PACIFIC JOURNAL OF MATHEMATICS
Vol. 54, No. 2, 1974

DIFFERENTIAL INEQUALITIES AND LOCAL VALENCY

Q. I. RAHMAN AND J. STANKIEWICZ

An entire function $f(z)$ is said to have bounded value distribution (b.v.d.) if there exist constants p, R such that the equation $f(z) = w$ never has more than p roots in any disk of radius R. It was shown by W. K. Hayman that this is the case for a particular p and some $R > 0$ if and only if there is a constant $C > 0$ such that for all z

$$|f^{(p+1)}(z)| \leq C \max_{1\leq\nu\leq p} |f^{(\nu)}(z)|,$$

so that $f'(z)$ has bounded index in the sense of Lepson.

The fact that $f'(z)$ has bounded index if $f(z)$ has b.v.d. follows readily from a classical result on p-valent functions. In the other direction Hayman proved that if

$$|f^{(n)}(z)| \leq \max_{0\leq\nu\leq n-1} |f^{(\nu)}(z)|,$$

then $f(z)$ cannot have more than $n-1$ zeros in $|z| \leq \sqrt{n}/e\sqrt{20}$. Here the order of magnitude is correct in the sense that $\sqrt{n}/e\sqrt{20}$ cannot be replaced by $\sqrt{3}\sqrt{n}$. The result when applied to $f(z) - w$ does show that $f'(z)$ has bounded index only if $f(z)$ has b.v.d. but it is clearly of interest to determine the largest disk containing at most $n-1$ zeros of $f(z)$. We are able to replace $\sqrt{n}/e\sqrt{20}$ by $\sqrt{n}/e\sqrt{10}$.

The above mentioned result of Hayman appeared in [2]. He did not assume $f(z)$ to be entire but simply regular in $|z| < 2n$. To be precise he proved [2, Theorem 3] the following:

THEOREM A. *If $f(z)$ is regular in $|z| < 2n$, where it satisfies*

$$|f^{(n)}(z)| \leq \max_{0\leq\nu\leq n-1} |f^{(\nu)}(z)|, \tag{1.1}$$

then $f(z)$ possesses at most $n-1$ zeros in

$$|z| \leq \frac{\sqrt{n}}{e\sqrt{20}}. \tag{1.2}$$

In his proof of Theorem A Hayman made use of the following lemma.

LEMMA A. *Let z_ν, $\nu = 1, 2, \ldots, n$ be complex numbers such that $\max_{1\leq\nu\leq n} |z_\nu| = \rho_0$. If*

(1.3) $$\varphi(z) = \Big\{\prod_{\nu=1}^{n}(1 - z_\nu z)\Big\}^{\varepsilon} = \sum_{0}^{\infty} b_k z^k$$

and $b_1 = \Sigma_{\nu=1}^{n} z_\nu = 0, \varepsilon = 1$ *or* -1, *then*

(1.4) $$|b_k| < (\sqrt{n})^k \rho_0{}^k, \quad k > 1.$$

The bound in (1.4) is not the best possible and this is one of the reasons why the conclusion of Theorem A is not precise. We observe that (1.4) can be considerably improved, viz. we have

LEMMA A′. *Under the hypotheses of Lemma A*

(1.5) $$|b_k| \leq \left\{\sqrt{\binom{n}{2}}\right\}^k \rho_0^k, \qquad k > 1$$

Now Hayman's reasoning itself gives us the following improvement of Theorem A.

THEOREM A′. *Under the hypothesis of Theorem A $f(z)$ possesses at most $n-1$ zeros in*

(1.6) $$|z| \leq \frac{\sqrt{n}}{e\sqrt{10}}.$$

This refined version of Theorem A gives corresponding refinements in several of the other theorems proved by Hayman in [2]. For example, Theorems 4 and 6 of his paper may respectively be replaced by

THEOREM 4′. *Suppose that $f(z)$ is regular in $|z - z_0| < R$ and satisfies there*

$$(CR)^{p+1}\left|\frac{f^{(p+1)}(z)}{(p+1)!}\right| \leq \max_{1\leq\nu\leq p}(CR)^{\nu}\left|\frac{f^{(\nu)}(z)}{\nu!}\right|$$

with $C \leq 1/2$. Then $f(z)$ is p-valent in $|z - z_0| \leq CR/\{e\sqrt{10}\,(p+1)^{1/2}\}$.

