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FIXED POINT THEOREMS FOR MULTIVALUED NONCOMPACT ACYCLIC MAPPINGS

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Let X be a Frechet space, D a closed convex subset of X, and $T: D \to 2^X$ an upper semicontinuous multivalued acyclic mapping. Using the Eilenberg-Montgomery Theorem and the earlier results of the authors, it is first shown that if $W \supset T(D)$ and $f: W \to D$ is a single-valued continuous mapping such that $fT: D \to 2^X$ is Φ-condensing, then fT has a fixed point. This result is then used to obtain various fixed point theorems for acyclic Φ-condensing mappings $T: D \to 2^X$ under the Leray-Schauder boundary conditions in case $D = \overline{\mathrm{Int}(D)}$ and under the outward and /or inward type conditions in case $\mathrm{Int}(D) = \phi$.

Introduction. Let X be a Frechet space and D an open or a closed convex subset of X. It is our object in this paper to establish fixed point theorems for not necessarily compact (e.g. condensing) multivalued acyclic mappings $T: D \to 2^X$ which need not satisfy the condition "$T(D) \subset D$" but instead are required to satisfy weaker conditions of the Leray-Schauder type. Our results are based upon the Eilenberg-Montgomery Theorem [4] and upon our Lemma 1 in [16]. The fixed point theorems presented in this paper for multivalued maps in infinite dimensional spaces strengthen and extend certain fixed point theorems of Górniewicz-Granas [7] and Powers [17] for acyclic compact maps, the results for star-shaped-valued maps of Halpern [8] for compact maps and our own [16] for condensing maps, and a number of fixed point theorems for convex-valued compact and noncompact maps (see Ky Fan [5], Browder [1], Reich [18], Ma [12], Walt [20], and [20, 8, 15] for related results and further references).

1. Let X be a Frechet space. If $D \subset X$, then we will denote by $\bar{D}$ and ∂D the closure and boundary of D, respectively.

DEFINITION 1. If C is a lattice with a minimal element, which we will denote by 0, then a mapping $\Phi: 2^X \to C$ is called a *measure of noncompactness* provided that the following conditions hold for any A, B in 2^X:

(1) $\Phi(A) = 0$ if and only if A is precompact.

(2) $\Phi(\overline{\mathrm{co}}A) = \Phi(A)$, where $\overline{\mathrm{co}}A$ denotes the convex closure of A.

(3) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

It follows that if $A \subset B$, then $\Phi(A) \leq \Phi(B)$. The above notation has been used in **[16, 19]** and is a generalization of the set-measure **[11]** and the ball-measure of noncompactness **[6]** defined either in terms of a family of seminorms or of a single norm when X is a Banach space. Specifically, if $\{P_\alpha | \alpha \in \mathscr{A}\}$ is a family of seminorms which determines the topology on X, then for each $\alpha \in \mathscr{A}$ and $\Omega \subset X$ we define $\gamma_\alpha(\Omega) = \inf\{d > 0 | \Omega$ can be covered by a finite number of sets each of which has P_α-diameter less than $d\}$, and $\chi_\alpha(\Omega) = \inf\{r > 0 | \Omega$ can be covered by a finite number of P_α-balls each of which has P_α-radius less than $r\}$.

Then letting $C = \{f: \mathscr{A} \to [0, \infty]\}$, with C ordered pointwise, we define the *set-measure of noncompactness* $\gamma: 2^X \to C$ by $(\gamma(\Omega))(\alpha) = \gamma_\alpha(\Omega)$ for each $\alpha \in \mathscr{A}$ and the *ball-measure of noncompactness* $\chi(\Omega)$ by $(\chi(\Omega))(\alpha) = \chi_\alpha(\Omega)$ for each $\alpha \in \mathscr{A}$ (see **[15]** for more details and properties of γ and χ).

The class of mappings considered here is given by the following.

DEFINITION 2. If Φ is a measure of noncompactness of X and $D \subset X$, an upper semicontinuous (u.s.c.) mapping $T: D \to 2^X$ is called Φ-*condensing* provided that if $\Omega \subset D$ and $\Phi(T(\Omega)) \geq \Phi(\Omega)$, then Ω is relatively compact.

