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STRUCTURAL CONSTANTS II

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The study of finite groups having a Self-Centralizing Sylow subgroup of prime order p is an important part of the theory of finite groups. In this paper these groups are studied under some arithmetical hypotheses. A rational number r, depending on the group and the prime p, is defined and some classification results are obtained by assuming that r is bounded as a function of the prime p.

Let G be a finite group, P a Sylow p-subgroup of G of order p, for an odd prime p.

Fix an element $\pi \in G$ such that $P = \langle \pi \rangle$, and assume $C_G(P) = P$, $q = |N_G(P) : P| = (p-1)/t \neq p-1$, where $C_G(P)$ and $N_G(P)$ denote the centralizer of P in G and the normalizer of P in G, respectively.

Let $\pi = \pi_1, \pi_2, \ldots, \pi_t$ be the representatives of conjugacy classes of elements of order p, where $\pi_i \in P$, $1 \leq i \leq t$. For $1 \leq i, j, k \leq t$, denote by s_{ijk} the number of times a product of a conjugate of π_i, in $N_G(P)$, by a conjugate of π_j, in $N_G(P)$, equals π_k.

Denote by C_{ijk} the number of times a product of a conjugate of π_i, in G, by a conjugate of π_j, in G, equals π_k.

This paper studies the relation between the numbers s_{ijk} and C_{ijk}, $1 \leq i, j, k \leq t$.

Suppose G satisfies the condition

$$(*) \qquad C_{i11} = 0 \quad \text{whenever} \quad s_{i11} = 0, \qquad 1 \leq i \leq t.$$

Define a rational number $r = r(G, p)$ and a rational number $a = a(p, t)$ (p-average of G) as follows:

$$r(G,p) = \max\left\{ \frac{C_{i11}}{s_{i11}} \,\middle|\, \begin{matrix} 1 \leq i \leq t \\ s_{i11} \neq 0 \end{matrix} \right\}$$

$$a(p,t) = \frac{\sum_{i=1}^{t} s_{i11}}{t}$$

This number $r = r(G, p)$ has some interesting properties. For instance, it is true that

(a) $r \equiv 1 (\operatorname{mod} p)$ as a rational number.

(b) $\lim_{p\to\infty} r(A_p, p) = \infty$ where A_p denotes the alternating group on p letters.

There is also a conjecture involving this number r.

Let x be the degree of the exceptional character in the principal p-block of G.

If the order of G is equal to $r \cdot p \cdot x$ and G is a simple group, does it follow that G is isomorphic to A_7?

Using the theory of R. Brauer for prime to the first power [2] and a theorem of W. Feit [4] the following results are proved:

THEOREM 1. *Let G be a simple group with $a(p, t) = 1$ and $r^2 < 20 \cdot 521 \cdot p$. Then G is isomorphic to one of the following groups:*

(i) PSL(2, 7) $(p = 7)$
(ii) A_7 $(p = 7)$
(iii) $U_3(3)$ $(p = 7)$
(iv) PSL(3, 4) $(p = 7)$
(v) A_8 $(p = 7)$

THEOREM 2. *Let G be a simple group with $a(p, t) = 2$ and $r^2 < 9310 \cdot p$. Then G is isomorphic to one of the following groups:*

(i) PSL(2, 11), $p = 11$, $r = 1$.

(ii) M_{11}, $p = 11$, $r = \frac{35}{2}$.

(iii) M_{12}, $p = 11$, $r = 320$.

1. *Proof of Theorem* 1. Let G be a counter example for Theorem 1, if one exists.

As $a(p, t) = 1$, then $p = t^2 + t + 1$ where $q = (p - 1)/t = |N_G(P) : P|$. Thus G cannot be of type (A) PSL(2, p) since here, $t = 2$, $p = 7$ and PS(2, 7) is in the list. G also cannot be of type (B) SL(2, p—1) where $p - 1 = 2^m$, $m \geq 2$ since for this type, $q = t + 1 = 2$ and $t = 1$. This gives $q = p - 1$, which is against the hypothesis.

