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MULTIPLICATIVITY-PRESERVING ARITHMETIC POWER SERIES

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In the Dirichlet algebra of arithmetic functions let the operator A be represented by an arithmetic power series $Af = \Sigma a(F)f^F$. A condition on the coefficients $a(F)$ is derived which is necessary and sufficient for Af to be multiplicative whenever f is multiplicative.

1. Introduction.

In [2] a *factorization* F was defined to be a nonnegative integer-valued arithmetic function having $F(1) = 0$ and $F(n) \neq 0$ for at most finitely n. The *index* of F was defined by $i(F) = \prod_{j=1}^{\infty} j^{F(j)}$. If f is any arithmetic function, we defined $f^F = \prod_{j=1}^{\infty} [f(j)]^{F(j)}$ with the understanding that $0^0 = 1$. If $a(F)$ is a mapping from factorizations into the real or complex numbers, we wrote

$$Af = \sum a(F)f^F \tag{1}$$

as an abbreviation for the arithmetic function Af whose value on n is equal to $\sum_{i(F)=n} a(F)f^F$. In [2] a series of the form (1) was called an *arithmetic power series*. Since for each n the series is terminating, there is never any question of convergence. Such a series defines an operator A on the Dirichlet algebra of arithmetic functions, and the theory of these operators has been investigated in [1] and [2].

In particular, if r is a real number, the Dirichlet rth power of an arithmetic function f is represented, when $f(1) = 1$, by an arithmetic power series $\sum \binom{r}{F} f^F$. The symbol $\binom{r}{F}$ was defined in [2]. It is known [1, Theorem 5] that f^r is multiplicative whenever f is, and therefore the series $\sum \binom{r}{F} f^F$ is an example of a multiplicativity-preserving arithmetic power series. The present paper is devoted to determining a necessary and sufficient condition on the coefficients $a(F)$ in order that the general series (1) preserve multiplicativity. The method, and the statement of the result (Theorem 1), depend on a certain equivalence relation between factorizations, to be introduced below.

2. Equivalent factorizations.

DEFINITION 1. If F and F' are two factorizations, we say F is *equivalent* to F', written $F \sim F'$, if $f^F = f^{F'}$ for every multiplicative arithmetic function f.

It is obvious that this is an equivalence relation. An example of a pair of nonequal but equivalent factorizations may be constructed by taking $F(2) = F(3) = F'(6) = 1$, with all other values being zero. Then $f^F = f(2)f(3) = f(6) = f^{F'}$ for every multiplicative f. Two equivalent factorizations F and F' necessarily have the same index, for if we choose the particular multiplicative function $f(n) = n$, we have $i(F) = f^F = f^{F'} = i(F')$.

DEFINITION 2. We shall use the letter C to denote an equivalence class of factorizations. The *index* $i(C)$ of an equivalence class C is defined to be the index of the factorizations F belonging to C. If f is multiplicative, we denote by f^C the common value of f^F for all $F \in C$. If $F_1 \in C_1$ and $F_2 \in C_2$, we define $C_1 + C_2$ to be the equivalence class containing the factorization $F_1 + F_2$.

It is obvious that the definition of $C_1 + C_2$ is unambiguous.

If the operator (1) is applied to a multiplicative f, the sum over all factorizations F of index n reduces to a sum over all classes C of index n, thus:

$$Af(n) = \sum_{i(F)=n} a(F)f^F = \sum_{i(C)=n} f^C \sum_{F \in C} a(F) .$$

Therefore, insofar as its action on multiplicative functions is concerned, an arithmetic power series is determined by the *sums* of its coefficients over equivalence classes of factorizations, and it is natural to make the following definition:

DEFINITION 3. $a^*(C) = \sum_{F \in C} a(F)$.

Thus, when f is multiplicative, we may write

$$Af(n) = \sum_{i(C)=n} a^*(C)f^C . \tag{2}$$

The main theorem may now be stated as follows.

THEOREM 1. *The arithmetic function* $Af = \sum a(F)f^F$ *is multiplicative whenever* f *is, if and only if the following pair of conditions holds*:

$$a^*(C_1 + C_2) = a^*(C_1)a^*(C_2) \tag{3}$$

for every pair of equivalence classes C_1 and C_2 having relatively prime indices, and

$$a^*(0) = 1 \tag{4}$$

where 0 *is the class containing the zero factorization.*

3. **Lemmas.** Let those positive integers which are prime powers be arranged in increasing order. Let $x_1, x_2, \cdots$ be an arbitrary sequence of complex numbers. We may construct a multiplicative function f by setting $f(1) = 1$ and, it p^ν is the kth prime power, defining

$$f(p^\nu) = x_k \, . \tag{5}$$

The requirement that f be multiplicative then defines $f(n)$ for all positive integers n. Furthermore, every multiplicative f arises from exactly one particular choice of the sequence $\{x_k\}$. (Following the usual convention, we do not consider the identically zero function to be multiplicative.)

