PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 1, 1974

ON (J, M, m)-EXTENSIONS OF BOOLEAN ALGEBRAS

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The class $\mathscr{K}$ of all (J, M, m)-extensions of a Boolean algebra $\mathscr{A}$ can be partially ordered and always contains a maximum and a minimal element, with respect to this partial ordering. However, it need not contain a smallest element. Should $\mathscr{K}$ contain a smallest element, then $\mathscr{K}$ has the structure of a complete lattice. Necessary and sufficient conditions under which $\mathscr{K}$ does contain a smallest element are derived. A Boolean algebra $\mathscr{A}$ is constructed for each cardinal m such that the class of all m-extensions of $\mathscr{A}$ does not contain a smallest element. One implication of this construction is that if a Boolean algebra $\mathscr{A}$ is the Boolean product of a least countably many Boolean algebras, each of which has more than one m-extension, then the class of all m-extensions of $\mathscr{A}$ does not contain a smallest element. The construction also has as implication that neither the class of all $(m, 0)$-products nor the class of all (m, n)-products of an indexed set $\{\mathscr{A}_t\}_{t \in T}$ of Boolean algebras need contain a smallest element.

1. Sikorski [2] has investigated the question of imbedding a given Boolean algebra $\mathscr{A}$ into a complete or m-complete Boolean algebra $\mathscr{B}$ and has shown that in the case where the imbedding map is not a complete isomorphism, the imbedding need not be unique up to isomorphism. He further has shown that if $\mathscr{K}$ is the class of all (J, M, m)-extensions of a Boolean algebra $\mathscr{A}$, then $\mathscr{K}$ has a naturally defined partial ordering on it and always contains a maximum and a minimal element. He has left as an open question whether it always contains a smallest element. La Grange [1] has given an example which implies that $\mathscr{K}$ need not always contain a smallest element. However, the question of when does $\mathscr{K}$ in fact contain a smallest element is of interest as it turns out that should $\mathscr{K}$ contain a smallest element, it has the structure of a complete lattice.

In § 2, necessary and sufficient conditions are given for $\mathscr{K}$ to contain a smallest element. In addition, the principle behind La Grange's example is generalized in Proposition 2.10 to show that if $\mathscr{A}$ is not m-representable then the class $\mathscr{K}$ of all (J, M, m')-extension of $\mathscr{A}$, where $\bar{\bar{J}}, \bar{\bar{M}} < \sigma$ and $m' > M$, will not contain a smallest element.

Since the proof of this result requires that J and M have cardinality $\leqq \sigma$, it is of interest to ask if the class of all m-extensions contain a smallest element in general, and the answer is no.

In § 3, a Boolean algebra $\mathscr{A}$ is constructed for each cardinal m such that the class $\mathscr{K}$ of all m-extensions of $\mathscr{A}$ does not contain a smallest element. The construction has as implication (Theorems 3.1 and 3.2; Corollary 3.1) that for each algebra in a rather broad group of Boolean algebras, the class of all m-extensions will not contain a smallest element. In particular, this group includes all Boolean algebras which are the Boolean product of at least countably many Boolean algebras each of which has more than one m-extension.

Finally, in the last section, Sikorski's result that there is an equivalence between the class $\mathscr{P}$ of all $(m, 0)$-products of an indexed set $\{\mathscr{A}_t\}_{t\in T}$ of Boolean algebras and the class of all (J, M, m)-extensions of the Boolean product $\mathscr{A}_0$ of $\{\mathscr{A}_t\}_{t\in T}$, for suitably defined J and M, is generalized to show there is an equivalence between the class $\mathscr{P}_n$ of all (m, n)-products of $\{\mathscr{A}_t\}_{t\in T}$ and all (J, M, m)-extensions of $\hat{\mathscr{F}}$, where $\hat{\mathscr{F}}$ is the field of sets generated by a certain set $\mathscr{S}$, for suitably defined J and M. Then the above results imply that neither $\mathscr{P}$ nor $\mathscr{P}_n$ need contain a smallest element.

The notation throughout follows that of Sikorski [2].

2. Let n be the cardinality of a set of generators for the Boolean algebra $\mathscr{A}$, let $\mathscr{A}_{m,n}$ be a free Boolean m-algebra with a set of n free m-generators, let $\mathscr{A}_{0,n}$ be the free Boolean algebra generated by this set of n free m-generators and let g be a homomorphism from $\mathscr{A}_{0,n}$ to $\mathscr{A}$. Let $\varDelta_0$ be the kernel of this homomorphism and let I be the set of all m-ideals $\varDelta$ in $\mathscr{A}_{m,n}$ such that:

a. $\varDelta \cap \mathscr{A}_{0,n} = \varDelta_0$;

b. $\varDelta$ contains all the elements

$$A_0 - \bigcup_{A\in S_1} A\,, \qquad \bigcup_{A\in S_1} A - A_0\,,$$
$$A_0 - \bigcap_{A\in S_2} A\,, \qquad \bigcap_{A\in S_2} A - A_0\,,$$

where $A_0 \in \mathscr{A}_{0,n}$ and $\mathscr{S}_1, \mathscr{S}_2$ are any subsets of $\mathscr{A}_{0,n}$ of cardinality $\leqq m$ such that:

$$g(\mathscr{S}_1) \in J\,, \qquad g(A_0) = \bigcup_{A\in\mathscr{S}_1} g(A)$$
$$g(\mathscr{S}_2) \in M\,, \qquad g(A_0) = \bigcap_{A\in\mathscr{S}} g(A)\,.$$

For each $\varDelta \in I$ let

$$\mathscr{A}_\varDelta = \mathscr{A}_{m,n}/\varDelta$$

and

$$g_\varDelta([A]_\varDelta) = g(\varDelta)\,, \quad \text{for all} \quad A \in \mathscr{A}_{0,n}\,.$$

Set $i_\varDelta = g_\varDelta^{-1}$. We need the following results due to Sikorski.