By Lemma 1.4 of [7], it follows that

$$(1) \qquad |G:N_G(P)| = \frac{g}{pq} \leqq r \cdot v \quad \text{where} \quad v = (q-1) \cdot \frac{p+q}{p-q}$$

Proposition 1.1. *Let G be a simple finite group as in the introduction and* $a(p, t) = 1$ (p-average of $G = 1$). *Then one of the following is true:*

(i) $s_{i11} = s_{211} = \ldots = s_{t11} = 1$

(ii) *G is of type* (A) PSL(2, p) *or type* (B) SL(2, $p-1$), $p - 1 = 2^m$, $m \geq 2$.

(iii) $t = 2$

(iv) $(p + q)$ *divides* $g = |G|$.

Proof. Assume neither (i), (ii), nor (iii) holds. Then it can be shown that (iv) holds.

If (i) is false, then some $s_{i_0 11} = 0$, $1 \leq i_0 \leq t$ since $a(p, t) = 1$.

By (*), $C_{i_0 11} = 0$.

Using character theory, the following formulas are obtained (see [3]):

$$s_{ijk} = \frac{pq}{p^2}\left[q + B_{ijk}\right], \quad 1 \leqq i, j, k \leqq t \tag{2}$$

$$C_{ijk} = \frac{g}{p^2}\left[l + A_{ijk}\right], \quad 1 \leqq i, j, k \leqq t \tag{3}$$

(4) $t \geq 3 \Rightarrow A_{ijk} = \epsilon q B_{ijk}/x$, where $x =$ degree of exceptional character in the principal p- block $\epsilon = \pm 1$, $x = bp + \epsilon q$, $b =$ integer, $1 \leq i, j, k \leq t$. Since it is assumed that $t \geq 3$, it follows that $B_{i_0 11} = -q$, $l = -A_{i_0 11} = q^2 \epsilon / x \Rightarrow \epsilon = +1$ and $x = bp + q$.

$$l(p - 1)l \geq p - q. \tag{5}$$

So $$(p - 1)l = \frac{q^2(p - 1)}{x} \quad \text{and} \quad x = \frac{(p - 1)q^2}{(p - 1)l}$$

Thus $$x \leq \frac{(p - 1)q^2}{p - q}, \quad bp \leq \frac{(p - 1)q^2 - (p - q)q}{p - q}$$

$$bp \leq \frac{pq^2 - q^2 - pq + q^2}{p - q} = \frac{pq(q - 1)}{(p - q)}$$

and $$b \leq \frac{q(q - 1)}{p - q} = \frac{(t + 1)t}{(t^2 + t + 1) - (t + 1)} = \frac{(t + 1)t}{t^2} = \frac{t + 1}{t} < 2.$$

Thus $b \leq 1$. Since G is not of type (A) nor of type (B), then by a theorem of Feit [4], $b = 1$ and $x = p + q$, dividing g. Hence the Proposition 1.1.

PROPOSITION 1.2. *There are the following possibilities for t and p:*
(i) $t = 2$, $p = 7$.
(ii) $t = 3$, $p = 13$.
(iii) $t = 5$, $p = 31$.
(iv) $t = 6$, $p = 43$.
(v) $t = 8$, $p = 79$.

Proof. First $t \leq 13$ is proved. For, if $t > 14$ is assumed, then $p = t^2 + t + 1 \geq 211$.

By (1) (see Lemma 1.4 of [7]), it follows that

$$|G{:}N_G(P)| = Mp + 1 \leqq r\cdot v = r(q-1)\left(\frac{p+q}{p-q}\right).$$

Now, $(p + q)/(p - q) < (t + 1)/(t - 1)$ since $p + q = p + (p - 1)/t = ((t + 1)p - 1)/t$ and $p - q = ((t - 1)p + 1)/t$.
Thus

$$Mp + 1 < r\cdot(q-1)\cdot\frac{t+1}{t-1}.$$

Since $(t + 1)/(t - 1)$ is a decreasing function of t, $Mp + 1 < r(q - 1)\ 15/13$.