These observations establish a one-to-one correspondence between the set of all multiplicative functions and the set of all sequences of variables $\{x_k\}$. Under this correspondence we may associate, with each factorization F, an expression f^F which is a monomial (with coefficient 1) in certain of the variables x_k. We note that a given variable x_k cannot appear in this monomial if it does not correspond, in (5), to a prime power divisor of $i(F)$, since, by definition of index $F(j) = 0$ if j does not divide $i(F)$.

LEMMA 1. *Two factorizations F and F' are equivalent if and only if the two corresponding monomials f^F and $f^{F'}$ are identical.*

Proof. It is familiar from algebra [3, Chapter 4] that if two polynomials always agree in value while each variable x_k is assigned infinitely many different values, holding the others fixed, then the two polynomials are identical. The converse part of the assertion is trivial.

Lemma 1 shows that equivalence classes of factorizations may be identified with monomials in an arbitrary finite number of variables. Also, it is clear that each equivalence class of prime power index p^ν consists of a single factorization.

LEMMA 2. *Let $F_1, \cdots, F_r$ be nonequivalent factorizations. Suppose that, for every multiplicative f, the linear combination $\sum_{j=1}^{r} b_j f^{F_j}$ is equal to zero. Then each of the coefficients b_j is zero.*

Proof. The linear combination referred to in the lemma is a polynomial in certain of the variables x_k, and the numbers b_j are precisely its coefficients, since by Lemma 1 no two of the monomials f^{F_j} are identical. As in the proof of Lemma 1, each of these coefficients must be zero.

LEMMA 3. *Let F, F', G, and G' be factorizations, with $i(F) = i(F') = m$ and $i(G) = i(G') = n$, and assume m and n are relatively prime. Suppose $F + G \sim F' + G'$. Then $F \sim F'$ and $G \sim G'$.*

Proof. As observed earlier, each variable x_k appearing in the monomial f^F corresponds, in (5), to a prime power divisor of m. Similarly, f^G contains only variables corresponding to prime power divisors of n. Since $(m, n) = 1$, these two sets of variables are disjoint. Applying the same reasoning to F' and G', we see that no variable appearing in either f^F or $f^{F'}$ can appear in either f^G or $f^{G'}$, and conversely. By hypothesis we have $f^F f^G = f^{F+G} = f^{F'+G'} = f^{F'} f^{G'}$ for all multiplicative f, or equivalently $f^F/f^{F'} = f^{G'}/f^G$. Since opposite sides of this identity are rational functions in disjoint sets of independent variables, both sides must be equal to a constant B. In the identity $f^F = Bf^{F'}$, putting $f(k) = 1$ for all k, we obtain $B = 1$. Therefore $f^F = f^{F'}$ and $f^G = f^{G'}$, meaning $F \sim F'$ and $G \sim G'$.

LEMMA 4. *Let $F_1, \cdots, F_r$ be nonequivalent factorizations of index m. Let $G_1, \cdots, G_s$ be nonequivalent factorizations of index n. Assume $(m, n) = 1$. Suppose that, for every multiplicative f, the linear combination $\sum_{j=1}^{r} \sum_{k=1}^{s} b_{jk} f^{F_j + G_k}$ is equal to zero. Then each of the coefficients b_{jk} is zero.*

Proof. By Lemma 3 the factorizations $F_j + G_k$ are all nonequivalent, and the result then follows from Lemma 2.

LEMMA 5. *Let F be a factorization of index mn, where $(m, n) = 1$. Then there exist factorizations F_1 and F_2, of indices m and n respectively, such that $F \sim F_1 + F_2$. Furthermore, if F_1' and F_2' also satisfy these conditions, then $F_1 \sim F_1'$ and $F_2 \sim F_2'$. In other words, if $(m, n) = 1$, then each equivalence class of index mn is the sum of a unique pair of classes of indices m and n respectively.*

Proof. The uniqueness part follows immediately from Lemma 3. As regards the existence of F_1 and F_2, we claim that the pair defined as follows will satisfy the requirements:

$$\begin{aligned} F_1(k) &= 0 && \text{if } k = 1 \\ &= \sum_{(j,m)=k} F(j) && \text{if } k > 1 \\ F_2(k) &= 0 && \text{if } k = 1 \\ &= \sum_{(j,n)=k} F(j) && \text{if } k > 1 \,. \end{aligned}$$

To check this, choose any multiplicative f. Then

$$\begin{aligned} f^{F_1+F_2} = f^{F_1}f^{F_2} &= \prod_{k=1}^{\infty} [f(k)]^{F_1(k)} \prod_{k=1}^{\infty} [f(k)]^{F_2(k)} \\ &= \prod_{j=1}^{\infty} [f((j, m))]^{F(j)} \prod_{j=1}^{\infty} [f((j, n))]^{F(j)} \\ &= \prod_{j=1}^{\infty} [f((j, m))f((j, n))]^{F(j)} \\ &= \prod_{j=1}^{\infty} [f((j, m)(j, n))]^{F(j)} \\ &= \prod_{j=1}^{\infty} [f((j, mn))]^{F(j)} = \prod_{j=1}^{\infty} [f(j)]^{F(j)} = f^F \,, \end{aligned}$$

where in the last step we use the fact that $F(j) = 0$ if j does not divide mn. Therefore $F \sim F_1 + F_2$. To find the indices of F_1 and F_2, we first observe that $i(F_1)i(F_2) = i(F_1 + F_2) = i(F) = mn$. Also, if we choose for f the identity function $f(k) = k$, we have $i(F_1) = f^{F_1} = \prod_{j=1}^{\infty} (j, m)^{F(j)}$, and each factor in the product is relatively prime to n, so $i(F_1)$ is relatively prime to n. Similarly, $i(F_2)$ is relatively prime to m. Therefore $i(F_1) = m$ and $i(F_2) = n$.