PROPOSITION 2.1. *The ordered pair* $\{i_\Delta, \mathscr{A}_\Delta\}$ *is a* (J, M, m)-*extension of the Boolean algebra* $\mathscr{A}$ *and if* $\{i, \mathscr{B}\}$ *is a* (J, M, m)-*extension of* $\mathscr{A}$ *there is a* $\Delta \in I$ *such that* $\{i_\Delta, \mathscr{A}_\Delta\}$ *is isomorphic to* $\{i, \mathscr{B}\}$. *Further, if* $\Delta, \Delta' \in I$ *then*

$$\{i_\Delta, \mathscr{A}_\Delta\} \leqq \{i_{\Delta'}, \mathscr{A}_{\Delta'}\} \quad \textit{if, and only if,} \quad \Delta \supseteqq \Delta' .$$

LEMMA 2.1. *If* S *is a set of elements in* $\mathscr{K}$ *then the least upper bound* (lub) *of* S *exists in* $\mathscr{K}$.

Now let $\mathscr{K}(J, M, m)$ denote the class of all (J, M, m)-extensions of $\mathscr{A}$.

THEOREM 2.1. *Let* $\mathscr{K}$ *be the class of all* (J, M, m)-*extensions of a Boolean algebra* $\mathscr{A}$. *The following are equivalent*:

1. $\mathscr{K}$ *contains a smallest element*;
2. $\mathscr{K}$ *is a lattice*;
3. $\mathscr{K}$ *is a complete lattice.*

Proof.

$1. \Rightarrow 3.$ It suffices to show that if S is a set of (J, M, m)-extensions of $\mathscr{A}$ then the greatest lower bound (glb) of S exists in $\mathscr{K}$, which follows from noting that if L is the set of all lower bounds for the set S then $L \neq 0$ and by Lemma 2.1 the lub of L exists in $\mathscr{K}$, hence is in L.

$3. \Rightarrow 2.$ By definition.

$2. \Rightarrow 1.$ If $\{i, \mathscr{B}\}$ is an m-completion of $\mathscr{A}$, $\{j, \mathscr{C}\} \in \mathscr{K}$, and $\mathscr{K}$ a lattice, then there is an element $\{j', \mathscr{C}'\} \in \mathscr{K}$ such that

$$\{j', \mathscr{C}'\} \leqq \{j, \mathscr{C}\} .$$

Thus

$$\{j', \mathscr{C}'\} \leqq \{i, \mathscr{B}\} ,$$

so

$$\{j', \mathscr{C}'\} = \{i, \mathscr{B}\} ,$$

implying

$$\{i, \mathscr{B}\} \leqq \{j, \mathscr{C}\} .$$

Hence $\{i, \mathscr{B}\}$ is a smallest element in $\mathscr{K}$.

COROLLARY 2.1. *If* $J' \supseteqq J$ *and* $M' \supseteqq M$ *then the following are equivalent*:

1. $\mathscr{K}(J, M, m)$ *contains a smallest element*;

2. *$\mathscr{K}(J', M', m)$ is a sublattice of $\mathscr{K}(J, M, m)$;*
3. *$\mathscr{K}(J', M', m)$ is a complete sublattice of $\mathscr{K}(J, M, m)$.*

Proof.
1. $\Rightarrow$ 3. Since $\mathscr{K}(J', M', m)$ contains a smallest element, so does $\mathscr{K}(J, M, m)$ hence $\mathscr{K}(J', M', m)$ and $\mathscr{K}(J, M, m)$ are complete lattices. If $\{\{i_t, \mathscr{B}_t\}\}_{t \in T} = S$ is a set of elements in $\mathscr{K}(J', M', m)$, $\{i, \mathscr{C}\}$ is the lub of S in $\mathscr{K}(J, M, m)$ and $\{i', \mathscr{C}'\}$ is the lub of S in $\mathscr{K}(J', M', m)$, then there is an m-homomorphism h mapping $\mathscr{C}'$ onto $\mathscr{C}$ such that $hi' = i$. Hence i is a (J', M', m)-isomorphism. Thus $\{i, \mathscr{C}\} \in \mathscr{K}(J', M', m)$, implying

$$\{i, \mathscr{C}\} = \{i', \mathscr{C}'\} .$$

If $\{i, \mathscr{C}\}$ is the glb of S in $\mathscr{K}(J, M, m)$ and $\{i', \mathscr{C}'\} \in S$, then by a similar argument, i is a (J', M', m)-isomorphism, which implies $\{i, \mathscr{C}\}$ is the glb of S in $\mathscr{K}(J', M', m)$.

3. $\Rightarrow$ 2. By definition.

2. $\Rightarrow$ 1. The proof is the same as that for showing 2. $\Rightarrow$ 1, in Theorem 2.1.