THEOREM 6′. *Consider the differential equation*

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0,$$

in the disk $D_0 = \{z \mid |z - z_0| < R\}$, *where* $0 < R \leq \infty$ *and the functions* a_1 *to* a_n *are supposed to be regular and bounded in* D_0. *Let* t_0 *be the positive root of the equation*

$$\sum_{\nu=1}^{n} \alpha_\nu t^\nu = 1$$

where

$$\alpha_\nu = \sup_{z \in D_0} |a_\nu(z)|.$$

If $y(z)$ *is a solution of the differential equation then* $y(z)$ *has at most* $n - 1$ *zeros in*

$$|z - z_0| \leq R'_1 = \min\left\{ t_0 \frac{\sqrt{n}}{e\sqrt{10}}, \frac{R}{2e(10n)^{1/2}} \right\},$$

i.e. *the differential equation is disconjugate in* $|z - z_0| < R'_1$.

Definition. Let $\mathscr{P}_n$ denote the class of polynomials

$$p_n(z) = \prod_{\nu=1}^{n} (1 - z_\nu z)$$

which do not vanish in $|z| < 1$ and for which $p_n'(0) = \sum_{\nu=1}^{n} - z_\nu = 0$.

Lemma A′ may now be stated in the following equivalent form.

Theorem 1. *If*

$$\varphi(z) = \{p_n(z)\}^\varepsilon = \sum_{0}^{\infty} b_{k,\varepsilon} z^k \tag{1.7}$$

where $p_n(z) \in \mathscr{P}_n$ *and* $\varepsilon = 1$ *or* -1, *then*

$$|b_{k,\varepsilon}| \leq \{\sqrt{(n/2)}\}^k. \tag{1.8}$$

If n is even and $p_n(z) = (1 - e^{i\gamma} z^2)^{n/2}$ where γ is real then $|b_{2,1}| = |b_{2,-1}| = n/2$ which shows that (1.8) is the best possible result of its kind.

The bound in (1.8) is not sharp for $k \geq 3$ and it is clearly of interest to get precise estimates for $|b_{k,\varepsilon}|$ for each k. We are able to do it for $k \leq 4$.

Theorem 2. *Under the hypothesis of Theorem* 1 *we have*

(1.9) $$|b_{2,\varepsilon}| \leq n/2,$$
(1.10) $$|b_{3,\varepsilon}| \leq n/3,$$
(1.11) $$|b_{4,1}| \leq (n^2 - 2n)/8,$$
(1.12) $$|b_{4,-1}| \leq (n^2 + 2n)/8.$$

The example $p_n(z) = (1 - z^2)^{n/2}$ where n is even shows that (1.9), (1.11) and (1.12) are sharp. To see that (1.10) is sharp we may consider $p_n(z) = (1 - z^3)^{n/3}$ where n is divisible by 3.

The following theorem shows that $|b_{2,\varepsilon}|$ and $|b_{3,\varepsilon}|$ cannot both be large at the same time.

THEOREM 3. *Under the hypothesis of Theorem* 1 *we have*

(1.13) $$|b_{2,\varepsilon}| + |b_{3,\varepsilon}| \leq \frac{25}{48}n$$

(1.14) $$|b_{2,\varepsilon}| + \frac{3}{4}|b_{3,\varepsilon}| \leq n/2$$

(1.15) $$\frac{2\sqrt{2}}{3}|b_{2,\varepsilon}| + |b_{3,\varepsilon}| \leq n/2\,.$$

If k is fixed, $k > 4$ and n is large, the bound in (1.8) can also be sharpened.

THEOREM 4. *Let* $p_n(z) \in \mathscr{P}_n$ *and* λ *a real number* $\neq 0$. *If*

(1.16) $$\varphi_\lambda(z) = \{p_n(z)\}^\lambda = \sum_0^\infty b_{k,\lambda} z^k$$

then for every given $0 < \delta < \pi$ *there exists an integer* n_0 *depending on* λ *and* δ *such that*

(1.17) $$|b_{k,\lambda}| \leq 2\frac{\left(1 + \sin\dfrac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin\dfrac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1}\left\{\sqrt{\left(\frac{|\lambda|n}{\delta}\right)}\right\}^k$$

provided $n > n_0$.

The proof of Theorem 4 depends on the fact that if $p_n(z) \in \mathscr{P}_n$ then

(1.18) $$\omega(z) = \frac{1 - \{p_n(z)\}^{1/n}}{z^2}$$

is analytic in $|z| < 1$ and there exists a positive number ρ_0 independent of n such that

(1.19) $$|\omega(z)| \leq \frac{1}{2} + \frac{1}{8}|z|^2 + |z|^4$$

for $|z| < \rho_0$. For the study of polynomials $p_n(z) \in \mathscr{P}_n$ it will be very helpful to get precise estimates for $|\omega(z)|$. The example

$$p(z) = (1 - z^2)^{n/2},\ n \text{ even}$$

shows that

$$\max_{p(z)\in\mathscr{P}_n} |\omega(z)| \geq \frac{1}{2} + \frac{1}{8}|z|^2 + \frac{1}{16}|z|^4 + \frac{5}{128}|z|^6.$$

We prove

THEOREM 5. *If $p_n(z) \in \mathscr{P}_n$ then*

(1.20) $$|\omega(z)| = \left|\frac{1 - \{p_n(z)\}^{1/n}}{z^2}\right| \leq \frac{1}{2} + \frac{1}{8}|z|^2 + \frac{1}{16}|z|^4 + \frac{3\sqrt{3}}{4}|z|^6$$

at least for $|z| \leq 1/2$.

