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CONTINUOUS MEASURES, BAIRE CATEGORY, AND UNIFORM CONTINUITY IN TOPOLOGICAL GROUPS

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In this paper we use an observation of M. Rajagopalan to show that each nondiscrete locally compact topological group can be written as a disjoint union of continuumly many closed nowhere dense G_δ sets. This observation also enables us to give a new constructive proof of a theorem of Kister. We show that in a nondiscrete noncompact locally compact group it is always possible to construct a bounded continuous function that is not left uniformly continuous. Finally this construction motivates a similar construction which yields examples of functions in LUC(G) but not in RUC(G) when G is a nondiscrete and nonunimodular locally compact topological group or when G is a nondiscrete locally compact metric group with inequivalent right and left uniform structures.

1. Introduction. In this paper we show that a simple observation concerning continuity of the Haar measure in a locally compact topological group may be used to derive some interesting set theoretic and topological properties of such groups. This observation has been used by M. Rajagopalan in [6] to show that extremally disconnected compact topological groups are discrete. Throughout this paper we shall assume that the continuum hypothesis holds.

Our underlying simple observation is the following one.

1.1 *Observation.* If G is a locally compact nondiscrete topological group, and if λ is left Haar measure then

(i) $\lambda\{x\} = 0$ for each $x \in G$,

(ii) there exists a nested sequence $\{U_n: n = 1, 2, \dots\}$ of neighborhoods of the identity $e \in G$ such that $\lambda(U_n) \to 0$.

In the sequel we will be employing the terms nowhere dense, first category, and second category, with their usual meaning.

DEFINITION. M is of the second category at $x \in M$ if $M \cap U$ is of the second category for each neighborhood U of x.

Obviously M is of the second category in every neighborhood in X if and only if M is of the second category at each $x \in X$.

2. Decomposition into nowhere dense G_δ sets. The author wishes to thank the referee for pointing out the following stronger form of a theorem of Kakutani and Kodaira [5] (see also 8.7, [4]) which is proved in Halmos 64G, [3]. His observation has as corollaries a strengthening of our original Theorem 2.2 from compactly generated groups to σ-compact groups, and the resulting more complete statement in Theorem 2.4 for all nondiscrete locally compact topological groups G. For completeness we give below a direct proof of the Kakutani-Kodaira Theorem suggested by the referee.

2.1. THEOREM (Kakutani-Kodaira). *Let G be a σ-compact, locally compact group with identity e. Then for every countable family $\{U_n : n = 1, 2, \ldots,\}$ of neighborhoods of e, there is a compact normal subgroup $N \subset G$ such that $N \subset \bigcap_{n=1}^{\infty} U_n$, and G/N is metrizable and has a countable base for its open sets.*

Proof. Write G as an increasing union $\bigcup_{n=1}^{\infty} F_n$ of compact sets F_n. *Select a sequence* $\{V_n: n = 1, 2, \ldots\}$ of neighborhoods of e so that $\bar{V}_1$ is compact, $V_n^2 \subset V_{n-1} \cap U_n$, and such that $xV_nx^{-1} \subset V_{n-1}$ for all $x \in F_n$. Now 5.6(iv) in [2] holds and so $N = \bigcap V_n$ is a closed normal subgroup and hence also compact. It now follows exactly as in 8.7[4] that $\{\phi(V_n) : n = 1, 2, \ldots\}$ is a countable base at N in G/N, where ϕ is the natural map of G onto G/N. Thus by 8.3 [4], G/N is metrizable. Also since $G = \bigcup_{n=1}^{\infty} F_n$ and ϕ is continuous, G/N must be σ-compact and hence Lindelöf. Thus G/N has a countable base for its open sets.