Thus it is of particular interest to know whether $\mathscr{K}(J, M, m)$ contains a smallest element, in general. Although, as it turns out, $\mathscr{K}(J, M, m)$ need not contain a smallest element in general, a minimal (J, M, m)-extension is always an m-completion, hence there is always a unique minimal (J, M, m)-extension in $\mathscr{K}$ (J, M, m).

PROPOSITION 2.2. *An m-completion $\{i, \mathscr{B}\}$ of the Boolean algebra $\mathscr{A}$ is a unique minimal element in $\mathscr{K}$.*

Proof. That a minimal element in $\mathscr{K}$ is an m-completion is clear.

If $\{i', \mathscr{B}'\}$ is another minimal element in $\mathscr{K}$, there are $\varDelta, \varDelta' \in I$ such that

$$\{i, \mathscr{B}\} = \{i_\varDelta, \mathscr{A}_\varDelta\}$$

and

$$\{i', \mathscr{B}'\} = \{i_{\varDelta'}, \mathscr{A}_{\varDelta'}\} .$$

Now $\{i, \mathscr{B}\}$ and $\{i', \mathscr{B}'\}$ minimal in $\mathscr{K}$ imply $\varDelta$ and $\varDelta'$ are maximal m-ideals in I, but if $\hat{\varDelta}$ is a maximal m-ideal in I then $g_{\hat{\varDelta}}(\mathscr{A}_{0,n})$ is dense in $\mathscr{A}_{\hat{\varDelta}}$. The ideal $\hat{\varDelta}' = \langle \hat{\varDelta}, A \rangle$ in $\mathscr{A}_{m,n}$ is an m-ideal and $\hat{\varDelta}' \in I$, contradicting the maximality of $\hat{\varDelta}$. So $\{i', \mathscr{B}'\}$ is an m-completion of $\mathscr{A}$, hence isomorphic to $\{i, \mathscr{B}\}$, implying

$$\{i', \mathscr{B}'\} = \{i, \mathscr{B}\} .$$

PROPOSITION 2.3. *If $\mathscr{A}$ is a Boolean m-algebra that satisfies the m-chain condition and*

$$\bigcup_{t \in T} A_t$$

is the join of an indexed set $\{A_t\}_{t \in T}$ in $\mathscr{A}$, then there is an indexed set $\{A'_t\}_{t \in T}$ of disjoint elements of $\mathscr{A}$ such that

1. $$\bigcup_{t \in T} A'_t = \bigcup_{t \in T} A_t ;$$
2. $$A'_t \subseteqq A_t \quad \textit{for all} \quad t \in T .$$

Proof. Let $\mathscr{S}$ be the collection of all sets S of disjoint elements in $\mathscr{A}$ such that for each $s \in S$ there is a $t \in T$ with $s \subseteqq A_t$. If

$$S_1 \subseteqq S_2 \subseteqq \cdots \subseteqq S_i \subseteqq \cdots$$

is a chain of sets in $\mathscr{S}$ indexed by I and ordered by set theoretical inclusion, then

$$\bigcup_{i \in I} S_i = S \in \mathscr{S} .$$

By Zorn's lemma there is a maximal set in $\mathscr{S}$, say $S' = \{A_r\}_{r \in R}$, and it immediately follows that

$$\bigcup_{r \in R} A_r \neq A .$$

Now let

$$\varphi \colon S' \longrightarrow T$$

be a mapping such that if $A_r \in S'$ then

$$A_r \subseteqq A_{\varphi(A_r)} .$$

For each $t \in T$ define

$$A'_t = \bigcup \{A_r \in S' \colon \varphi(A_r) = t\}$$

if there is an $A_r \in S'$ such that $\varphi(A_r) = t$, otherwise define

$$A'_t = \bigwedge .$$

Then

$$\{A'_t\}_{t \in T}$$

is the desired set.

PROPOSITION 2.4. *Let $\mathscr{A}$ be a Boolean algebra. The following are equivalent:*

1. *$\mathscr{A}$ satisfies the m-chain condition:*
2. *for all sets S in $\mathscr{A}$ such that $\bigcup_{s \in S} s$ exists,*

$$\bigcup_{s \in S} s = \bigcup_{s \in S'} s$$

for some set $S' \subseteqq S$ with $S' \leqq m$; and dually for meets.

Proof.

1\. $\Rightarrow$ 2. Suppose $\mathscr{A}$ satisfies the m-chain condition. It suffices to show that if

$$S = \{A_t\}_{t \in T} \text{ and } \mathsf{V} = \bigcup_{t \in T} A_t\ , \quad \overline{\overline{T}} = m' > m\ ,$$

then there is a set $T' \subseteqq T$, $\overline{\overline{T'}} \leqq m$, such that

$$\bigcup_{t \in T'} A_t = \mathsf{V}\ .$$

Let $\{i, \mathscr{B}\}$ be an m'-completion of $\mathscr{A}$. Then $\mathscr{B}$ satisfies the m-chain condition and

$$\begin{aligned} \mathsf{V}_{\mathscr{B}} &= i(\mathsf{V}_{\mathscr{A}}) \\ &= \bigcup_{t \in T}^{\mathscr{B}} i(A_t)\ . \end{aligned}$$

By Proposition 2.3, there is a set $\{\mathscr{B}_t\}_{t \in T}$ of disjoint elements in $\mathscr{B}$ such that

$$B_t \subseteqq i(A_t) \quad \text{and} \quad \bigcup_{t \in T}^{\mathscr{B}} B_t = \bigcup_{t \in T}^{\mathscr{B}} i(A_t)\ .$$