The following corollary is obtained by applying Theorem 5 to the reciprocal polynomial $z^n p_n(1/z)$, and setting $\alpha = z^{-1}\omega(z^{-1})$.

COROLLARY 1. *Let*

$$p_n(z) = \prod_{k=1}^{n} (z - \alpha_k)$$

be a polynomial of degree n having all its zeros in $|z| \leq 1$. If the centre of gravity of the zeros lies at the origin then for $|z| > 2$ the equation

(1.21) $$\sum_{k=1}^{n} \frac{1}{n} \log\left(1 - \frac{\alpha_k}{z}\right) = \log\left(1 - \frac{\alpha}{z}\right)$$

has a solution which satisfies

(1.22) $$|\alpha| \leq \frac{1}{2|z|} + \frac{1}{8|z|^3} + \frac{1}{16|z|^5} + \frac{3\sqrt{3}}{4|z|^7}.$$

If α_k, $k = 1, 2, \ldots, n$ are complex numbers of absolute value ≤ 1 and $m_k = p_k/q_k$, $k = 1, 2, \ldots, n$ are positive rational numbers such that $\Sigma_{k=1}^n m_k = 1$, $\Sigma_{k=1}^n m_k\alpha_k = 0$ then

$$\left\{\prod_{k=1}^{n}(z-\alpha_k)^{p_k/q_k}\right\}^{q_1q_2\dots q_n}$$

is a polynomial of degree $q_1q_2\cdots q_n$ having all its zeros in $|z|\leq 1$. Besides, the centre of gravity of the zeros (taking into account their multiplicity) lies at the origin. Hence by the above corollary the equation in α

$$\sum_{k=1}^{n}\frac{p_kq_1q_2\cdots q_{k-1}q_{k+1}\cdots q_n}{q_1q_2\cdots q_n}\log\left(1-\frac{\alpha_k}{z}\right)\equiv$$
$$\sum_{k=1}^{n}m_k\log\left(1-\frac{\alpha_k}{z}\right)=\log\left(1-\frac{\alpha}{z}\right)$$

has a solution α which satisfies (1.22) at least for $|z|\geq 2$. It is clear that if some or all the numbers m_k are irrational then we get the same conclusion by a limiting process. Thus we have

COROLLARY 1′. *If we have* $m_k>0$, $\Sigma m_k=1$, $|\alpha_k|\leq 1$, $\Sigma m_k\alpha_k=0$, $|z|\geq 2$ (*where* $k=1,2,3,\dots,n$) *then there exists an* α *such that*

$$|\alpha|\leq\frac{1}{2|z|}+\frac{1}{8|z|^3}+\frac{1}{16|z|^5}+\frac{3\sqrt{3}}{4|z|^7} \tag{1.22}$$

with

$$\sum m_k\log\left(1-\frac{\alpha_k}{z}\right)=\log\left(1-\frac{\alpha}{z}\right). \tag{1.23}$$

This result was proved by Walsh (see [4], Lemma 2 and (1.10) on p. 358) except that he had

$$|\alpha|\leq\frac{1}{2|z|}+\frac{3}{2|z|^2}$$

for $|z|>3$ instead of (1.22) which we prove to be valid for $|z|\geq 2$. As illustrated by Walsh (see [4], pp. 358–360) such a result is very useful for applications.

2. LEMMAS. We shall need the following subsidiary results.

LEMMA 1. *If*

$$f(z)=\sum_0^\infty a_kz^k$$

is analytic in $|z|<1$, *where* $|f(z)|\leq 1$ *then*

$$|a_0|^2 + |a_k| \leq 1, \qquad k \geq 1 \tag{2.1}$$

and

$$\sum_0^\infty |a_k|^2 \leq 1 . \tag{2.2}$$

For (2.1) we refer to [3, p. 172, exer. #9]. Inequality (2.2) follows from the fact that for $0 < r < 1$

$$\sum_0^\infty |a_k|^2 r^{2k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta \leq 1 .$$

LEMMA 2. *Under the hypothesis of Lemma* 1 *we have*

$$\left|\sum_0^\infty \frac{a_k}{k+2} z^k\right| \leq \frac{1}{2}|a_0| + \frac{1}{3}(1 - |a_0|^2)|a_0||z| \quad \textit{for} \quad |z| < 1. \tag{2.3}$$