It follows immediately that a compact mapping is Φ-condensing with respect to any measure of noncompactness Φ. Classes of Φ-condensing mappings which are not compact have been considered in **[19, 13, 14, 18]**. In particular, if X is a Banach space, $D \subset X$ is closed, $C: D \to 2^X$ is compact, and $S: X \to 2^X$ is such that $S(x)$ is compact for each $x \in X$, and $d^*(S(x), S(y)) \leq kd(x, y)$ for all $x, y \in X$ and some $k \in (0, 1)$, where d^* denotes the Hausdorff metric on the compact subsets of 2^X generated by the norm d, then $S + C: D \to 2^X$ is γ-condensing.

By homology we mean Čech homology with rational coefficients, and call a compact metric space Y *acyclic* if it has the same homology as a one point space. In particular, any contractable space is acyclic and thus any convex or star-shaped subset of X is acyclic. A mapping $T: D \to 2^X$ is called acyclic if $T(x)$ is compact and acyclic for each $x \in D$.

The following theorem of Eilenberg and Montgomery **[4]** together with the succeeding result from **[16]** will form the basis from which we will deduce our results.

THEOREM A. **[4]** *Let M be an acyclic absolute neighborhood retract (ANR), N a compact metric space, $r: N \to M$ a continuous single-valued mapping and $T: M \to 2^N$ a* u.s.c. *acyclic mapping. Then the mapping*

$rT\colon M \to 2^M$ has a fixed point, i.e., there exist $x \in M$ such that $x \in r(T(x))$.

LEMMA A. [16] *Let $D \subset X$ be closed and convex and $T\colon D \to 2^X$. Then for each $\Omega \subset D$ there exists a closed convex set K, depending on T, D, and Ω, with $\Omega \subset K$ and $\overline{\text{co}}\{T(D \cap K) \cup \Omega\} = K$.*

Our first result is the following fixed point theorem.

THEOREM 1. *Let X be a Frechet space with $D \subset X$ closed and convex. Suppose $T\colon D \to 2^X$ is* u.s.c. *and acyclic and $f\colon W \to D$ is single-valued and continuous, where $W \supset T(D)$. If $fT\colon D \to 2^X$ is Φ-condensing, then fT has a fixed point. In particular, if $T(D) \subset D$ and T is Φ-condensing, then T has a fixed point.*

Proof. Choose $x_0 \in D$. By Lemma A, we obtain a closed convex set K such that $x_0 \in K$ and $\overline{\text{co}}\{f(T(K \cap D)) \cup \{x_0\}\} = K$. Since $f(T(D)) \subset D$, we see that $K \cap D = K$ and so $\overline{\text{co}}\{f(T(K)) \cup \{x_0\}\} = K$. By the defining properties of the measure of noncompactness Φ, and, since fT is Φ-condensing, K must be compact. In view of the results in [3, 10], every compact convex subset of a Frechet space is an ANR, and is acyclic. Consequently, letting $M = K$, $N = T(K)$, and $f = r$ we may invoke Theorem A to conclude that fT has a fixed point. The last part of the theorem follows by letting $f =$ identity.

REMARK 1. Using the above result, it is clear that a theorem analogous to Theorem 3.4 in [15] is valid for acyclic 1-set and 1-ball contractive mappings.

The second part of Theorem 1 has been obtained in [7, 17] for the case when T is compact and X is a Banach space.

THEOREM 2. *Let X be a Frechet space and $D \subset X$ open and convex with $0 \in D$. If $T\colon \bar{D} \to 2^X$ is a Φ-condensing and acyclic mapping such that*

$$T(x) \cap \{\lambda x | \lambda > 1\} = \phi \quad \text{for } x \in \partial D, \tag{4}$$

then T has a fixed point. In particular, if $T(\partial D) \subset \overline{D}$, T has a fixed point.

Proof. Let $\rho\colon X \to \bar{D}$ be the single-valued mapping defiined by: $\rho(x) = x$ if $x \in \bar{D}$, and $\rho(x) = x/p(x)$ if $x \in X \setminus \bar{D}$, where p is the support function of $\bar{D}$. Since $0 \in D$, it follows that ρ is continuous. Furthermore, for each $A \subset X$, $\rho(A) \subset \overline{\text{co}}\{A \cup \{0\}\}$, so that, by the defining properties of Φ,

$\Phi(\rho(A)) \le \Phi(A)$. Hence, ρT is a Φ-condensing mapping of $\bar{D}$ into $\bar{D}$ because if $\Omega \subset \bar{D}$ and $\Phi(\rho(T(A))) \ge \Phi(\Omega)$, Ω must be relatively compact. Thus, by Theorem 1, we may choose $x \in \bar{D}$, with $x = \rho(z)$ and $z \in T(x)$, i.e., $x \in \rho T(x)$. It follows from (4) that $x \in T(x)$. Indeed, if $z \in \bar{D}$, then $\rho(z) = z = x$ and so $x \in T(x)$, and if $z \notin \bar{D}$, then $\rho(z) = \beta z$ for some $\beta < 1$ and so $(1/\beta)x \in T(x)$, in contradiction to (4). The last assertion follows from the fact that, for each $y \in \partial D$ and $\beta < 1$, $\beta y \in D$ and so $T(\partial D) \subset \bar{D}$ implies (4).