By a theorem of Brauer-Nagai [5], it may be assumed that $M \geq p + 3$. Thus it follows that

$$(p+3)\cdot p < r(q-1)\cdot\frac{15}{13}.$$

Now,
$$q - 1 = \frac{p-1}{t} - 1 = \frac{p-(t+1)}{t}$$

$$(p+3)q < r\cdot\frac{[p-(t+1)]}{t}\cdot\frac{15}{13} \leqq r\cdot\frac{[p-(t+1)]\cdot 15}{14\cdot 13}$$

$$p + 3 < \frac{20521 \times (15)^2}{(14)^2 \times (13)^2} < 140$$

$$p + 3 < \frac{r\cdot 15}{14\cdot 13}$$

$$(p+3)^2 < \frac{r^2\cdot(15)^2}{(14)^2\times(13)^2} \leqq \frac{20521\cdot p\cdot(15)^2}{(14)^2\times(13)^2}$$

and this contradicts $p \geq 211$. Thus $t \leq 13$.

For the cases $t = 4, 7, 9, 10, 11, 13$, $p = t^2 + t + 1$ is not a prime number.

For case $t = 12$, it follows that $p = 157$ and

$$Mp + 1 \leqq r \cdot v = r(q-1)\frac{p+q}{p-q} \quad \text{giving} \quad M^2 p < \frac{20521}{157} \cdot \frac{(85)^2}{36}$$

$$M < \sqrt{\frac{20521}{157}} \cdot \frac{85}{6}$$

and $$M < 163.$$

But by Brauer-Nagai [5], $160 \leq M < 163$ and there are only three possibilities for $M = 160, 161, 162$. Here $t = 12$, $q = 13$, $p = 157$. Hence

$$g = pq(Mp + 1) = \text{odd} \times (M \times 157 + 1).$$

If M is even, g becomes odd and this case is out by Thompson-Feit odd order paper.

If $M = 161$, then it becomes $161 \times 157 + 1$ and in this case,

$$g = (13) \times 157 \times (161 \times 157 + 1) = 13 \times 157 \cdot 298$$

$g = 2 \times$ (odd number), and G is not simple by Burnside. Hence the Proposition 1.2.

PROPOSITION 1.3.

(i) *For case* $t = 8, p = 73, q = 9$, *it follows that* M *is odd and of type* $79 + 4K = M$, $K = 0, 1, 2, \ldots, 23$ and $g = 657 \cdot (73M + 1)$.

(ii) *For case* $t = 6, p = 43, q = 7$, *it follows that* M *is odd and of type* $49 + 4K = M$, $K = 0, 1, 2, \ldots, 33$, $g = 301 \cdot (43 \cdot M + 1)$ *and* $(5)^2$ *divides* G.

(iii) *For case* $t = 5, p = 31, q = 6$, *it follows that* M *is odd and of type* $35 + 2K = M$, $K = 0, 1, \ldots, 72$, $g = 186 \cdot (31 \cdot M + 1)$ *and* $p + q = 37$ *divides* g.

(iv) *For case* $t = 3$, $p = 13$, $q = 4$, *it follows that* $16 \leq M \leq 225$, $g = 52 \cdot (13 \cdot M + 1)$ *and* $p + q = 17$ *divides* g.

(v) *For case* $t = 2, p = 7, q = 3$, *it follows that* M *is odd and of type* $13 + 4K = M$, $K = 0, 1, \ldots, 64$, $g = 21 \cdot (7 \cdot M + 1)$.

Proof. All these bounds are obtained by using

$$Mp + 1 \leqq r \cdot v = r \cdot (q - 1)\frac{p + q}{p - q} \quad \text{and} \quad r \leqq \sqrt{20521 \cdot p},$$

Thompson-Feit odd order paper, Burnside Theorem for $g = 2 \times$ (odd number) and Brauer-Nagai [**5**], $M \geq p + 3$, are also used.

In order to have $p + q$ divides g in cases $t = 6$, $t = 5$ and $t = 3$, Lemma 2.1 [**7**] and Proposition 1.1. may be used.

Hence Proposition 1.3.

Case $t = 3, p = 13, q = 4$.

Using a simple computer program, only 10 possibilities arise, and all of them may be eliminated by using the results of Burnside [**9**], Fong [**6**], Alperin-Brauer-Gorenstein [**1**], Gorenstein-Walter [**8**] and Walter [**11**].

Case $t = 5, p = 31, q = 6$.