Proof of Lemma 2. By Schwarz's lemma

$$|f(\zeta)| \leq \frac{|a_0| + |\zeta|}{1 + |a_0||\zeta|}$$

for $|\zeta| < 1$. Hence

$$\left|\sum_0^\infty \frac{a_k}{k+2} z^k\right| = \left|\frac{1}{z^2}\int_0^z \zeta f(\zeta)\, d\zeta\right| \leq \frac{1}{|z|^2}\int_0^{|z|} |\zeta| \frac{|a_0| + |\zeta|}{1 + |a_0||\zeta|} d|\zeta|$$

$$= \frac{1}{2}|a_0| + (1 - |a_0|^2) \sum_{k=1}^\infty (-1)^{k+1} \frac{|a_0|^{k-1}}{k+2} |z|^k$$

$$\leq \frac{1}{2}|a_0| + \frac{1}{3}(1 - |a_0|^2)|a_0||z| .$$

LEMMA 3. *If*

$$g(z) = 1 + \sum_{k=1}^\infty \alpha_k z^k$$

is analytic in $|z| < 1$, *where*

$$\operatorname{Re} g(z) > 0 \tag{2.4}$$

and

(2.5) $$|g(z)| < M$$

then

(2.6) $$|\alpha_k| \le 2\frac{M^2 - 1}{M^2 + 1}.$$

Proof of Lemma 3. The function $G(z) = F^{-1}(w)$ where

$$F(w) = \left\{\left(\frac{iM - w}{iM + w}\right)^2 - \left(\frac{iM - 1}{iM + 1}\right)^2\right\} \Big/ \left\{\left(\frac{iM - w}{iM + w}\right)^2 - \left(\frac{iM + 1}{iM - 1}\right)^2\right\}$$

$$= \frac{1}{2}\frac{M^2 + 1}{M^2 - 1}(w - 1) + \ldots$$

maps the unit disk $|z| < 1$ onto the semicircular disk

$$D^+ = \{w : \operatorname{Re} w > 0, |w| < M\}$$

such that $G(0) = 1$, $G'(0) = 2(M^2 - 1)/(M^2 + 1)$. Since the function $g(z)$ maps the unit disk into D^+ and the function $G(z)$ is convex univalent it follows from a well-known result (see e.g. [3], p. 238, exer. #6) that

$$|\alpha_k| \le |G'(0)| = 2\frac{M^2 - 1}{M^2 + 1}, \qquad k \ge 1.$$

LEMMA 4. *If*

$$p_3(z) = \prod_{\nu=1}^{3}(1 - z_\nu z) = \sum_0^3 b_{k,1} z^k \in \mathscr{P}_3$$

then

(2.7) $$|b_{2,1}|^2 + |b_{3,1}|^2 \le 1.$$

Proof of Lemma 4. Let $|z_1| = \max_{1 \le \nu \le 3} |z_\nu|$. The polynomial

$$\hat{p}_3(z) = p_3\left(\frac{z}{z_1}\right) = 1 + \hat{b}_{2,1} z^2 + \hat{b}_{3,1} z^3$$

also belongs to $\mathscr{P}_3$ and $|b_{2,1}| \le |\hat{b}_{2,1}|$, $|b_{3,1}| \le |\hat{b}_{3,1}|$. Hence it is enough to prove (2.7) for $\hat{p}_3(z)$. We have

$$\hat{p}_3(z) = 1 - (1 - \hat{z}_2\hat{z}_3)z^2 - \hat{z}_2\hat{z}_3 z^3$$

where $|\hat{z}_2| \leq 1$, $|\hat{z}_3| \leq 1$ and $1 + \hat{z}_2 + \hat{z}_3 = 0$. Since $\hat{z}_2 + \hat{z}_3 = -1$ we may suppose

$$\hat{z}_2 = -a + ib, \hat{z}_3 = -1 + a - ib, \qquad 0 \leq a \leq 1/2.$$

Since $|\hat{z}_3| \leq 1$ we have $(1-a)^2 + b^2 \leq 1$, i.e.

$$b^2 \leq 2a - a^2. \tag{2.8}$$

We write $\hat{z}_2\hat{z}_3 = (-1 + a - ib)(-a + ib) = x + iy$, where

$$x = a(1-a) + b^2, y = b(2a-1).$$

Then

$$\begin{aligned} |\hat{b}_{2,1}|^2 + |\hat{b}_{3,1}|^2 &= |1 - x - iy|^2 + |x + iy|^2 \\ &= 2(x^2 + y^2 - x) \\ &= 2\{(b^2 + a(1-a))^2 + b^2(2a-1)^2 - b^2 - a(1-a)\} \\ &= 2\{(b^2 - a(1-a))^2 - a(1-a)\}. \end{aligned}$$

In view of (2.8) and since $0 \leq a \leq 1/2$, we have

$$(b^2 - a(1-a))^2 \leq a^2 \leq a(1-a),$$

and now Lemma 4 follows.