In case $T(x)$ is convex for each $x \in \bar{D}$, the above result has been obtained in **[15]** by use of a topological degree argument, without the assumption that D is convex.

1. In case X is a Banach space, whose norm has certain additional properties, we will now prove some results for acyclic mappings $T\colon D \to 2^X$, where D is closed and convex, without the assumption that $T(D) \subset D$. In particular, we strengthen the results of **[8, 16]** for mappings satisfying the so-called "nowhere normal outward" condition and without the assumptions (as in **[8, 16]**) that D contains a set with a nonempty core and that X is equipped with a collection of approximation maps (see **[8]** for definitions of these concepts).

We recall that a Banach space X is said to have Property (H) if X is strictly convex and whenever $\langle x_n \rangle \subset X$ is such that $\langle \|x_n\| \rangle \to \|x\|$ and $\langle x_n \rangle$ converges weakly to x, then $\langle x_n \rangle \to x$. Every locally uniformly convex Banach space has this property. We will use the following lemma concerning such spaces, and use the notation $\langle x_n \rangle \rightharpoonup x$ to denote the weak convergence of the sequence $\langle x_n \rangle$ to x.

LEMMA 1. *Let X be a reflexive Banach space with Property* (H), *and suppose $D \subset X$ is closed and convex. Then to each $x \in X$ there exists a unique point $N(x)$ in D such that $\|x - N(x)\| = \inf_{y\in D}\|y - x\|$. Furthermore, the mapping $x \to N(x)$ is continuous.*

Proof. Let $x \in X$ and let $d = \inf_{y\in D}\|y - x\|$. Choose $\langle u_n \rangle \subset D$ such that $\langle \|u_n - x\| \rangle \to d$. Then $\langle u_n \rangle$ is a bounded subset of D and since X is reflexive and D is weakly complete we may choose a subsequence $\langle u_{n_k} \rangle$ of $\langle u_n \rangle$ with $\langle u_{n_k} \rangle \rightharpoonup z \in D$. Since $\langle u_{n_k} - x \rangle \rightharpoonup z - x$,

$$d = \lim_k \|u_{n_k} - x\| = \liminf_k \|u_{n_k} - x\| \ge \|z - x\|.$$

But $\|z - x\| \ge d$, and so $\langle \|u_{n_k} - x\| \rangle \to \|z - x\|$. Since X has Property (H) we must have $\langle u_{n_k} \rangle \to z$. The point z with $z \in D$ and $\|z - x\| = d$ is unique because X is strictly convex, and since, by the above argument, any subsequence of $\langle u_n \rangle$ will in turn have a subsequence which converges to z, we see that $\langle u_n \rangle \to z = N(x)$.

We now show that N is continuous. Let $y \in X$ with $\langle y_n \rangle \subset X$ such that $\langle y_n \rangle \to y$. For each n we have $\|y_n - N(y_n)\| \leq \|y_n - N(y)\|$, so that $\limsup \|y_n - N(y_n)\| \leq \|y - N(y)\|$. Since $\langle N(y_n)\rangle$ is a bounded subset of D we may choose $\langle N(y_{n_k})\rangle$ such that $\langle N(y_{n_k})\rangle \rightharpoonup z \in D$. Then

$$\|y - N(y)\| \leq \|y - z\| \leq \liminf \|y_{n_k} - N(y_{n_k})\|$$

$$\leq \limsup \|y_{n_k} - N(y_{n_k})\| \leq \|y - N(y)\|.$$

Consequently, $\lim \|y_{n_k} - N(y_{n_k})\| = \|y - N(y)\|$, and so by the first part of the proof, $\langle N(y_{n_k})\rangle \to N(y)$. This argument shows that any subsequence of $\langle N(y_n)\rangle$ in turn has a subsequence which converges to $N(y)$, so that $\langle N(y_n)\rangle \to N(y)$.

We point out that any uniformly convex Banach space is reflexive and has Property (H).