Since this set contains at most m-distinct elements,

$$\bigcup_{t \in T}^{\mathscr{B}} B_t = \bigcup_{t \in T'}^{\mathscr{B}} B_t\ ,$$

$T' \subseteqq T$ and $\overline{\overline{T'}} \leqq m$. Thus

$$\mathsf{V}_{\mathscr{B}} = \bigcup_{t \in T'}^{\mathscr{B}} i(A_t)$$

or

$$\mathsf{V}_{\mathscr{A}} = \bigcup_{t \in T'}^{\mathscr{A}} A_t\ .$$

2\. $\Rightarrow$ 1. Suppose $\{A_t\}_{t \in T}$ is an m'-indexed set of disjoint elements of $\mathscr{A}$, $m' > m$. It may be assumed that $\{A_t\}_{t \in T}$ is a maximal set of disjoint elements of $\mathscr{A}$. Then for some $T' \subseteqq T$, $\overline{\overline{T'}} \leqq m$,

$$\mathsf{V}_{\mathscr{A}} = \bigcup_{t \in T'}^{\mathscr{A}} A_t\ .$$

Since $\overline{\overline{T'}} \neq \overline{\overline{T}}$, there is a $t_0 \in T - T'$ such that

$$A_{t_0} \in \{A_t\}_{t \in T} - \{A_t\}_{t \in T'} \quad \text{and} \quad A_{t_0} \neq \bigwedge_{\mathscr{A}} .$$

Thus

$$\bigcup_{t \in T'}^{\mathscr{A}} A_t \neq \bigvee_{\mathscr{A}} ,$$

a contradiction. Hence $\overline{\overline{T}} \leqq m$.

This gives, as an immediate corollary, the following result due to Sikorski [2].

COROLLARY 2.2. *If $\mathscr{A}$ is a Boolean m-algebra and satisfies the m-chain condition, it is a complete Boolean algebra.*

PROPOSITION 2.5. *The class $\mathscr{K}(J, M, m')$ contains a smallest element if $\mathscr{K}(J, M, m)$ contains a smallest element, $m' < m$.*

Proof. Let $\{i, \mathscr{B}\}$ be the smallest element in $\mathscr{K}(J, M, m)$. If $\{j', \mathscr{C}'\} \in \mathscr{K}(J, M, m')$, let $\{k, \mathscr{C}\}$ be an m-completion of $\mathscr{C}'$. Then $\{kj, \mathscr{C}\} \in \mathscr{K}(J, M, m)$.

By the fact that $\{i, \mathscr{B}\}$ is the smallest element in $\mathscr{K}(J, M, m)$, there is an m-homomorphism h such that

$$h\colon \mathscr{C} \longrightarrow \mathscr{B} \quad \text{and} \quad hkj = i .$$

Also $\{i, \mathscr{B}\}$ an m-completion of $\mathscr{A}$ implies that there is an m'-completion $\{i, \mathscr{B}'\}$ of $\mathscr{A}$ such that $\mathscr{B}' \subseteqq \mathscr{B}$. Thus $hk(\mathscr{C}')$ is an m-subalgebra of $\mathscr{B}$, hence $\mathscr{B}' \subseteqq hk(\mathscr{C}')$ and is an m-subalgebra of $\mathscr{C}$.

Now $kj(\mathscr{A})$ m-generates $k(\mathscr{C}')$ in $\mathscr{C}$ and $kj(\mathscr{A}) \subseteqq h^{-1}(\mathscr{B}')$, hence

$$h^{-1}(\mathscr{B}') \supseteqq k(\mathscr{C}') ,$$

or

$$h(h^{-1}(\mathscr{B}')) \supseteqq hk(\mathscr{C}') .$$

But

$$h(h^{-1}(\mathscr{B}')) = \mathscr{B}' ,$$

thus

$$\mathscr{B}' \supseteqq hk(\mathscr{C}') ,$$

so

$$\mathscr{B}' = hk(\mathscr{C}') .$$

Since $hkj = i$,

$$\{i, \mathscr{B}'\} \leqq \{kj, k(\mathscr{C}')\} .$$

But k a complete isomorphism implies that

$$\{kj, k(\mathscr{C}')\} \cong \{j, \mathscr{C}'\} ,$$

and since isomorphic elements in $\mathscr{K}(J, M, m)$ have been identified,

$$\{i, \mathscr{B}'\} = \{j, \mathscr{C}'\} .$$

LEMMA 2.2. *If $\bar{\bar{J}} \leqq \sigma$ and $\bar{\bar{M}} \leqq \sigma$ then there is a (J, M, m)-isomorphism i of a Boolean algebra $\mathscr{A}$ into the field $\mathscr{F}$ of all subsets of a space.*

PROPOSITION 2.6. *If the Boolean algebra $\mathscr{A}$ is m-representable but not m^+-representable, m^+ the smallest cardinal greater than m, then $\mathscr{K}(J, M, m^+)$ does not contain a smallest element if*

$$\mathscr{K}_r(J, M, m^+) \neq \varnothing .$$

If $\bar{\bar{J}} \leqq \sigma$, $\bar{\bar{M}} \leqq \sigma$ then $\mathscr{K}_r(J, M, m^+) \neq \varnothing$.

