^2 + 1, q + 1, 1)$.

Section 1 is devoted to preliminaries on strongly regular graphs and group divisible designs. We also give a brief description of the problem under consideration and a statement of our main result Theorem 1.1. Section 2 contains the proof of Theorem'1.1.

We refer to [3] for the necessary background. Throughout this paper I will denote an identity matrix and J a square matrix of all ones. Also j and O will denote row vectors of all ones and zeros respectively. Finally, $|S|$ denotes the cardinality of the set S.

1. Preliminary results and the statement of the main result Theorem 1.1.

A strongly regular graph [1] is a graph on v vertices, without loops or multiple edges and whose standard (0, 1) adjacency matrix A satisfies

$$AJ = JA = n_1 J \tag{1.1}$$

and

$$A^2 = n_1 I + \lambda A + \mu(J - I - A). \tag{1.2}$$

The parameters of a strongly regular graph are then denoted by

$$v, n_1, \lambda, \mu. \tag{1.3}$$

Let $v = mn$ objects (= treatments) be partitioned into m disjoint sets S_i $(i = 1, 2, \dots, m)$, each containing n objects. Let two objects be called adjacent if and only if they belong to the same set S_i. We then get a strongly regular graph, which is traditionally called a group divisible (G.D.) association scheme. The parameters of a G.D. scheme are given by

$$(1.4) \qquad v = mn,\ n_1 = n - 1,\ \lambda = n - 2,\ \mu = 0 \quad (n \geq 2).$$

We observe that for a G.D. scheme, the $mn \times mn$ adjacency matrix (= association matrix) C has the form

$$(1.5) \qquad C = \text{diag}[J_n - I_n, J_n - I_n, \dots, J_n - I_n].$$

Suppose now that we have a G.D. scheme on $v = mn$ treatments as above. A G.D. design $D(v, b, r, k, m, n, \lambda_1, \lambda_2)$ is an arrangement of these v treatments $t_1, t_2, \dots, t_v$ into b distinct subsets $B_1, B_2, \dots, B_b$ (called blocks) satisfying the following conditions:

(1) $|B_i| = k \quad (i = 1, 2, \dots, b)$
(2) Each treatment occurs in exactly r blocks.
(3) Two treatments from the same set S_i appear together in exactly λ_1 blocks and two treatments from distinct sets S_i and S_j occur together in exactly λ_2 blocks.

The parameters of a G.D. design are denoted by

$$(1.6) \qquad v, b, r, k, m, n, \lambda_1, \lambda_2.$$

A G.D. design D is called semi-regular group divisible (S.R.G.D.) if $r > \lambda_1$ and $rk = \lambda_2 v$. Bose and Connor [2] have shown that for a S.R.G.D. design, m divides k and each block contains k/m treatments from each set S_i $(i = 1, 2, \dots, m)$.

We now indicate the problem considered in the present paper. Let D be a S.R.G.D. design with parameters (1.6). Let $t_1, t_2, \dots, t_v$ and $B_1, B_2, \dots, B_b$ denote the treatments and blocks of D respectively. Suppose D has the additional property that there exist distinct nonnegative integers μ_1 and μ_2 satisfying $|B_i \cap B_j| \in \{\mu_1, \mu_2\}$ $(i \neq j)$. We construct the block graph B of D as follows. Take the vertices of B to be the blocks of D. Define blocks B_i, B_j $(i \neq j)$ to be adjacent if and only if $|B_i \cap B_j| = \mu_1$.

Let N denote the usual $v \times b$ (0, 1) incidence matrix of D. Let C be given by (1.5). Define

$$A = \begin{bmatrix} 0 & j_v & O_b \\ j'_v & C & N \\ O'_b & N' & B \end{bmatrix}. \tag{1.7}$$

We note that A is a symmetric (0, 1) matrix of size $b + v + 1$, and has zero trace. Therefore A is the adjacency matrix of a graph. We wish to find necessary and sufficient conditions on the parameters of D, so that A is strongly regular.

In [3], the converse situation was investigated. There, one starts with a very specific strongly regular graph, namely a pseudo-geometric graph $(q^2 + 1, q + 1, 1)$ $(q \geq 2)$. (See [1] for a general discussion of geometric and pseudo-geometric graphs (r, k, t)). The adjacency matrix A of this graph can be brought to the form (1.7), where C, N, B are now (0, 1) matrices of the appropriate form. If further, A has the properties (P) and (P^*) as in the notation of [3], then it was shown that N is the incidence matrix of a S.R.G.D. design D and C is given by (1.5). Moreover the blocks of D have two intersection cardinalities μ_1, μ_2 and B is the block graph of D.