3. Proofs of theorems.

Proof of Theorems 1, 2, 3. It has been proved by Dieudonné [**1**, p. 7] that if

$$p_n(z) = \prod_{\nu=1}^{n} (1 - z_\nu z)$$

is a polynomial of degree n with all its zeros in $|z| \geq 1$ then in $|z| < 1$

$$\frac{p'_n(z)}{p_n(z)} = \frac{n}{z - \dfrac{1}{\Psi(z)}} \tag{3.1}$$

where $\Psi(z)$ is analytic and $|\Psi(z)| \leq 1$. We observe that if $p_n(z) \in \mathscr{P}_n$, i.e. $\Sigma_{\nu=1}^{n} z_\nu = 0$ then $\Psi(0) = 0$ and hence by Schwarz's lemma $\Psi(z) = z\psi(z)$ where $\psi(z)$ is analytic and $|\psi(z)| \leq 1$ in $|z| < 1$. Thus for polynomials $p_n(z) \in \mathscr{P}_n$ the representation (3.1) takes the form

$$\frac{p'_n(z)}{p_n(z)} = \frac{-nz\psi(z)}{1 - z^2\psi(z)}. \tag{3.2}$$

If $\varphi(z) = \{p_n(z)\}^\varepsilon = \Sigma_0^\infty b_{k,\varepsilon} z^k$ then

$$\frac{\varphi'(z)}{\varphi(z)} = \frac{-\varepsilon n z\psi(z)}{1 - z^2\psi(z)} \tag{3.3}$$

$$z\phi'(z) = \{z^3\phi'(z) - n\varepsilon z^2\phi(z)\}\,\psi(z). \tag{3.4}$$

Setting $\psi(z) = \Sigma_{\nu=0}^{\infty} c_\nu z^\nu$ and comparing coefficients on the two sides of (3.4) we get

$$kb_{k,\varepsilon} = \sum_{\substack{\nu=0\\ \nu\neq 1}}^{k-2} (-n\varepsilon + \nu)\, b_{\nu,\varepsilon} c_{k-2-\nu}, \qquad k \geq 2. \tag{3.5}$$

In particular

$$2b_{2,\varepsilon} = -n\varepsilon c_0, \qquad 3b_{3,\varepsilon} = -n\varepsilon c_1 \tag{3.6}$$

which give (1.9) and (1.10) immediately since the coefficients of a function $\psi(z)$ analytic and bounded by 1 in $|z| < 1$ are themselves bounded by 1.

Again from (3.5) we have

$$\begin{aligned} 4b_{4,\varepsilon} &= -n\varepsilon c_2 + (-n\varepsilon + 2)\, b_{2,\varepsilon} c_0 \\ &= -n\varepsilon c_2 - \frac{1}{2} n\varepsilon(-n\varepsilon + 2)\, c_0^2 \qquad \text{using (3.6)} \\ &= -n\varepsilon\{c_2 - \frac{1}{2}(n\varepsilon - 2)\, c_0^2\}. \end{aligned}$$

By (2.1)

$$|b_{4,\varepsilon}| \leq \frac{n}{8}\{(|n\varepsilon - 2| - 2)\,|c_0|^2 + 2\}$$

which readily gives (1.11), (1.12) and completes the proof of Theorem 2.

Theorem 3 is an immediate consequence of (3.6) and (2.1).

Now we come to the proof of Theorem 1. From inequalities (1.9)–(1.12) it follows that Theorem 1 holds for $k \leq 4$. For a given $n \geq 4$ let (1.8) hold for $k \leq j - 1$. We shall show that it then holds for $k = j$ and (for $n \geq 4$) the theorem will follow by the principle of mathematical induction. By formula (3.5) we have

$$\begin{aligned} j|b_{j,\varepsilon}| &\leq \sum_{\substack{\nu=0\\ \nu\neq 1}}^{j-2} |-n\varepsilon + \nu|\,|b_{\nu,\varepsilon}|\,|c_{j-2-\nu}| \\ &\leq (n + j - 2) \sum_{\nu=0}^{j-2} |b_{\nu,\varepsilon}|\,|c_{j-2-\nu}| \end{aligned}$$

$$\leq (n+j-2)\left(\sum_{\nu=0}^{j-2}|b_{\nu,\varepsilon}|^2\right)^{1/2}\left(\sum_{\nu=0}^{j-2}|c_{j-2-\nu}|^2\right)^{1/2}.$$

Using (2.2) and the induction hypothesis we deduce

$$j|b_{j,\varepsilon}| \leq (n+j-2)\left\{\sum_{\nu=0}^{j-2}(n/2)^{\nu}\right\}^{1/2}$$

$$= (n+j-2)(n/2)^{j/2}\left\{\sum_{\nu=0}^{j-2}(2/n)^{j-\nu}\right\}^{1/2}$$

$$< (n+j-2)(n/2)^{j/2}\frac{2/n}{\sqrt{1-(2/n)}}$$

$$< j\left(\sqrt{\frac{n}{2}}\right)^j \qquad \text{if } j \geq 5.$$

This completes the proof of (1.8) *for* $n \geq 4$. If $n = 2$ or 3 we argue as follows.