Proof. Suppose $\{j, \mathscr{C}\} \in \mathscr{K}_r(J, M, m^+)$. Then $\mathscr{C}$ is m-representable and if an m^+-completion $\{i, \mathscr{B}\}$ of $\mathscr{A}$ is a smallest element in $\mathscr{K}(J, M, m^+)$, there is a surjective m^+-homomorphism

$$h: \mathscr{C} \longrightarrow \mathscr{B} ,$$

which implies $\mathscr{B}$ is m^+-representable, hence $\mathscr{A}$ is m^+-representable, a contradiction. Thus $\mathscr{K}(J, M, m^+)$ does not contain a smallest element if $\mathscr{K}_r(J, M, m^+) \neq \varnothing$.

If $\bar{\bar{J}} \leqq \sigma$ and $\bar{\bar{M}} \leqq \sigma$ then $\mathscr{A}$ is (J, M, m^+)-representable by Lemma 2.2, hence $\mathscr{K}_r(J, M, m^+) \neq \varnothing$.

The next proposition is an easy generalization of Sikorski's [2] Proposition 25.2 and will be needed for the last theorem in this section.

PROPOSITION 2.7. *A Boolean algebra $\mathscr{A}$ is completely distributive, if, and only if, it is atomic.*

COROLLARY 2.3. *A Boolean algebra $\mathscr{A}$ is completely distributive, if, and only if, $\mathscr{A}$ is m-distributive, $m = \bar{\bar{\mathscr{A}}}$.*

The following proposition is due to Sikorski [2] and will be given without proof.

PROPOSITION 2.8. *If the Boolean algebra $\mathscr{A}$ is m-distributive, then $\mathscr{K}(J, M, m)$ contains a smallest element for arbitrary J and M.*

LEMMA 2.3. *If $\{i, \mathscr{B}\}$ is an m-extension of the Boolean algebra $\mathscr{A}$ and $\mathscr{B}$ is m-representable, then $\mathscr{A}$ is m-representable.*

Proof. This follows immediately from the fact that $\mathscr{A}$ is m-regular in $\mathscr{B}$.

Now to prove the main theorem of this section.

THEOREM 2.2. *Let $\mathscr{A}$ be a Boolean algebra. Then the following are equivalent:*

1. *$\mathscr{K}$ contains a smallest element for arbitrary J, M, and m;*
2. *$\mathscr{A}$ is m-representable for all m;*
3. *$\mathscr{A}$ is completely distributive;*
4. *$\mathscr{A}$ is atomic;*
5. *an m-completion of $\mathscr{A}$ is atomic for all m;*
6. *an m-completion of $\mathscr{A}$ is in $\mathscr{K}_r(J, M, m)$ for arbitrary J, M, and m;*
7. *$\mathscr{K}(J, M, 2^{m^*})$ contains a smallest element, where $J = M = \varnothing$ and $\overline{\overline{\mathscr{A}}} = m^*$.*

Proof.

$1. \Rightarrow 2.$ If $\mathscr{A}$ is m-representable but not m^*-representable, then Proposition 2.6 implies $\mathscr{K}(J, M, m^*)$ does not contain a smallest element if $\overline{\overline{J}}, \overline{\overline{M}} < \sigma$.

$2. \Rightarrow 3.$ This follows from the fact that if a Boolean algebra $\mathscr{A}$ is 2^m-representable, it is m-distributive.

$3. \Leftrightarrow 4.$ This follows from Proposition 2.7.

$3. \Rightarrow 1.$ This follows from Proposition 2.8.

$4. \Leftrightarrow 5.$ If $\{i, \mathscr{B}\}$ is an m-completion of $\mathscr{A}$ then $i(\mathscr{A})$ is dense in $\mathscr{B}$, so $\mathscr{B}$ is atomic, and conversely.

$2. \Rightarrow 6.$ This follows from noting that $2. \Rightarrow 3.$ and $\mathscr{A}$ completely distributive implies an m-completion of $\mathscr{A}$ is completely distributive, hence m-representable for all cardinals m.

$6. \Rightarrow 2.$ This follows from Lemma 2.3.

$3. \Leftrightarrow 7.$ If $J = M = \varnothing$ and $\mathscr{K}(J, M, 2^{m^*})$ contains a smallest element, then by Proposition 2.6, $\mathscr{A}$ is 2^{m^*}-representable, hence m^*-distributive. Since $m^* = \overline{\overline{\mathscr{A}}}$, $\mathscr{A}$ is completely distributive, by Corollary 2.3. The converse is clear.

3. The example in § 2 of a Boolean algebra $\mathscr{A}$ such that the class of all (J, M, m)-extensions of $\mathscr{A}$ does not contain a smallest element depends on the assumption that $\bar{\bar{J}}, \bar{\bar{M}} \leqq \sigma$. Thus it is of interest to know whether an example can be found showing that the class of all m-extensions of $\mathscr{A}$ does not contain a smallest element, since this corresponds to the case where J and M are as large as possible. As it turns out, there are Boolean algebras $\mathscr{A}$ such that the class of all m-extensions $\mathscr{K}$ does not contain a smallest element. In this section such an example will be constructed for each infinite cardinal m and several general types of Boolean algebras such that $\mathscr{K}$ does not contain a smallest element will be given.

Throughout this section $\mathscr{K}$ will denote the class of all m-extensions of a Boolean algebra $\mathscr{A}$ and $\mathscr{K}(J, M, m)$ the class of all (J, M, m)-extensions.