4. **Proof of Theorem 1.** First assume conditions (3) and (4) hold. Choose any multiplicative f, and let m and n be relatively prime. We are to show that $Af(mn) = Af(m)Af(n)$ and $Af(1) = 1$. By Lemma 5, each equivalence class C of index mn is the sum of a unique pair of classes $C_1 + C_2$ where $i(C_1) = m$ and $i(C_2) = n$. Remembering (2), we may evaluate $Af(mn)$ as follows:

$$\begin{aligned} Af(mn) &= \sum_{i(C)=mn} a^*(C)f^C = \sum_{i(C_1)=m} \sum_{i(C_2)=n} a^*(C_1 + C_2)f^{C_1+C_2} \\ &= \sum_{i(C_1)=m} a^*(C_1)f^{C_1} \sum_{i(C_2)=n} a^*(C_2)f^{C_2} = Af(m)Af(n) \,. \end{aligned}$$

Also, $Af(1) = a^*(0)f0 = 1$.

To prove the converse, assume the operator A preserves multiplicativity. Choose m and n relatively prime, and let f be any multiplicative function. Proceeding as in the last computation above, we have

$$\begin{aligned} 0 &= Af(mn) - Af(m)Af(n) \\ &= \sum_{i(C_1)=m} \sum_{i(C_2)=n} f^{C_1+C_2}[a^*(C_1 + C_2) - a^*(C_1)a^*(C_2)] \,. \end{aligned}$$

This double sum is a linear combination of the type considered in Lemma 4, and therefore, by the result of that lemma, the expression in square brackets is equal to zero for all C_1 and C_2 in the sum. That is, equation (3) is satisfied. Also, (4) is satisfied because $1 = Af(1) = a^*(0)f0 = a^*(0)$. This completes the proof of Theorem 1.

5. **Further consequences.** We wish to show how to construct all solutions $a^*(C)$ of (3) which also satisfy (4) (and which we shall refer to as *nontrivial* solutions of (3)). Given a nontrivial solution $a^*(C)$ of (3), we can recover (nonuniquely) by Definition 3 the coefficients $a(F)$ of an arithmetic power series (1) which preserves multiplicativity, and the class of such series will then be completely characterized.

LEMMA 6. *Let C be an equivalence class whose index is greater than 1 and has prime factorization $i(C) = p_1^{\nu_1}, \cdots, p_r^{\nu_r}$. Then there are unique classes $C_1, \cdots, C_r$, of indices $p_1^{\nu_1}, \cdots, p_r^{\nu_r}$ respectively, such that $C = C_1 + \cdots + C_r$.*

Proof. Apply Lemma 5 repeatedly to the r maximal prime power divisors $p_1^{\nu_1}, \cdots, p_r^{\nu_r}$ of $i(C)$.

LEMMA 7. *$a^*(C)$ is a nontrivial solution of (3) if and only if $a^*(0) = 1$ and*

$$a^*(C) = \prod_{k=1}^{r} a^*(C_k) \tag{6}$$

whenever $i(C) > 1$, where the classes $C_1, \cdots, C_r$ are related to C as in Lemma 6.

Proof. Equation (6) is obtained from (3) by applying the latter repeatedly to the maximal prime power divisors of $i(C)$. Conversely, (3) is obtained from (6) by applying (6) to the prime decomposition of mn, separating the maximal prime power divisors of m from those of n.

Lemma 7 gives us a process for constructing all nontrivial solutions of (3). The method is analogous to that used at the beginning of § 3 to construct all multiplicative functions, namely:

THEOREM 2. *The nontrivial solutions $a^*(C)$ of (3) are exactly those which take the value 1 on the zero class and are defined arbitrarily on classes of prime power index, the definition then being extended to all C by the product formula (6).*

Finally, we shall determine the number of equivalence classes of index n. Let this number be denoted by $E(n)$. It follows from Lemma 5 that $E(n)$, as an arithmetic function, is multiplicative. Therefore, it suffices to evaluate this function on prime powers p^ν. Since each class of index p^ν contains only one factorization, $E(p^\nu)$ is equal to the number of factorizations of index p^ν, and this is evidently just the number of unrestricted partitions of ν. These observations yield the following explicit formula for $E(n)$:

THEOREM 3.

$$E(1) = 1$$
$$E(n) = \prod_{p^\nu || n} p(\nu) \quad \text{if } n > 1 ,$$

where $p(\nu)$ is the partition function, and the product is extended over all maximal prime power divisors p^ν of n.

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