2.2 THEOREM. *Every infinite σ-compact locally compact nondiscrete topological group G may be written as a disjoint union of continuumly many nowhere dense compact G_δ sets.*

Proof. Choose a nested sequence $\{U_n: n = 1, 2, \ldots\}$ of open neighborhoods of $e \in G$, such that $\lambda(U_n) \to 0$. By Theorem 2.1 there is a compact normal subgroup $N \subset G$ such that $N \subset \bigcap U_n$, and G/N is metrizable and has a countable base for its open sets. Therefore card$(G/N) \leq \mathfrak{c}$. However G/N is not discrete since N is not open ($\lambda(N) = 0$). But then card $(G/N) \geq \mathfrak{c}$. (See Hewitt and Ross[4], 4.26, for an interesting proof of this fact.). Clearly each $\tilde{x} \in G/N$ is a compact G_δ. Hence so is each coset of N in G. Finally since

$$G = \bigcup \{xN : x \in \varphi^{-1}(\tilde{x}),\ \tilde{x} \in G/N,\ \varphi \text{ the quotient map}\},$$

the theorem follows.

Our method of proof allows us to deduce the following.

2.3. THEOREM. *Let G be a locally compact topological group. Then the following are equivalent:*

(i) *G is not discrete*

(ii) *G contains a nowhere dense compact G_δ subgroup N.*

(iii) *There exists a nested sequence $\{U_n\}$* of neighborhoods of $e \in G$ *such that $\lambda(U_n) \to 0$, where λ is Haar measure.*

Proof. (i) $\Rightarrow$ (iii) is our observation 1.1, and (ii) $\Rightarrow$ (i) since $e \in N$ is nowhere dense and hence not open. Thus we need only show that (ii) follows from (iii).

Choose a sequence of neighborhoods $\{V_n : n = 1, 2, \ldots\}$ of e such that $V_n^2 \subset V_{n-1} \cap U_n$. Then since $\bar{V}_n \subset V^2$ it follows that

$$N = \bigcap_{n=2}^{\infty} V_n \subset \bigcap_{n=2}^{\infty} \bar{V}_n \subset \bigcap_{n=2}^{\infty} V_n^2 \subset \bigcap_{n=2}^{\infty} V_{n-1} = N,$$

so that N is closed. Clearly $N \subset \cap U_n$ so that $\lambda(N) = 0$. Thus N is nowhere dense. By Theorem 5.6 in [4] N is a subgroup. Since we may assume that $\bar{U}_1$ is compact, our theorem is true.

2.4. THEOREM. *If G is a nondiscrete locally compact topological group then G may be written as a disjoint union of continuumly many closed nowhere dense G_δ sets.*

Proof. G contains an open nondiscrete σ-compact subgroup J to which Theorem 2.2 applies. Thus J may be written as a disjoint union $\cup_{\alpha \in A} F_\alpha$ where card $(A) = \mathfrak{c}$ and each F_α is a compact nowhere dense G_δ set. Let B be a subset of G such that $BJ = G$ and such that the family $\{bJ : b \in B\}$ is disjoint. For each $\alpha \in A$, let $E_\alpha = \cup_{b \in B} bF_\alpha$. Then each E_α is a closed nowhere dense G_δ set in G and $\cup_{\alpha \in A} E_\alpha = G$.

That Theorem 2.4 cannot be extended to read "G may be written as a disjoint union of continuumly many compact nowhere dense G_δ sets" is evident from the following example.

EXAMPLE. Let $G = R \times G_\circ$ where $G_\circ$ is discrete, card $(G_\circ) > \mathfrak{c}$, and R is the real numbers with the usual topology. Clearly any compact subset of G is contained in a union of only finitely many of the disjoint open sets $\{R \times \{x\} : x \in G_\circ\} \subset G$. Thus any $\mathfrak{c}$ compact subsets of G are contained in a union of the form $E = \cup_{x \in A} R \times \{x\}$, where card $(A) \leq \mathfrak{c}$. Thus E is a proper subset of G and hence so is a union of any $\mathfrak{c}$ compact subsets of G.

3. Decompositions into continuumly many disjoint sets of second category. S. Ulam in 1933 [10] (and [9]), proved the following assuming the continuum hypothesis holds.

3.1. THEOREM. *Let X be a perfect space and let $Z \subset X$ be of the second category in X where* card $(X) = \mathfrak{c}$. *Then there exists a collection of* $\mathfrak{c}$ *mutually disjoint sets in Z which are all of the second category.*

DEFINITION. A perfect space is a complete metric space with no isolated points.

For topological groups the above theorem takes the following form.