If $\mathscr{A}$ is a Boolean algebra and $\{i, \mathscr{C}\} \in \mathscr{K}(J, M, m)$, let

$$K(\mathscr{C}) = \{C \in \mathscr{C}\colon \text{ if } i(A) \subseteqq C,\ A \in \mathscr{A}, \text{ then } A = \Lambda_{\mathscr{A}}\}\ ,$$

and

$$K_P(\mathscr{C}) = \{C \in \mathscr{C}\colon \text{ if } P = \{A \in \mathscr{A}\colon i(A) \supseteqq C\} \text{ then } \bigcap_{A \in P}^{\mathscr{A}} A = \Lambda_{\mathscr{A}}\}\ .$$

Note that $K_P(\mathscr{C}) \subseteqq K(\mathscr{C})$.

LEMMA 3.1. *The set $K_P(\mathscr{C})$ is an ideal and $K(\mathscr{C}) = K_P(\mathscr{C})$, if, and only if, $K(\mathscr{C})$ is an ideal.*

Proof. It follows easily that $K_P(\mathscr{C})$ is an ideal.

If $K(\mathscr{C})$ is an ideal and $\mathscr{C} \in K(\mathscr{C})$ let

$$P = \{A \in \mathscr{A}\colon i(A) \supseteqq C\}\ .$$

If $A' \in \mathscr{A}$ and $A' \subseteqq A$ for all $A \in P$, then

$$i(A') - C \in K(\mathscr{C})\ .$$

Now $i(A') \cap C \in K(\mathscr{C})$, hence

$$i(A') = (i(A') - C) \cup (i(A') \cap C) \in K(\mathscr{C})\ ,$$

which implies $i(A') = \Lambda_{\mathscr{C}}$ or $A' = \Lambda_{\mathscr{A}}$. Thus

$$\bigcap_{A \in P}^{\mathscr{A}} A = \Lambda_{\mathscr{A}}\ ,$$

so $C \in K_P(\mathscr{C})$, and

$$K_P(\mathscr{C}) = K(\mathscr{C})\ .$$

Since $K_P(\mathscr{C})$ is an ideal, the converse is true.

PROPOSITION 3.1. *If $\mathscr{A}$ is a Boolean algebra the following are equivalent*:

1. *$\mathscr{K}(J, M, m)$ contains a smallest element*;
2. *$K(\mathscr{C}) = K_P(\mathscr{C})$ for all $\{i, \mathscr{C}\} \in \mathscr{K}(J, M, m)$*;
3. *$K(\mathscr{C}) = K_P(\mathscr{C})$ if $\{i, \mathscr{C}\}$ is the maximum element in $\mathscr{K}(J, M, m)$.*

Proof.

1. $\Rightarrow$ 2. Suppose $\mathscr{K}(J, M, m)$ contains a smallest element $\{i, \mathscr{B}\}$, and there is an element

$$\{j, \mathscr{C}\} \in \mathscr{K}(J, M, m)$$

with the property that

$$K(\mathscr{C}) \neq K_P(\mathscr{C}) .$$

Let h be the unique m-homomorphism mapping $\mathscr{C}$ onto $\mathscr{B}$ such that $hj = i$. Let $\ker h$ be the kernel of this mapping. Then

$$K_P(\mathscr{C}) \subseteqq \ker h \subseteqq K(\mathscr{C}) ,$$

and

$$\ker h \neq K(\mathscr{C}) .$$

Pick $x \in K(\mathscr{C}) - \ker h$ and let

$$\varDelta = \langle x \rangle ,$$

so $\varDelta$ is a complete ideal. Thus

$$\{i_\varDelta, \mathscr{C}/\varDelta\} \in \mathscr{K}(J, M, m) ,$$

where

$$i_\varDelta : \mathscr{A} \rightarrow \mathscr{C}/\varDelta$$

is defined by

$$i_\varDelta(A) = [i(A)]_\varDelta .$$

Consequently, there are unique homomorphisms $h_\varDelta$ and h' mapping $\mathscr{C}$ onto $\mathscr{C}/\varDelta$, $\mathscr{C}/\varDelta$ onto $\mathscr{B}$, and satisfying $h_\varDelta j = i_\varDelta$, $h' i_\varDelta = i$, respectively. Hence

$$h' h_\varDelta j = h' i_\varDelta = i$$

and by the uniqueness of h,

$$h = h' h_\varDelta .$$

This implies

$$h(x) = h' h_\varDelta(x) = \bigwedge\nolimits_{\mathscr{B}} ,$$

a contradiction. Thus

$$K(\mathscr{C}) = K_P(\mathscr{C}) .$$

2. $\Rightarrow$ 3. Obvious.