In this case, only 2 possibilities arise, and they are eliminated by using Fong [**6**], Alperin-Brauer-Gorenstein [**1**], Gorenstein-Walter [**8**] and Walter [**11**].

Case $t = 6, p = 43, q = 7$.

In this case there is only one possibility and it is eliminated by using Fong [**6**], Alperin-Brauer-Gorenstein [**1**], Gorenstein-Walter [**8**] and Walter [**11**].

Case $t = 8, p = 73, q = 9$.

Then $g = 657 \cdot (73 \cdot M + 1)$, $M = 79 + 4K$ and $K = 0, 1, 2, \dots, 23$.

Using some characterization's theorems previously mentioned, all but one case are eliminated. That case is:

$$m = 135, \; g = 2^7 \cdot 3^2 \cdot 7 \cdot 11 \cdot 73.$$

This case is eliminated by using character theory as follows:

$$M = 135, \; g = 2^7 \cdot 3^2 \cdot 7 \cdot 11 \cdot 73.$$

Let S be a Sylow 11 subgroup of G. Let $n = |G : N(S)|$. Assume $n \neq 1$. By Burnside w.m.a., $|N(S)/C(S)| = 2$. Now $73|n$, and by calculation the possibilities for $n \neq 1$, $n \equiv 1 \pmod{11}$ are:

(a) 73×8
(b) $73 \times 8 \times 32$
(c) $73 \times 7 \times 64$
(d) $73 \times 7 \times 9$,

where $|N(S)/C(S)| = 2$.

Let f_0 be the degree of exceptional character in $B_0(11)$ (= principal 11-block), and let $1 \neq f_1$ be the degree of nonexceptional character in $B_0(11)$.

$f_0 \equiv \pm 2 \pmod{11}$; $f_1 \equiv \pm 1 \pmod{11}$. There are signs ϵ_0, ϵ_1 such that:

$$1 + \epsilon_0 f_0 + \epsilon_1 f_1 = 0. \text{ Also, } f_0, f_1 \text{ both divide } 2n.$$

Note also that $C(S) \neq S$ is in all the possibilities (a), (b), (c), and (d).

Case (a). $n = 73 \times 8$. Here $2n = 16 \times 73$, $f_0, f_1 \mid 16 \times 73$ and $(f_0, f_1) = 1$. Since $73 \equiv \pm 1 \pmod{11} \Rightarrow f_1 \mid 16$, $f_1 \equiv \pm 1 \pmod{11}$ $\Rightarrow f_1 \equiv 1$.

Case (b). $n = 32 \times 3 \times 73$; here $f_0, f_1 \mid 2n = 64 \times 3 \times 73$.

Assume first $73 \mid f_0$. Then, by calculation, f_0 becomes even $\Rightarrow f_1 \mid 3 \Rightarrow$ $f_1 = 1$. Thus, $73 \nmid f_0$. Hence, $f_0 \mid 64 \times 3$.
By calculation there are two possibilities for f_0:

(1) $f_0 = 64 \equiv -2 \pmod{11} \Rightarrow f_1 = 65 \nmid 64 \times 3 \times 73$.

(2) $f_0 = 8 \times 3 = 24 \equiv +2 \pmod{11} \Rightarrow \epsilon_0 = -1: \Rightarrow f_0 = 24 = 1 + f_1 \Rightarrow f_1 = 23$.

Case (c). $2n = 128 \times 7 \times 73; f_0, f_1 \mid 2n$.

Assume first $73 \mid f_0$. By calculation f_0 becomes even $\Rightarrow f_1 \mid 7 \Rightarrow f_1 = 1$. Thus, $73 \nmid f_0$. Hence, $f_0 \mid 128 \times 7$. By calculation the possibilities for f_0 are:

(1) $f_0 = 64 \equiv -2 \pmod{11} \Rightarrow f_1 = 65 \nmid 2n$.

(2) $f_0 = 7 \times 16 \equiv 2 \pmod{11} \Rightarrow f_1 = -1 + 7 \times 16 = 111 \nmid g$.

Case (d). $2n = 73 \times 9 \times 7 \times 2; f_0, f_1 \mid 2n$.