It follows from (2.7) that if

$$p_3(z) = \sum_0^\infty b_{k,1}z^k \in \mathscr{P}_3$$

then

$$|b_{2,1}| \leq 1, \; |b_{3,1}| \leq 1.$$

Since $|b_{k,1}| = 0$ for $k \geq 4$ we trivially have

$$|b_{k,1}| < \left(\sqrt{\frac{3}{2}}\right)^k, \qquad k \geq 2.$$

From (1.9), (1.10) and (1.12) we have

$$|b_{k,-1}| \leq \left(\sqrt{\frac{3}{2}}\right)^k \qquad \text{for } k \leq 4. \tag{3.7}$$

Hence (3.7) will be proved for all k if we show that it holds for $k = j$ provided it holds for $k \leq j-2$. So let (3.7) be true for $k \leq j-2$. From the identity

$$\frac{1}{1+b_{2,1}z^2+b_{3,1}z^3} \equiv \sum_0^\infty b_{k,-1}z^k$$

we have

$$b_{j,-1} + b_{j-2,-1}b_{2,1} + b_{j+3,-1}b_{3,1} = 0.$$

Using this, Lemma 4, and the induction hypothesis, we deduce

$$\begin{aligned}|b_{j,-1}| &\le (|b_{j-2,-1}|^2 + |b_{j-3,-1}|^2)^{1/2}(|b_{2,1}|^2 + |b_{3,1}|^2)^{1/2} \\ &\le (|b_{j-2,-1}|^2 + |b_{j-3,-1}|^2)^{1/2} \\ &\le \left(\left(\frac{3}{2}\right)^{j-2} + \left(\frac{3}{2}\right)^{j-3}\right)^{1/2} \\ &< \left(\sqrt{\frac{3}{2}}\right)^j .\end{aligned}$$

This completes the proof of (1.8) for $n = 3$.

If

$$p_2(z) = \prod_{\nu=1}^{2} (1 - z_\nu z) \in \mathscr{P}_2$$

then $z_2 = -z_1$. Hence $p_2(z) = 1 - (z_1 z)^2$ and

$$|b_{k,\varepsilon}| \le |z_1|^k \le 1 = \left(\sqrt{\frac{2}{2}}\right)^k, \qquad k \ge 2.$$

Next we prove Theorem 5 since we shall need it (in a weaker form) for the proof of Theorem 4.

Proof of Theorem 5. It was shown by Dieudonné (see [1], p. 7) that if

$$p_n(z) = \prod_{\nu=1}^{n} (1 - z_\nu z)$$

is a polynomial of degree n having all its zeros in $|z| \ge 1$ then

$$\Omega(z) = \frac{1 - \{p_n(z)\}^{1/n}}{z}$$

is analytic in $|z| < 1$ and $|\Omega(z)| \le 1$. If $p_n(z) \in \mathscr{P}_n$ then $\Omega(0) = 0$ and hence by Schwarz's lemma

$$\omega(z) = \frac{\Omega(z)}{z} = \frac{1 - \{p_n(z)\}^{1/n}}{z^2} \tag{3.8}$$

is analytic in $|z| < 1$ and $|\omega(z)| \le 1$. From (3.8) we get

$$\frac{p_n'(z)}{p_n(z)} = \frac{-n\{2z\omega(z) + z^2\omega'(z)\}}{1 - z^2\omega(z)} . \tag{3.9}$$

The two representations (3.2) and (3.9) for $p'_n(z)/p_n(z)$ give us the identity

(3.10) $$\{2\omega(z) + z\omega'(z)\} \equiv \{1 + z^2\omega(z) + z^3\omega'(z)\}\psi(z).$$

Setting

$$\omega(z) = \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu, \quad \psi(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu,$$

and comparing coefficients on the two sides of (3.10) we get

$$\alpha_0 = \frac{1}{2} c_0, \quad \alpha_1 = \frac{1}{3} c_1,$$

(3.11) $$\alpha_\nu = \frac{1}{\nu+2} c_\nu + \frac{1}{\nu+2} \sum_{\mu=0}^{\nu-2} (\mu+1)\,\alpha_\mu c_{\nu-2-\mu}, \quad \nu \geq 2.$$