3.2. THEOREM. *Each nondiscrete σ-compact locally compact metric group G may be written as a disjoint union of* $\mathfrak{c}$ *sets each of the second category in G.*

W. Sierpinski in [8] was able to use Ulam's theorem to prove a considerably strengthened version for the real line which we state below.

3.3. THEOREM. *Each subset of the line of the second category in each interval of the line may be written as a disjoint union of* $\mathfrak{c}$ *sets each of the second category in each interval.*

We will show that the method of proof employed by Sierpinski is valid in the more general setting of locally compact, σ-compact metric spaces with no isolated points.

3.4. LEMMA. *Let X be a locally compact σ-compact metric space with no isolated points. If $M \subset X$ is of the second category, then there exists a neighborhood $U \subset X$ such that M is of the second category at each point of U.*

Proof. Since X is a σ-compact metric space it has a countable dense set $\{x_i\}$ and the open balls $B_{r,i} = \{y : d(y, x_i) < r, r \text{ rational}\}$ form a base for the topology. If the Lemma is false, then for each $B_{r,i}$ there is some $B_{r',j} \subset B_{r,i}$ such that $B_{r',j} \cap M$ is of the first category. Let $\{B_i\}$ be the collection of all $B_{r,i}$ such that $B_{r,i} \cap M$ is of the first category. Then

$$D = \complement\left(\bigcup_{i=1}^{\infty} B_i\right)$$

is a closed set which contains no open set (since each open set contains a B_i). Thus D is of the first category. But then $D \cap M$ is of the first category so that

$$M = (D \cap M) \cup \left[\bigcup_{i=1}^{\infty} (M \cap B_i) \right]$$

is a set of the first category, a contradiction.

Convention. Let $\{B_i\}$ denote the basis of open balls $\{B_{r,i}\}$ of rational radii for X.

3.5. LEMMA. *Let X be a locally compact σ-compact metric space with no isolated points. Let $M \subset X$ be of the second category in each open subset of X and let $U \subset X$ be open. Then there exists $Q \subset \bar{U} \cap M$ of the second category such that $M - Q$ is of the second category in each neighborhood of X.*

Proof. By the theorem of S. Ulam the set $\bar{U} \cap M$ contains $\mathfrak{c}$ mutually disjoint sets each of the second category. Let Φ denote this family of sets and let $A \in \Phi$. Let $\{C_i\} \subset \{B_i\}$ be the relative basis for the open sets in U. Since A is of the second category, there is C_k such that $A \cap C_k$ is of the second category at each point in C_k.

Observe now that the family $\{C_i\}$ is countable though Φ is uncountable. Thus there is a closed ball $\bar{C}_l$ such that uncountably many members of Φ are of the second category at each point of $\bar{C}_l$. Let $A_1, A_2 \in \Phi$ be two such sets.

Let $Q = C_l \cap A_2$. *Clearly the set $Q \subset \bar{U}$* is of the second category in $\bar{U}$. Also

$$\begin{aligned} M - Q &= (M - C_l) \cup (M \cap C_l - Q) \\ &= (M - C_l) \cup (C_l \cap (M - A_2)) \supset (M - C_l) \cup (C_l \cap A_1) \end{aligned}$$

since $A_1 \cap A_2 = \phi$, $A_1 \subset M$, $A_2 \subset M$. Thus $M - Q$ is of the second category at each point of C_l and at each point of $X - C_l$. *This proves the lemma.*

3.6. THEOREM. *Let X be a locally compact σ-compact metric space with no isolated points. Then each set $M \subset X$ of the second category in every neighborhood of X contains c mutually disjoint sets, each of the second category in every neighborhood of X.*

Proof. Let $B_1, B_2, \ldots, B_n, \ldots$ be the countable basis of open sets with rational radii. We proceed by induction.

By Lemma 3.5 given U an open set in X there exists a set $Q_1 \subset M \cap \bar{U}$ such that $M - Q_1$ is of the second category in every neighborhood in X. Let $\bar{U} = \bar{B}_1$.