Assume first $73 \mid f_0$. By calculation f_0 becomes even $\Rightarrow f_1 \mid 9 \times 7, f_1 \equiv \pm 1 \pmod{11} \Rightarrow f_1 = 21 \equiv -1 \pmod{11} \Rightarrow \epsilon_1 = -1 \Rightarrow \epsilon_0 = +1$ and $f_1 = 1 + f_0 \Rightarrow f_0 = 20 \nmid 2n$.

Thus $73 \nmid f_0$. Hence $f_0 \mid 9 \times 7 \times 2$. By "Stanton condition" it may be assumed that $f_0 \geq 2 \times 11 - 1 = 21$.

By calculation $f_0 = 2 \times 3 \times 7$ is the only possibility for $f_0 \geq 21$. But here $f_0 = 42 \equiv -2 \pmod{11}$, $\epsilon_0 = +1 \Rightarrow f_1 = 43 \nmid g$. Hence, this case is out.

Case $t = 2, p = 7, q = 3$.

In this case $g = 21 \cdot (7M + 1)$, $M = 13 + 4K$, $K = 0, 1, \dots, 64$.

The following is true:

(i) $M = 1, g = 21 \times 8 = |\mathrm{PSL}(2, 7)|$.

(ii) $M = 17, g = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = |A_7|$.
(iii) $M = 41, g = 2^5 \cdot 3^3 \cdot 7 = |U_3(3)|$.
(iv) $M = 137, g = 2^6 \cdot 3^2 \cdot 5 \cdot 7 = |A_8| = |\mathrm{PSL}(3, 4)|$.

All these groups appear in the list and they are not counter-examples for the proposed Theorem.

The other possibilities are eliminated by using previously mentioned theorems or character theory as in the preceding case.

This finishes Theorem 1.

2. *Proof of Theorem* 2. Let G be a counter example for Theorem 2. From $a(p, t) = 2$, it follows that $p = 2t^2 + t + 1$ and $q = 2t + 1$, where $q = (p-1)/t = |N_G(P) : P|$.

G cannot be of type (A) since here $t = 2, p = 11$ and $G \approx \mathrm{PSL}(2, 11)$ is not a counter example. Also G cannot be of type (B) since $q = 2t + 1 \neq 2$.

By Lemma 1.4 of [7], there is

$$(1) \qquad |G : N_G(P)| = \frac{g}{pq} \leq r \cdot v \quad \text{where} \quad v = (q-1)\frac{p+q}{p-q}.$$

PROPOSITION 2.1. *There are the following possibilities for t and p:*
(i) $t = 2, \quad p = 11$
(ii) $t = 4, \quad p = 37$
(iii) $t = 6, \quad p = 79$
(iv) $t = 8, \quad p = 137$

Proof. $p = 2t^2 + t + 1$ prime number implies that t is even. First, assume $t > 10$, hence $t \geq 12$.

As before, by Brauer-Nagai Theorem [5].

It may be assumed that

$$(p+3)p + 1 \leqq Mp + 1 \leqq r \cdot (q-1)\frac{p+q}{p-q}$$

$$(p+3)p < r \cdot \frac{p-(t+1)}{t} \cdot \frac{t+1}{t-1} \quad \text{and}$$

since $(t+1)/(t^2 - t)$ is a decreasing function, then

$$(p+3)^2 < \left(\frac{r \cdot 13}{12 \cdot 11}\right)^2 \leq \frac{9310 \cdot p \cdot (13)^2}{(11)^2 \cdot (12)^2}$$

$$(p+3) < \frac{9310 \cdot (13)^2}{(11)^2 \cdot (12)^2} < 91$$

and $p < 88$.

But $t \geq 12 \Rightarrow p \geq 2 \times (12)^2 + 12 + 1 > 88$, a contradiction. Thus $t \leq 10$.

Now, if $t = 10$, then $p = 211$, $q = 21$.

$$Mp + 1 \leq r \cdot v = r(q-1) \cdot \frac{p+q}{p-q}$$

$$\text{and} \quad Mp < r \cdot 20 \cdot \frac{232}{190} = \frac{464}{19} \cdot r.$$

This implies $M < 180 < p + 3$ and this situation is eliminated by the theorem of Brauer-Nagai [5].