In particular

$$\alpha_2 = \frac{1}{4} c_2 + \frac{1}{4}\alpha_0 c_0 = \frac{1}{4} c_2 + \frac{1}{8} c_0^2,$$

$$\alpha_3 = \frac{1}{5} c_3 + \frac{7}{30} c_0 c_1,$$

$$\alpha_4 = \frac{1}{6} c_4 + \left(\frac{5}{24} c_0 c_2 + \frac{1}{9} c_1^2 + \frac{1}{16} c_0^3\right),$$

$$\alpha_5 = \frac{1}{7} c_5 + \left(\frac{13}{70} c_0 c_3 + \frac{17}{84} c_1 c_2 + \frac{157}{840} c_0^2 c_1\right).$$

Thus

(3.12) $$\omega(z) = \sum_{\nu=0}^{\infty} \frac{1}{\nu+2} c_\nu z^\nu + \frac{1}{8} c_0^2 z^2 + \frac{7}{30} c_0 c_1 z^3$$

$$+ \left(\frac{5}{24} c_0 c_2 + \frac{1}{9} c_1^2 + \frac{1}{16} c_0^3\right) z^4 + \left(\frac{13}{70} c_0 c_3 + \frac{17}{84} c_1 c_2 + \frac{157}{840} c_0^2 c_1\right) z^5$$

$$+ \sum_{\nu=6}^{\infty} \left(\alpha_\nu - \frac{c_\nu}{\nu+2}\right) z^\nu.$$

Now let $|z| \leq 1/2$. By (2.1) we have

(3.13) $$\left|\frac{1}{8} c_0^2 z^2 + \frac{7}{30} c_0 c_1 z^3 + \frac{1}{30} c_0 c_2 z^4\right|$$

$$\leq \left\{\left(\frac{7}{60}+\frac{1}{120}\right)|c_0|^2+\frac{7}{60}|c_1|+\frac{1}{120}|c_2|\right\}|z|^2 \leq \frac{1}{8}|z|^2,$$

$$\left|\left(\frac{1}{16}c_0c_2+\frac{1}{16}c_0^3\right)z^4\right| \leq \frac{1}{16}|z|^4, \tag{3.14}$$

$$\left|\left(\frac{9}{80}c_0c_2+\frac{1}{9}c_1^2\right)z^4+\left(\frac{13}{70}c_0c_3+\frac{17}{84}c_1c_2+\frac{157}{840}c_0^2c_1\right)z^5\right|$$

$$\leq \left(\frac{9}{640}|c_2|+\frac{1}{72}|c_1|+\frac{13}{1120}|c_3|+\frac{17}{1344}|c_1|+\frac{157}{13440}|c_1|\right)|z|$$

$$\leq \left(\frac{9}{640}+\frac{1}{72}+\frac{13}{1120}+\frac{17}{1344}+\frac{157}{13440}\right)(1-|c_0|^2)|z|$$

$$\leq \frac{1}{6}(1-|c_0|^2)|z|. \tag{3.15}$$

Using (3.12)–(3.14) and Lemma 2 in (3.12) we get

$$|\omega(z)| \leq \frac{1}{8}|z|^2+\frac{1}{16}|z|^4+\frac{1}{2}|c_0|+\frac{1}{2}(1-|c_0|^2)|z|+\sum_{\nu=6}^{\infty}\left(|\alpha_\nu|+\frac{|c_\nu|}{\nu+2}\right)|z|^\nu$$

$$\leq \frac{1}{2}+\frac{1}{8}|z|^2+\frac{1}{16}|z|^4+\sum_{\nu=6}^{\infty}|\alpha_\nu||z|^\nu+\frac{1}{8}\sum_{\nu=6}^{\infty}|c_\nu||z|^\nu.$$

But by (2.2)

$$\sum_{\nu=6}^{\infty}|\alpha_\nu||z|^\nu \leq \left(\sum_{\nu=0}^{\infty}|\alpha_\nu|^2\right)^{1/2}\left(\sum_{\nu=6}^{\infty}|z|^{2\nu}\right)^{1/2} \leq \frac{|z|^6}{(1-|z|^2)^{1/2}} \leq \frac{2}{\sqrt{3}}|z|^6,$$

$$|\omega(z)| \leq \frac{1}{2}+\frac{1}{8}|z|^2+\frac{1}{16}|z|^4+\frac{3\sqrt{3}}{4}|z|^6.$$

Hence

$$\sum_{\nu=6}^{\infty}|c_\nu||z|^\nu \leq \left(\sum_{\nu=0}^{\infty}|c_\nu|^2\right)^{1/2}\left(\sum_{\nu=6}^{\infty}|z|^{2\nu}\right)^{1/2} \leq \frac{|z|^6}{(1-|z|^2)^{1/2}} \leq \frac{2}{\sqrt{3}}|z|^6.$$

This completes the proof of Theorem 5.