Suppose now that sets $Q_1, \ldots, Q_{n-1}$ have been defined so that $M - (Q_1 \cup \ldots \cup Q_{n-1})$ is of the second category in each neighborhood in X. By 3.5 there exists a set Q_n of the second category contained in $\{M - (Q_1 \cup \ldots \cup Q_{n-1})\} \cap \bar{B}_n$ such that

$$\{M - (Q_1 \cup \ldots \cup Q_{n-1})\} - Q_n = M - \{Q_1 \cup \ldots \cup Q_n\}$$

is of the second category in every neighborhood. The sets $\{Q_n\}$ thus defined obviously satisfy

(i) $Q_n \subset \bar{B}_n \cap M \quad n = 1, 2, \ldots$

and are mutually disjoint.

By the theorem of S. Ulam, each Q_n contains $\mathfrak{c}$ mutually disjoint sets each of the second category. Let Ω be the first uncountable ordinal. We denote the subsets of Q_n, whose existence we've shown, by $\{Q_n^\xi : \xi < \Omega\}$. Then

(ii) $Q_n^\xi \subset Q_n \subset \bar{B}_n \cap M$, for $\xi < \Omega \quad n=1, 2, \ldots$

(iii) $Q_n^\xi \cap Q_n^\eta = \Phi$ for $\xi < \eta < \Omega$ $n = 1, 2, 3, \ldots$

We define

(iv) $M^\xi = \bigcup_{n=1}^{\infty} Q_n^\xi$ for each $\xi < \Omega$.

The set $Q_n^\xi \subset \bar{B}_n$ is of the second category. Thus M^ξ is of the second category in every neighborhood in X since B_n is a basis for the topology of X. Obviously $M^\xi \subset M$ for $\xi < \Omega$.

Since each of the sets $Q_1, \ldots, Q_n, \ldots$ are mutually disjoint, (ii) implies that $Q_m^\xi \cap Q_n^\eta = \Phi$ for $m \neq n, \xi < \Omega, \eta < \Omega$. Thus $M^\xi \cap M^\eta = \Phi$ for $\xi < \eta < \Omega$, proving the theorem.

3.7. Corollary. *Each nondiscrete σ-compact locally compact metric group may be written as a disjoint union of* $\mathfrak{c}$ *mutually disjoint sets each of the second category in every open set.*

4. Haar measure and uniform continuity in topological groups. In this section we show that the continuity of the Haar measure on nondiscrete locally compact groups takes the part of the metric in a metric space. This fact should be evident from the work of the first two sections. Below we give a new proof of the theorem of J. M. Kister [6]. The method of proof has been used by others in metric spaces, and was suggested to the author in the case of metric spaces by Professor C. Chou. The proof for metric spaces rests on the fact that it is possible to choose a descending sequence of open balls whose intersection contains no open sets (actually only a point).

4.1. THEOREM. *Let G be a locally compact nondiscrete noncompact topological group. Then there exists a bounded real valued function f that is continuous on G but not left uniformly continuous on G.*

Proof. Let U be a neighborhood of the identity such that $\bar{U}$ is compact. By induction choose a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $x_n \notin \cup_{i=1}^{n-1} x_i U$. Let B be a symmetric neighborhood of e satisfying $V \subset U$. Then the sets $\{x_i V : i \geq 1\}$ are pairwise disjoint. Since G is not discrete there is a sequence $V_1, V_2, \ldots, V_n, \ldots$ of nested neighborhoods of e satisfying $\lambda(V_i) \rightarrow 0$. We may choose $V_1 = V$ without loss of generality. We define now functions f_i, $i \geq 1$, as follows:

$$f_i(x_i) = 1, \qquad f_i(\complement\, x_i V_i) = \{0\}, \qquad 0 \leq f_i \leq 1.$$

Such functions always exist because G is completely regular. Let $f = \Sigma_{i=1}^{\infty} f_i$. Then f is continuous. However f is not left uniformly continuous. To see this, suppose that given $\varepsilon > 0$ we have that there exists W a neighborhood of e such that $x^{-1}y \in W$ implies that $|f(x) - f(y)| < \varepsilon$. Without loss of generality we may suppose $W \subset V$. Observe that if $\varepsilon < 1$, and if $x \in x_n W$ then $|f(x) - f(x_n)| < \varepsilon$. This means that $x \in x_n V_n$ so that $x_n W \subset x_n V_n$ and hence $W \subset V_n$. But then $W \subset \cap V_n$, contradicting the fact that $\lambda(V_i) \rightarrow 0$.