Following Halpern [**8**], for a subset D of a Banach space X, we define the *outward* set of a point $x \in D$, denoted by $n_D(x)$, to be

$$n_D(x) = \{y \in X \mid y \neq x, \|y - x\| \leq \|y - z\| \text{ for all } z \in D\}.$$

We add in passing that, as was shown in [**9**], if $I_D(x)$ is the *inward set* of $x \in X$, i.e., $I_D(x) = \{y \in X \mid \lambda x + (1 - \lambda) y \in D \text{ for some } \lambda \in [0, 1)\}$, then $n_D(x) \cap \overline{I_D(x)} = \Phi$.

THEOREM 3. *Let X be a Banach space with $D \subset X$ closed and convex. Suppose that T: $D \to 2^X$ is acyclic and "nowhere normal outward," i.e.,*

$$T(x) \cap n_D(x) = \Phi \quad \text{for } x \in D. \tag{5}$$

Furthermore, suppose that one of the following conditions holds:

(i) *X is strictly convex and D is compact.*

(ii) *X is reflexive, satisfies condition* (H), *and $T(D)$ is compact.*

Then T has a fixed point.

Proof. (i) Since X is strictly convex and D is compact, the mapping N: $X \to D$ defined by the inequality $\|N(x) - x\| \leq \|y - x\|$ for all $y \in D$, is well defined and continuous [**8**]. Since D is an acyclic ANR, we use

Theorem A to conclude that NT has a fixed point in D. Since T satisfies (5), the fixed point of NT must also be a fixed point of T.

(ii) By Lemma 1, the above mapping N is continuous. Since $T(D)$ is relatively compact, NT is condensing, and so NT has a fixed point by Theorem 1. Again, using (1), this fixed point must also be a fixed point of T.

COROLLARY 1. *Theorem 3 holds with the hypothesis "T is nowhere normal outward" replaced by either of the stronger conditions, "$T(x) \subset \overline{I_D(x)}$ for all $x \in D$" or "$T(x) \subset I_D(x)$ for all $x \in D$."*

In case $T(x)$ is star-shaped for each $x \in \bar{D}$, Theorem 3 has been proved in [**8**, Theorem 20] under the additional condition that X is equipped with a collection of approximation maps and that the core $(D) \neq \phi$.

THEOREM 4. *Let X be a Banach space with $D \subset X$ closed and convex. Suppose $T: D \rightarrow 2^X$ is acyclic and Φ-condensing. Furthermore, assume that one of the following conditions holds:*

(i) *X is strictly convex and $T(x) \subset I_D(x)$ for x in D.*

(ii) *X is a Hilbert space, $T(x) \cap n_D(x) = \phi$ for each $x \in D$, and Φ is either the ball-measure or the set-measure of noncompactness defined in* §1. *Then T has a fixed point.*

Proof. (i) Let $x_0 \in D$. By Lemma A, we may choose a closed convex set K which contains x_0 and such that $\overline{\text{co}}\{T(D \cap K) \cup \{x_0\}\} = K$. By previously used arguments, K must be compact. Let $x \in K \cap D$ with $z \in T(x)$. Then $z \in I_D(x)$, so that for some $\lambda \in [0, 1)$, $\lambda x + (1 - \lambda)z \in D \cap K$. This shows that $T(x) \subset I_{D \cap K}(x)$ for each $x \in D \cap K$. Hence, by Corollary 1, T has a fixed point.

(ii) Let $N: X \rightarrow D$ be defined by $\|N(x) - x\| = \inf\{\, \|z - x\|$ for each $x \in D\}$. Now, X is a Hilbert space, and Cheney and Goldstein [**2**] have shown that $\|N(x) - N(y)\| \leq \|x - y\|$ for each x and y in X. It is not hard to show that this implies that for each $A \subset X$, $\Phi(N(A)) \leq \Phi(A)$. Consequently, $NT: D \rightarrow 2^D$ is Φ-condensing, and hence, by Theorem 1, NT has a fixed point. Since $T(x) \cap n_D(x) = \phi$, this fixed point must also be a fixed point of T.

Under hypothesis (i) the above result strengthens Theorem 3 in [**16**] and, in particular, Theorem 24 in [**8**].

REMARK 2. If X is a Hilbert space and $D = \overline{B(0, 1)}$, then for $x \in \partial D$, $n_D(x) = \{\lambda x | \lambda > 1\}$. Hence for a mapping $T: D \rightarrow 2^X$ the Leray-Schauder condition (4) of Theorem 2 coincides with the requirement that $T(x) \cap n_D(x) = \Phi$ for all $x \in D$.

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