Proof of Theorem 4. By Theorem 5

$$p_n(z) = \{1 - z^2\omega(z)\}^n$$

where $\omega(z)$ is analytic in $|z| < 1$ and

$$|\omega(z)| \leq \frac{1}{2} + \frac{1}{4}|z|^2 \quad \text{for} \quad |z| \leq \frac{1}{2}.$$

If λ is a real number $\neq 0$ and $n > 6/|\lambda|$ then by simple geometrical considerations $\operatorname{Re}\phi_\lambda(z) > 0$ if $|z| < \rho_0$ where ρ_0 is the only positive root of the equation

$$\rho^4 + 2\rho^2 = 4\sin\frac{\pi}{2|\lambda|n}. \tag{3.16}$$

In other words, $\operatorname{Re}\phi_\lambda(\rho_0 z) > 0$ for $|z| < 1$. Besides, in $|z| < 1$

$$|\varphi_\lambda(\rho_0 z)| \leq \left(1 + \sin\frac{\pi}{2|\lambda|n}\right)^{|\lambda|n}.$$

Hence by Lemma 3

$$|b_{k,\lambda}|\rho_0^k \leq 2\frac{\left(1 + \sin\frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin\frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1}.$$

This gives

$$|b_{k,\lambda}| \leq 2\frac{\left(1 + \sin\frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin\frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1}\left(\frac{1}{\sqrt{\sqrt{1 + 4\sin\frac{\pi}{2|\lambda|n}} - 1}}\right)^k$$

from which the desired result follows at once.

It may be noted that for fixed λ

$$\frac{\left(1 + \sin\frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin\frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1} \to \frac{e^\pi - 1}{e^\pi + 1} \quad \text{as} \quad n \to \infty.$$

4. Some remarks.

REMARK 1. Theorem 2 can be easily extended to read as follows.

THEOREM 2′. *Let*

$$p_n(z) = \prod_{\nu=1}^{n} (1 - z_\nu z)$$

be a polynomial of degree n not vanishing in $|z| < 1$ *and let*

$$p'_n(0) = p''_n(0) = \ldots = p_n^{(l)}(0) = 0.$$

If

$$\varphi(z) = \{p_n(z)\}^\varepsilon = \sum_0^\infty b_{k,\varepsilon} z^k$$

where $\varepsilon = 1$ *or* -1 *then*

$$|b_{k,\varepsilon}| \leq n/k \qquad (l + 1 \leq k \leq 2l + 1),$$

and

$$|b_{2l+2,1}| \leq \frac{n}{2(l+1)^2}(n - l - 1), \quad |b_{2l+2,-1}| \leq \frac{n}{2(l+1)^2}(n + l + 1).$$

For the proof we simply need to observe that in $|z| < 1$

$$\frac{p'_n(z)}{p_n(z)} = \frac{-nz^l\psi(z)}{1 - z^{l+1}\psi(z)} \tag{4.1}$$

where $\psi(z)$ is analytic and $|\psi(z)| \leq 1$ for $|z| < 1$.

REMARK 2. The radii of starlikeness and of convexity of the family

$$\{z[p_n(z)]^\alpha : p_n(z) \neq 0 \quad \text{in} \quad |z| < 1, p_n(0) = 1\}$$

were determined by Dieudonné [1] with the help of the representation (3.1) for $p'_n(z)/p_n(z)$. In precisely the same way we may use (4.1) to determine the radii of starlikeness and of convexity of the family

$$\{z[p_n(z)]^\alpha : p_n(z) \neq 0 \text{ in } |z| < 1, p_n(0) = 1, p'_n(0) = p''_n(0) = \ldots = p_n^{(l)}(0) = 0\}.$$

We are thankful to Professor Hayman for giving a series of very inspiring lectures on the subject of his paper at the University of Montreal.

REFERENCES

1. J. Dieudonné, *Sur quelques propriétés des polynomes*, Actualités Sci. Indust. No. 114, Hermann, Paris, 1934.
2. W. K. Hayman, *Differential inequalities and local valency*, Pacific J. Math., **44** (1973), 117–137.
3. Z. Nehari, *Conformal Mapping*, 1st ed., McGraw-Hill, New York, 1952.
4. J. L. Walsh, *A theorem of Grace on the zeros of polynomials, revisited*, Proc. Amer. Math. Soc., **15** (1964), 354–360.

Received June 1972. The work of both authors was supported by National Research Council of Canada Grant A-3081 and by a grant of le Ministère de l'Education du Gouvernment du Québec.

UNIVERSITE DE MONTREAL