REMARK. It was pointed out by the referee that the above proof is similar to though simpler than the proof of Theorem 4.10, page 68, in R. B. Burkel [**1**]. This theorem is as follows: "If G is a locally compact topological group then $W(G) = C(G)$ if and only if G is compact." Here $W(G)$ is the set of weakly almost periodic functions on G, and $C(G)$ is the set of continuous functions on G. Our construction yields a simpler proof of the part of Burkel's theorem that deals with the nondiscrete case. We also note that the construction in Burkel also proves our Theorem 4.1.

Finally we observe that the method of proof used in 4.1 has actually been anticipated by Comfort and Ross [**2**]. Specifically their Theorems 1.2 and 2.2 employ a similar construction of a continuous function.

4.2. EXAMPLE. The function $f = \Sigma_i f_i$, where

$$f_i(x_i) = i, \qquad f_i(\complement\, x_i V_i) = \{0\}, \qquad 0 \le f_i \le i,$$

is an example of a function that is continuous, unbounded, and not uniformly continuous, if $\{V_i\}$, V are as above.

4.3. COROLLARY (Kister). *The following are equivalent for a nondiscrete locally compact topological group G:*

(a) *G is compact*

(b) *Every continuous real valued function on G is left uniformly continuous.*

The above theorem is concerned only with existence of a continuous function that is not left uniformly continuous. A natural question to ask at this point is whether there is a left uniformly continuous real valued function that is not right uniformly continuous. It is clear that if the right and left structure are equivalent then the answer is obviously negative. However, if the two structures are not equivalent this is a meaningful question. We show next that if the group is not unimodular then such functions always exist. The construction is similar to the above one but the technical details are slightly more complicated.

Let R^* denote the multiplicative group of positive real numbers.

DEFINITION. Let λ be the Haar measure on G. The modular function $\triangle : G \to R^*$ is defined by

$$\Delta(x) = \frac{\int_G f(tx^{-1})\lambda(dt)}{\int_G f(t)\lambda(dt)}, \qquad f \in C_{00}^{+}(G).$$

G is unimodular iff $\triangle \equiv 1$.

PROPERTIES.

(i) $\triangle$ is a continuous homomorphism

(ii) $\triangle(x) = \lambda(Ex)/\lambda(E)$ where $E \subset G$ is such that $\lambda(E) > 0$.

4.4. THEOREM. *Each locally compact, nondiscrete, nonunimodular group admits a function that is left uniformly continuous but not right uniformly continuous.*

Proof. Since G is not unimodular we have that $\triangle$ and $1/\triangle$ are unbounded on G. Let U be a compact symmetric neighborhood of e. Inductively we can choose a sequence $\{x_n\} \subset G$ such that

$$\text{(a)} \quad x_n \notin U_{n-1} = \bigcup_{i=1}^{n-1} U x_i U$$

and

$$\text{(b)} \quad \triangle(x_n) > n.$$

To see this observe that U_{n-1} is compact so $G \not\subset U_{n-1}$. Also $\triangle(U_{n-1})$ is a compact subset of $(0, \infty)$, so there is a point $x_n \in G$ such that $\triangle(x_n) > n$ and $\triangle(x_n) \notin \triangle(U_{n-1})$.

Now let $V \subset U$ be a symmetric neighborhood of e satisfying $V^2 \subset U$. Then it is easily seen that each of the following hold

(i) $x_i V \cap x_j V = \Phi \quad i \neq j$

(ii) $V x_i \cap V x_j = \Phi \quad i \neq j$

(iii) $V x_i \cap x_j V = \Phi \quad i \neq j.$

Let $f_1(x)$ be any continuous function such that

$$f_1(x_1) = 1, \qquad f_1(\complement\,[x_1 V]) = \{0\}, \qquad 0 \leq f_1 \leq 1.$$