PROPOSITION 2.2.

(ii) *For case* $t = 6, p = 79, q = 13, M = 85 + 4K, K = 0, \ldots, 26$; *and* $g = 2329 \cdot (137 \cdot M + 1)$.

(ii) *For case* $t = 6, p = 79, q = 13, M = 85 + 4K, K = 0, \ldots, 26$; *and* $g = 1027 \cdot (79M + 1)$.

(iii) *For case* $t = 4, p = 37, q = 9, M = 43 + 4K, K = 0, \ldots, 41$; *and* $g = 333 \cdot (37 \cdot M + 1)$.

(iv) *For case* $t = 2, p = 11, q = 5, M = 17 + 4K, K = 0, \ldots, 73$; *and* $g = 55 \cdot (11 \cdot M + 1)$.

Proof. All bounds for M are obtained by using the same argument, $Mp + 1 \leq r \cdot v$ and to complete the proof the odd order paper of Feit-Thompson and Burnside's result for $g = 2 \times$ (odd number) are used. Brauer-Nagai [5] is also used to get $M \geq p + 3$.

Case $t = 8, p = 137, q = 17$.

These 7 possibilities are eliminated by using the already mentioned characterization theorems.

Case $t = 6, p = 79, q = 13$.

All possibilities but one are eliminated, by using directly the characterization results previously mentioned. This case is finished by eliminating $g = 2^6 \cdot 13 \cdot 79 \cdot 179$ which is a N-group but not a Suzuki group.

Case $t = 4, p = 37, q = 9$.

All cases but one are eliminated by using directly characterization results previously mentioned, the possibility is,

$$g = 2^6 \cdot 3^2 \cdot 5 \cdot 17 \cdot 37.$$

and this is eliminated as follows:

$$m = 147, g = 2^6 \cdot 3^2 \cdot 5 \cdot 17 \cdot 37.$$

Let $S =$ Sylow 17 subgroup of G. Let $n = |G : N(S)|$. Assume $n \neq 1$. By Burnside (see [9]) $|N(S) / C(S)| = 2, 4, 8,$ or 16. Also, $37 \mid n$.

By calculation we have the following possibilities for n:

(a) $n = 37 \times 3 \times 2$
(b) $n = 37 \times 5 \times 8$
(c) $n = 37 \times 5 \times 9 \times 16$.

Case (a). $n = 37 \times 3 \times 2$, $|N(S) / C(S)| = q_0 = 2, 3, 8,$ or 16.

The degrees f_0 (degree of exceptional character), $f_1, f_2, \dots, f_{q_0-1}$, all not equal to 1, in $B_0(17) =$ principal 17-block, must divide $q_0 \cdot n$. Thus, they must divide $32 \times 3 \times 37$.

The nonexceptional ones, $f_1, \dots, f_{q_0-1}$ are all $\equiv \pm 1 (\text{mod } 17)$ and $f_0 \equiv \pm q_0$ (mod 17).

Now, by calculation the possibilities for numbers $u \equiv \pm 1$ (mod 17) dividing $32 \times 3 \times 37$ are:

(1) $37 \times 3 \times 2 \equiv +1$ (mod 17)
(2) $37 \times 3 \times 32 \equiv -1$ (mod 17)
(3) $16 \equiv -1$ (mod 17).

In any case all $f_1, f_2, \dots, f_{q_0}$ are even. By relation $1 + \epsilon_0 f_0 + \epsilon_1 f_1 + \dots + \epsilon_{q_0-1} f_{q_0-1} = 0$, where $\epsilon_i = \pm 1$, $i = 0, \dots, q_0 - 1$, f_0 becomes odd dividing $32 \times 3 \times 37 \Rightarrow f_0 \mid 3 \times 37$. Then the only possibility for f_0 is $f_0 = 3 \times 37 = 111 \equiv -8$ (mod 17) $q_0 = 8$.

Thus, $\epsilon_0 = +1$, and some $\epsilon_i = -1$ say $\epsilon_1 = -1$. Then $f_1 = 37 \times 3 \times 32$ or 16.