Specifically, the parameters of the S.R.G.D. design D were shown to be

$$\begin{cases} v = q(q^2 + 1), \ b = q^4, \ r = q^3, \ k = q^2 + 1, \ m = q^2 + 1, \\ n = q, \ \lambda_1 = 0, \ \lambda_2 = q^2, \ \mu_1 = 1, \ \mu_2 = q + 1 \end{cases} \tag{1.8}$$

In this paper, we shall show that there are only two parametrically possible strongly regular graphs A, of the form (1.7), which can be obtained from S.R.G.D. designs in the above manner. One of these graphs is pseudo-geometric $(q^2 + 1, q + 1, 1)$.

The full content of our main result is the following:

THEOREM 1.1 *Let N be the incidence matrix of a S.R.G.D. design D with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ having m sets of n treatments each. Suppose any two distinct blocks of D intersect in μ_1 or μ_2 $(\neq \mu_1)$ treatments. Let C be the association matrix of D and let B be the adjacency matrix of the blocks of D.*

Then,

$$A = \begin{bmatrix} 0 & j_v & O_b \\ j'_v & C & N \\ O'_b & N' & B \end{bmatrix}$$

represents a strongly regular graph if and only if the parameters of D are given by

$$(1)\quad v = q(q^2+1), b = q^4, r = q^3, k = q^2+1, m = q^2+1, \\ n = q, \lambda_1 = 0, \lambda_2 = q, \mu_1 = 1, \mu_2 = q+1 \quad (q \geq 2)$$

or

$$(2)\quad v = 2n, b = n^2, r = n, k = 2, m = 2, n, \\ \lambda_1 = 0, \lambda_2 = 1, \mu_1 = 1, \mu_2 = 0 \quad (n \geq 2).$$

Moreover, the corresponding strongly regular graphs A are respectively pseudo-geometric $(q^2+1, q+1, 1)$ or pseudo-geometric $(2, n+1, 1)$.

2. *Proof of Theorem* 1.1. Let $D(v, b, r, k, m, n, \lambda_1, \lambda_2)$ be a S.R.G.D. design based on m sets of n treatments each. Let $t_1, t_2, \ldots, t_v$ and $B_1, B_2, \ldots, B_b$ denote the treatments and blocks of D. We assume further that any two distinct blocks of D intersect in μ_1 or μ_2 $(\neq \mu_1)$ treatments. Then the parameters of D can be taken to be

$$v = mn, b, r, k, m, n, \lambda_1, \lambda_2, \mu_1, \mu_2. \tag{2.1}$$

Let N, B, C and A be as in the statement of Theorem 1.1. Let m_i denote the number of blocks intersecting a given block in μ_i treatments $(i = 1, 2)$. Then clearly

$$m_1 + m_2 = b - 1 \tag{2.2}$$

and

$$m_1\mu_1 + m_2\mu_2 = k(r-1). \tag{2.3}$$

Therefore, B has constant row sum m_1 given by

$$m_1 = \frac{k(r-1) + \mu_2(1-b)}{\mu_1 - \mu_2}. \tag{2.4}$$

Now, since D is a S.R.G.D. design with incidence matrix N, we know from [2], that NN' has eigenvalues rk, $r - \lambda_1$ and $rk - \lambda_2 v = 0$, with multiplicities 1, $m(n-1)$ and $m - 1$ respectively. Hence $N'N$ has eigenvalues rk, $r - \lambda_1$ and 0 with multiplicities 1, $m(n-1)$ and $b - m(n-1) - 1$ respectively. But, we have

(2.5) $$N'N = kI + \mu_1 B + \mu_2 (J - I - B).$$

Hence, from Frobenius' theorem on commuting matrices, B has eigenvalues $\theta_0, \theta_1, \theta_2$ given by

(2.6) $$\theta_0 = \frac{k(r-1) + \mu_2(1-b)}{\mu_1 - \mu_2} = m_1, \quad \text{with multiplicity 1}$$

(2.7) $$\theta_1 = \frac{(r - \lambda_1) + (\mu_2 - k)}{\mu_1 - \mu_2}, \qquad \text{with multiplicity } m(n-1)$$