3. $\Rightarrow$ 1. To show that $\mathscr{K}(J, M, m)$ contains a smallest element, let $\{j, \mathscr{C}\}$ be the largest element in $\mathscr{K}(J, M, m)$ and suppose $\{j', \mathscr{C}'\} \in \mathscr{K}(J, M, m)$. Let $\{i, \mathscr{B}\}$ be an m-completion of $\mathscr{A}$. Then there is an m-homomorphism h' mapping $\mathscr{C}$ onto $\mathscr{C}'$ such that $h'j = j'$ and an m-homomorphism h mapping $\mathscr{C}$ onto $\mathscr{B}$ such that $hj = i$. Thus

$$K_P(\mathscr{C}) \subseteqq \ker h \subseteqq K(\mathscr{C}) ,$$

which implies, by assumption, that

$$K_P(\mathscr{C}) = \ker h = K(\mathscr{C}) ,$$

so $K_P(\mathscr{C})$ and $K(\mathscr{C})$ are m-ideals in $\mathscr{C}$. Further,

$$h'(K_P(\mathscr{C})) \subseteqq K_P(\mathscr{C}') \subseteqq K(\mathscr{C}') \subseteqq h'(K(\mathscr{C})) .$$

This implies that

$$h'(K_P(\mathscr{C})) = K_P(\mathscr{C}') = K(\mathscr{C}') = h'(K(\mathscr{C})) ,$$

hence $K(\mathscr{C}')$ is an m-ideal. Let

$$\Delta = K(\mathscr{C}') .$$

Then $\mathscr{C}'/\Delta$ is an m-algebra and

$$j'_\Delta(\mathscr{A}) = \{[j'(A)]_\Delta : A \in \mathscr{A}\}$$

m-generates $\mathscr{C}'/\Delta$. Finally, $j'_\Delta(\mathscr{A})$ is dense in $\mathscr{C}'/\Delta$. Thus $\{j'_\Delta, \mathscr{C}'/\Delta\}$ is an m-completion of $\mathscr{A}$, hence is equal to $\{i, \mathscr{B}\}$, as isomorphic elements of $\mathscr{K}(J, M, m)$ have been identified. The m-homomorphism

$$h_\Delta : \mathscr{C}' \longrightarrow \mathscr{C}'/\Delta$$

defined by

$$h_\Delta(C') = [C']_\Delta$$

has the property that

$$h_\Delta j = j'_\Delta \quad \text{for all} \quad A \in \mathscr{A} ,$$

implying that

$$\{i_\Delta, \mathscr{C}'/\Delta\} \leqq \{j', \mathscr{C}'\} .$$

Hence $\mathscr{K}(J, M, m)$ contains a smallest element.

This, then, gives a way to construct a Boolean algebra $\mathscr{A}$ such that $\mathscr{K}$ does not contain a smallest element. Namely, by finding a Boolean algebra $\mathscr{A}$ with an m-extension $\{i, \mathscr{C}\}$ such that $K_P(\mathscr{C}) \neq K(\mathscr{C})$. The next task is to construct such a Boolean algebra.

If $\overline{\overline{T}} = m$ and $\mathscr{A} = \mathscr{A}_t$ for all $t \in T$, the Boolean product of $\{\mathscr{A}_t\}_{t \in T}$ will be called the *m-fold product of $\mathscr{A}$.* Note that if $\mathscr{A}$ is a subalgebra of the Boolean algebra $\mathscr{A}'$, $\mathscr{F}$ is the m-fold product of $\mathscr{A}$ and $\mathscr{F}'$ is the m-fold product of $\mathscr{A}'$, then $\mathscr{F} \subseteqq \mathscr{F}'$.

LEMMA 3.2. *If $\mathscr{A}$ is an m-regular subalgebra of the Boolean algebra $\mathscr{A}'$ then the Boolean m-fold product $\mathscr{F}$ of $\mathscr{A}$ is isomorphic to an m-regular subalgebra of the Boolean m-fold product $\mathscr{F}'$ of $\mathscr{A}'$.*

Proof. Since $\mathscr{A}$ is a subalgebra of $\mathscr{A}'$, $\mathscr{F} \subseteqq \mathscr{F}'$. Let $\mathscr{S}(\mathscr{S}')$ be the set of all $\varphi_t(A)$, $A \in \mathscr{A}$ and $t \in T(A \in \mathscr{A}'$ and $t \in T)$. Then $F \in \mathscr{S}(F \in \mathscr{S}')$ implies $-F \in \mathscr{S}(-F \in \mathscr{S}')$ and $\mathscr{S}(\mathscr{S}')$ are sets of generators for $\mathscr{F}(\mathscr{F}')$. For elements $F \in \mathscr{F}'$ of the form

$$F = \bigcap_{i=1}^{N} F_i \,, \qquad F_i \in \mathscr{S} \,,$$

define

$$\lambda_t(F) = \left\{\pi_t(x)\colon x \in \bigcap_{i=1}^{N} F_i\right\} .$$

Note that if $F \in \mathscr{S}'$ and $t \in T$ is such that $\lambda_t(F) \neq \bigvee_{\mathscr{A}'}$ then $\varphi_t(\lambda_t(F)) = F$.

In order to show $\mathscr{F}$ is m-regular in $\mathscr{F}'$, it suffices to prove that if $\{F_t\}_{t \in T}$ is an m-indexed set of elements of $\mathscr{F}$ such that

$$\bigcap_{t \in T}^{\mathscr{F}} F_t = \bigwedge\nolimits_{\mathscr{F}}$$

then

$$\bigcap_{t \in T}^{\mathscr{F}'} F_t = \bigwedge\nolimits_{\mathscr{F}'} .$$

Now $F_t \in \mathscr{F}$ so F_t may be rewritten as

$$F_t = \bigcap_{p=1}^{P_t} \bigcup_{q=1}^{Q_t} F_{p,q,t} \,,$$

where P_t, Q_t are finite numbers and $F_{p,q,t} \in \mathscr{S}$, for all $p \in P_t$, $q \in Q_t$, and $t \in T$. Thus