Assume $f_1 = 37 \times 3 \times 32 = 3552$.

$$1 + 3 \times 37 - 37 \times 3 \times 32 + \epsilon_2 f_2 + \dots + \epsilon_7 f_7 = 0$$
$$112 - 3552 + \epsilon_2 f_2 + \dots + \epsilon_7 f_7 = 0.$$

Now since $|112 - 3552| > 6 \times 37 \times 3 \times 2$, it is not possible that $f_1 = 3552$.

Hence, the situation is:

$1 + 37 \times 3 - 16 - 16 - \dots - 16 = 0$. Hence,

$$f_i = 16 < 2 \times 17 - 1.$$

By "Stanton condition" [10] we have $C(S) = S$.

But $|N(S)/C(S)| = 8$.

$n = 37 \times 3 \times 2 \Rightarrow 2 \mid |C(S)|$. Hence, we finish Case (a).

Case (b). $n = 37 \times 5 \times 8$. $|N(S)/C(S)| = q_0 = 2, 4$, or 8 since $|G|_2 = 64$.

Here, $f_0, f_1, \ldots, f_{q_0-1}$ divide $64 \times 5 \times 37$. By calculation the only possibilities for f_i, $i \geq 1$ are:

(1) $37 \times 5 \times 8 \equiv +1 \pmod{17}$

(2) $16 \equiv -1 \pmod{17}$.

Since both are even we have f_0 odd $| 37 \times 5$. Hence, by calculation $f_0 = 37 \times 5 \equiv -2 \pmod{17}$, $q_0 = 2$, $\epsilon_0 = +1$. Now, $(f_0, f_1) = 1 \Rightarrow f_1 = 16 < 2 \times 17 - 1 = 33$, by "Stanton condition" [10] $C(S) = S$. But $|N(S)/C(S)| = 2$ and $n = 37 \times 5 \times 8 \Rightarrow 2 | \, |C(S)|$.

Case (c). $n = 37 \times 5 \times 9 \times 16$, $|N(S)/C(S)| = q_0 = 2$ or 4, since $|G|_2 = 64$.

Here $f_0, f_1, \ldots, f_{q_0-1}$ divide $64 \times 9 \times 5 \times 37$.

By calculation the possibilities for the degree of nonexceptional characters in $B_0(17)$ are:

(1) $37 \times 2 \times 3 = 222 \equiv +1 \pmod{17}$

(2) $73 \times 32 \times 3 = 1332 \equiv -1 \pmod{17}$

(3) $37 \times 8 \times 5 = 1480 \equiv +1 \pmod{17}$

(4) $37 \times 9 \times 5 = 1665 \equiv -1 \pmod{17}$

(5) $37 \times 9 \times 5 \times 16 = 246640 \equiv -1 \pmod{17}$ (1)

(6) $5 \times 8 \times 3 = 120 \equiv +1 \pmod{17}$

(7) $32 \times 9 = 288 \equiv -1 \pmod{17}$

(8) $18 = 18 \equiv +1 \pmod{17}$

(9) $16 = 16 \equiv -1 \pmod{17}$

By calculation the possibilities for $f_0 \equiv \pm 2 \pmod{17}$ ($q_0 = 2$) are:

(1)′	$37 \times 5 \equiv -2 \pmod{17}$	$\epsilon_0 = +1$	
(2)′	$37 \times 3 \times 4 \equiv +2 \pmod{17}$	$\epsilon_0 = -1$	
(3)′	$37 \times 3 \times 64 \equiv -2 \pmod{17}$	$\epsilon_0 = +1$	
(4)′	$37 \times 5 \times 16 \equiv +2 \pmod{17}$	$\epsilon_0 = -1$	
(5)′	$37 \times 9 \times 5 \times 2 \equiv -2 \pmod{17}$	$\epsilon_0 = +1$	
(6)′	$37 \times 9 \times 5 \times 32 \equiv +2 \pmod{17}$	$\epsilon_0 = -1$	(1)′
(7)′	$5 \times 3 \equiv -2 \pmod{17}$	$\epsilon_0 = +1$	
(8)′	$5 \times 3 \times 16 \equiv +2 \pmod{17}$	$\epsilon_0 = -1$	
(9)′	$32 \equiv -2 \pmod{17}$	$\epsilon_0 = +1$	
(10)′	$36 \equiv +2 \pmod{17}$	$\epsilon_0 = -1$	
(11)′	$9 \times 64 \equiv -2 \pmod{17}$	$\epsilon_0 = +1$	