Define, for each $i > 1$,

$$f_i(x) = f_1(x_1 x_i^{-1} x),$$

so that $f_i(x_i) = f_1(x_1) = 1$ and

$$f_i(\complement\, x_i V) = \{0\}.$$

Observe that the function $f = \Sigma_{i=1}^{\infty} f_i$ is continuous and even left uniformly continuous on G. However f is not right uniformly continuous. For if f were right uniformly continuous then given $1 > \varepsilon > 0$ there is W a neighborhood of e such that $xy^{-1} \in W$ implies that

$$|f(x) - f(y)| < \varepsilon.$$

Without loss of generality we may suppose that $W \subset V$. Then if $x \in Wx_n$ we have $x \notin Vx_m$, $x \notin x_mV (m \neq n)$ and $|f(x) - f(x_n)| < \varepsilon$. Defining $V_n = x_nVx_n^{-1}$ we see that $x \in x_nV = V_nx_n$ and hence $W \subset V_n$. *Since n* is arbitrary we have $W \subset \cap_{n\geq 1} V_n$. Observe now that

$$0 < \lambda(V) = \lambda(x_nV) = \lambda(V_nx_n) = \lambda(V_n)\Delta(x_n).$$

Since $\Delta(x_n) \to \infty$, we must have $\lambda(V_n) \to 0$. But then $\lambda(W) = 0$, a contradiction. Therefore f is not right uniformly continuous.

REMARK. This construction shows that in Theorem 4.1. the function f might possibly be right uniformly continuous if G is not unimodular.

4.5. THEOREM. *If G is a nondiscrete locally compact metric group with inequivalent uniform structures then G admits a function f that is left uniformly continuous but not right uniformly continuous.*

Proof. It is well-known (see [2], 8.18) that G has equivalent uniform structures iff given $x_n \to e$ and $\{y_n : n \geq 1\}$ any sequence in G then $y_n^{-1}x_ny_n \to e$. Thus if G has inequivalent uniform structures, there is a neighborhood U of e, a sequence $\{y_n\} \subset G$, and a sequence $x_n \to e$, such that $y_n^{-1}x_ny_n \notin U$ for each n.

We observe now that $\{y_n : n \geq 1\}$ is not contained in any compact subset F of G. This follows from the well known theorem (see [4], 4.9) that given any neighborhood U of e and compact set F there is a neighborhood V of e such that $x^{-1}Vx \subset U$, for all $x \in F$. (The condition $y_n^{-1}x_ny_n \in U$ for each n, implies that there is no neighborhood V of e such that $y_n^{-1}Vy_n \subset U$, for all n).

Without loss of generality we may assume that $\bar{U}$ is compact. Passing to subsequences if necessary we may assume that y_n satisfies

$$y_n \notin \bigcup_{i=1}^{n-1} \bar{U}y_i\bar{U}, \qquad n = 2, 3, 4, \ldots$$

Let $V \subset U$ satisfy $V^2 \subset U$. Then as in 4.4 we have

(i) $\{Vy_i : i \geq 1\}$ are pairwise disjoint

(ii) $\{y_iV : 1 \geq 1\}$ are pairwise disjoint

(iii) $y_iV \cap Vy_j = \Phi$ if $i \neq j$

Define f as in 4.4. As before let V_n be defined by $y_nV = V_ny_n$ and suppose f is right uniformly continuous. Then given $\varepsilon > 0$ there is a neighborhood W such that $xy^{-1} \in W$ implies that $|f(x) - f(y)| < \varepsilon$. Since we may suppose

$W \subset V$ we see as in 4.4 that $W \subset V_n$, for each n. Thus $W \subset \cap V_n$. *Clearly this implies*

$$y_n^{-1} W y_n \subset y_n^{-1} V_n y_n = V \subset U,$$

for all n. But then this contradicts the statement that there is no neighborhood V of e such that $y_n^{-1} V y_n \subset U$ for all n. Therefore f is not right uniformly continuous.

REMARK. It appears from this last theorem that the classes of left and right uniformly continuous real valued functions on a locally compact topological group with inequivalent uniform structures cannot coincide. However the author was not able to use this construction to prove this. What is clear is that the construction here may be used provided either

(i) $\lambda(\cap V_n) \to 0$

or

(ii) $\cap V_n$ contains no open sets for some sequence of points $\{x_n\} \subset G$ such that

$$V_n = x_n V x_n^{-1}.$$

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