(2.8) $$\theta_2 = \frac{(\mu_2 - k)}{(\mu_1 - \mu_2)}, \qquad \text{with multiplicity } b - m(n-1) - 1.$$

Thus, from Lemma 5, [4], B is strongly regular (b, m_1, α, β), where

(2.9) $$\alpha = m_1 + \theta_1 + \theta_2 + \theta_1\theta_2, \beta = m_1 + \theta_1\theta_2.$$

Let

(2.10) $$A = \begin{bmatrix} 0 & j_v & O_b \\ j'_v & C & N \\ O'_b & N' & B \end{bmatrix}.$$

Suppose A is strongly regular $(b + v + 1, n_1, \lambda, \mu)$. Any row sum of A is either v, $n + r$ or $k + m_1$. Hence, for regularity we must have

(2.11) $$n_1 = v = n + r = k + m_1.$$

Next, by considering any two treatments or any two blocks which are adjacent or nonadjacent, easy counting arguments in (2.10) give

(2.12) $$\lambda = n - 1 = (n - 1) + \lambda_1 = \mu_1 + \alpha$$

and

(2.13) $$\mu = k = 1 + \lambda_2 = \mu_2 + \beta.$$

From (2.12), we see that $\lambda_1 = 0$. This together with the Bose-Connor property mentioned in §1, implies that every block contains exactly one treatment from each set. Hence the parameters (2.1) of D can be taken as

(2.14) $v = mn, b = n^2\lambda_2, r = \lambda_2 n, k = m, m, n, \lambda_1 = 0, \lambda_2, \mu_1, \mu_2.$

Next, consider a treatment t_i and a block B_j such that $t_i \in B_j$. Denoting $N = (n_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, we have from (2.10),

$$\lambda = |\{l : c_{il} = 1 = n_{lj}, 1 \le l \le v\}| + |\{l : n_{il} = 1 = b_{jl}, 1 \le l \le b\}|.$$

Using the Bose-Connor property, we get

(2.16) $$\lambda = |\{B_l : l \neq j, t_i \in B_l \text{ and } |B_l \cap B_j| = \mu_1\}|.$$

Let $B_j = \{t_i, y_1, y_2, \ldots, y_{k-1}\}$, $B_l = \{t_i, x_1, x_2, \ldots, x_{k-1}\}$(say). Since $\lambda_1 = 0$, each pair (t_i, y_p), $1 \le p \le k - 1$ occurs λ_2 times in the blocks of D. Counting the distribution of these pairs in two ways, we get

(2.17) $$\lambda = \frac{(k-1)(\lambda_2 - 1) - (\mu_2 - 1)(r-1)}{\mu_1 - \mu_2}.$$

Next, consider a treatment t_i and a block B_j such that $t_i \notin B_j$. Then using the Bose-Connor property, a similar type of counting yields

(2.18) $$\mu = \frac{(k-1)\lambda_2 + (\mu_1 - \mu_2) - r\mu_2}{\mu_1 - \mu_2}.$$

Then, (2.12), (2.17) and (2.13), (2.18) imply that

(2.19) $$(n-1)(\mu_1 - \mu_2) = (k-1)(\lambda_2 - 1) - (\mu_2 - 1)(r-1)$$

and

(2.20) $$(k-1)(\mu_1 - \mu_2) = (k-1)\lambda_2 - r\mu_2.$$

Then, (2.19) and (2.20) give

(2.21) $$\mu_1 + r - k = (n - k + 1)(\mu_1 - \mu_2)$$

and

$$\mu_2 + r - k = (n - k)(\mu_1 - \mu_2). \tag{2.22}$$

Next, using (2.13), (2.12), (2.11) and (2.9), we obtain

$$(\mu_1 + r - k)\{(\mu_1 - \mu_2)^2 + \mu_1 - k\} = 0 \tag{2.23}$$

and

$$(\mu_2 + r - k)\{(\mu_1 - \mu_2)^2 + \mu_2 - k\} + (\mu_1 - \mu_2)^2(n - k) = 0. \tag{2.24}$$