By calculation the possibilities for $f_0 \equiv \pm 4 \pmod{17}$, $q_0 = 4$, are:

(1)″	$37 \times 3 \times 8 \equiv +4 \pmod{17}$	$\epsilon_0 = -1$
(2)″	$37 \times 5 \times 2 \equiv -4 \pmod{17}$	$\epsilon_0 = +1$
(3)″	$37 \times 5 \times 32 \equiv +4 \pmod{17}$	$\epsilon_0 = -1$

$$
\begin{array}{lll}
(4)'' & 37 \times 9 \times 5 \times 4 \equiv -4 \pmod{17} & \epsilon_0 = +1 \\
(5)'' & 37 \times 9 \times 5 \times 64 \equiv +4 \pmod{17} & \epsilon_0 = -1 \\
(6)'' & 5 \times 3 \times 2 \equiv -4 \pmod{17} & \epsilon_0 = +1 \\
(7)'' & 5 \times 3 \times 32 \equiv +4 \pmod{17} & \epsilon_0 = -1 \\
(8)'' & 64 \equiv -4 \pmod{17} & \epsilon_0 = +1 \\
(9)'' & 8 \times 9 \equiv +4 \pmod{17} & \epsilon_0 = -1
\end{array} \qquad (1)''
$$

Case. $|N(S)/C(S)| = q_0 = 2, f_0$ one of values in list $(1)', (2)', \dots, (11)'$.

Here $\epsilon_1 f_1 = -1 - \epsilon_0 f_0$. Note that: $q_0 = 2 \Rightarrow |C(S)|$ is divided by $2 \Rightarrow C(S) \neq S$. Also, f_1 divides: $64 \times 9 \times 5 \times 37$. Using condition $(f_0, f_1) = 1$ and "Stanton condition" (see [**10**]), the following possibilities are eliminated at once for f_0 in list $(1)', (2)', (3)', (4)', (5)', (6)', (7)'$.

Now, by calculation and using formula $\epsilon_1 f_1 = -1 - \epsilon_0 f_0$ and list (1) and $(1)'$ all other cases are eliminated.

Case. $q_0 = 4$. Here $|N(S)/C(S)| = 4$, $n = 16 \times 9 \times 5 \times 37$, $|C(S)| = |S|$.

$$1 + \epsilon_0 f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3 = 0.$$

Looking at list $(1)''$ we see that f_0 is even. Hence,

$$\epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3 \text{ is odd.}$$

Assume first: f_1, f_2 and f_3 are all odd. Looking at list (1), it follows that $f_1 = f_2 = f_3 = 37 \times 5 \times 9 \equiv -1 \pmod{17}$. This implies: $\epsilon_0 = +1$ and $f_0 = 3 \times 37 \times 5 \times 9 - 1 = 4994$ and this possibility is out by looking at list $(1)''$.

Thus, only one of f_1, f_2, f_3, say, f_1, is odd. This implies $f_1 = 37 \times 5 \times 9 = 1665 \equiv -1 \pmod{17}$. Then $\epsilon_0 f_0 + \epsilon_2 f_2 + \epsilon_3 f_3 = -1664$.

NOTE. f_2, f_3 are even in list (1).

Now, by calculation (analyzing each case) and using the signs $\epsilon_0, \epsilon_2, \epsilon_3$ and checking list (1) and $(1)''$, all possibilities are eliminated. Hence, this case is out.

Case $t = 2, p = 11, q = 5$.

We have $g = 55 \cdot (11M + 1)$.

The following is true:

(i) $M = 1, g = |\mathrm{PSL}(2, 11)|$

(ii) $M = 13, g = |M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$
(iii) $M = 157, g = |M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$

and all these groups appear in the proposed list.

The other possibilities are eliminated by using either characterization results or character theory methods as before.

This finishes the proof of Theorem 2.

NOTE. M_{11} and M_{12} are the Mathieu groups on 11 and 12 symbols.

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