Thus,

$$(\mu_1 - \mu_2)(n - k + 1)\{(\mu_1 - \mu_2)^2 + \mu_1 - k\} = 0. \tag{2.25}$$

and

$$(\mu_1 - \mu_2)(n - k)\{(\mu_1 - \mu_2)^2 + \mu_1 - k\} = 0. \tag{2.26}$$

Since $\mu_1 \neq \mu_2$, this gives

$$(\mu_1 - \mu_2)^2 + \mu_1 - k = 0. \tag{2.27}$$

Putting $\mu_1 - \mu_2 = g$ in (2.27) gives

$$\mu_1 = k - g^2 \tag{2.28}$$

$$\mu_2 = k - g^2 - g. \tag{2.29}$$

Substituting these values in (2.22) we get

$$(n + g)\lambda_2 = (n + g)g. \tag{2.30}$$

Hence, either

$$\lambda_2 = g \qquad (> 0) \qquad \text{case (a)}$$

or

$$n = -g \qquad (\geq 2) \qquad \text{case (b)}$$

If case (a) holds, then $k = m = 1 + \lambda_2 = 1 + g$ and $\mu_1 = g + 1 - g^2$, $\mu_2 = 1 - g^2$. But $\mu_1 \neq \mu_2$ and $\mu_1 \geq 0$, $\mu_2 \geq 0$ then imply that D has parameters

$$\text{(2.31)} \qquad \begin{cases} v = mn = 2n, \quad b = n^2, \quad r = n, \quad k = 2, \quad m = 2, n \\ \lambda_1 = 0, \quad \lambda_2 = 1, \quad \mu_1 = 1, \quad \mu_2 = 0. \end{cases}$$

Also, the parameters of A are then

$$\text{(2.32)}\, b + v + 1 = (n + 1)^2, n_1 = 2n, \lambda = n - 1, \mu = 2 \quad (n \geq 2).$$

Thus A is pseudo-geometric $(2, n + 1, 1)$.

Finally if case (b) holds, put

$$\text{(2.33)} \qquad n = -g = q \qquad \text{(say)}.$$

Then (2.20), together with $\lambda_2 \neq 0$, $n \geq 2$ implies that D has parameters

$$\text{(2.34)} \qquad \begin{cases} v = q(q^2 + 1), \quad b = q^4, \quad r = q^3, \quad k = q^2 + 1, \quad m = q^2 + 1, \\ n = q, \quad \lambda_1 = 0, \quad \lambda_2 = q^2, \quad \mu_1 = 1, \quad \mu_2 = q + 1, \quad (q \geq 2). \end{cases}$$

And in this case, it is easily seen that A has parameters

$$\text{(2.35)} \qquad \begin{cases} b + v + 1 = (q + 1)(q^3 + 1), \quad n_1 = q(q^2 + 1), \\ \lambda = q - 1, \quad \mu = q^2 + 1. \end{cases}$$

Thus, in this case A is pseudo-geometric $(q^2 + 1, q + 1, 1)$.

We have therefore established that if A is strongly regular, then D has parameters given by (2.31) or (2.34). Moreover A is then pseudo-geometric $(2, n + 1, 1)$ or $(q^2 + 1, q + 1, 1)$ respectively.
Conversely it can be easily shown that if D has parameters given by (2.31) or (2.34), then A is strongly regular and is pseudo-geometric $(2, n + 1, 1)$ or $(q^2 + 1, q + 1, 1)$ respectively.

This completes the proof of Theorem 1.1.

REMARKS. (i) The existence of S.R.G.D. designs D with parameters of case (1) in Theorem 1.1 and partial geometries $(q^2 + 1, q + 1, 1)$ is known for q a prime or prime power (See **[1]** and **[3]**).

(ii) The design D with parameters of case (2) in Theorem 1.1 is known for any integer n and is constructed as follows: Arrange n^2 treatments in an $n \times n$ array as

$$L = \begin{bmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \therefore & \therefore \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{bmatrix}$$

Write down $2n$ blocks corresponding to the rows and columns of L. We get a design E where the blocks are the columns in

$$\overbrace{\begin{matrix} 1 & n+1 & \cdots & n^2-n+1 \\ 2 & n+2 & \cdots & n^2-n+2 \\ \vdots & \vdots & \cdots & \vdots \\ n & 2n & \cdots & n^2 \end{matrix}}^{S_1} \left| \overbrace{\begin{matrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \cdots & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{matrix}}^{S_2} \right.$$

The required design D is the dual of E. It is easily seen that in this case the line graph $L_2(n+1)$ of the complete bipartite graph $K(n+1, n+1)$ has the same parameters as the graph A.

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