$$\bigwedge_{\mathscr{F}} = \bigcap_{t\in T}^{\mathscr{F}} \bigcap_{p=1}^{P_t} \bigcup_{q=1}^{Q_t} F_{p,q,t}$$

$$= \bigcap_{s\in S}^{\mathscr{F}} \bigcup_{q=1}^{Q_s} F_{s,q}$$

after a suitable re-indexing, where $\bar{\bar{S}} \leqq m$ and $F_{s,q} = F_{p,q,t}$ for suitable $p \in P_t$, $t \in T$. Without loss of generality, assume that for each $s \in S$, $\lambda_t(F_{s,q}) \neq \bigwedge_{\mathscr{A}'}$ implies $\lambda_t(F_{s,q'}) = \bigvee_{\mathscr{A}'}$ for all $t \in T$ and $q' \neq q$, and that $F_{s,q} \neq \bigvee_{\mathscr{F}'}$ for all q, $1 \leqq q \leqq Q_s$, and all $s \in S$. Suppose $F' \in \mathscr{F}'$ and $F' \subseteqq F_t$ for all $t \in T$. Then

$$F' = \bigcup_{m=1}^{M} \bigcap_{n=1}^{N} F'_{m,n} , \qquad F'_{m,n} \in \mathscr{S}' ,$$

so

$$\bigcap_{n=1}^{N} F'_{m,n} \subseteqq \bigcup_{q=1}^{Q_s} F_{s,q}$$

for $1 < m \leqq M$, and all $s \in S$. Thus to show $F' = \bigwedge_{\mathscr{F}'}$, it suffices to prove that if

$$\bigcap_{n=1}^{N} F'_n \subseteqq \bigcup_{q=1}^{Q_s} F_{s,q} ,$$

for all $s \in S$, where $F'_n \in \mathscr{S}'$, then

$$\bigcap_{n=1}^{N} F'_n = \bigwedge_{\mathscr{F}'} .$$

It may be assumed that for each n, $1 \leqq n \leqq N$, $\lambda_t(F'_n) \neq \bigwedge_{\mathscr{A}'}$ implies $\lambda_t(F'_{n'}) = \bigvee_{\mathscr{A}'}$ for all $t \in T$ and $n' \neq n$, and that $F'_n \neq \bigvee_{\mathscr{F}'}$ for all n, $1 \leqq n \leqq N$.

Now

$$\bigcap_{n=1}^{N} F'_n \subseteqq \bigcup_{q=1}^{Qs} F_{s,q}$$

implies

$$\bigcap_{n=1}^{N} F'_n \cap \bigcup_{q=1}^{Q_s} -F_{s,q} = \bigwedge_{\mathscr{F}'} ,$$

and as each F'_n and $-F_{s,q}$ is of the form $\varphi_t(A)$ for some $A \in \mathscr{A}'$ and $t \in T$, the independence of the indexed set $\{\varphi_t(\mathscr{A}')\}_{t\in T}$ of subalgebras of $\mathscr{F}'$ implies that for some n_s, $1 \leqq n_s \leqq N$, and some q_s, $1 \leqq q_s \leqq Q_s$,

$$F'_{n_s} \cap -F_{s,q_s} = \bigwedge_{\mathscr{F}'} ,$$

which implies $F'_{n_s} \subseteqq F_{s,q_s}$. This argument may be repeated for each $s \in S$.

The set $\{n_s: s \in S\}$ is finite so let $\{n_s: s \in S\} = \{n_i: 1 \leqq i \leqq N'\}$. Let $S_i = \{s \in S: F'_{n_i} \subseteqq F_{s,q_s}\}$. If $t_s \in T$ is such that

$$\lambda_{t_s}(F_{s,q_s}) \neq \bigvee_{\mathscr{A}'} \quad \text{for all} \quad s \in S$$

then $\lambda_{t_s}(F_{s,q_s}) \in \mathscr{A}$ and

$$\bigcap_{s \in S_i}^{\mathscr{A}} \lambda_{t_s}(F_{s,q_s}) \neq \bigwedge_{\mathscr{A}'} .$$

Thus

$$\bigcap_{s \in S_i}^{\mathscr{A}'} \lambda_{t_s}(F_{s,q_s}) \neq \bigwedge_{\mathscr{A}'} ,$$

or

$$\bigcap_{s \in S_i}^{\mathscr{A}} \lambda_{t_s}(F_{s,q_s}) \neq \bigwedge_{\mathscr{A}'} ,$$

hence there is an $A_i \in \mathscr{A}$, $A_i \neq \bigwedge_{\mathscr{A}'}$, with

$$A_i \subseteqq \lambda_{t_s}(F_{s,q_s}) \quad \text{for all} \quad s \in S_i .$$

Let $A_{t,i}$ be the set of all $x \in X$ such that $\pi_{t_s}(x) \in A_i$. Thus $A_{t,i} \in \mathscr{F}$ and this argument may be repeated for each i, $1 \leqq i \leqq N'$. Now

$$\bigwedge_{\mathscr{F}'} \neq \bigcap_{i=1}^{N'} A_{t,i}$$

and

$$\bigcap_{i=1}^{N'} A_{t,i} \subseteqq \bigcup_{q=1}^{Q_s} F_{q,s}$$

for all $s \in S$. But then

$$\bigcap_{i=1}^{N'} A_{t,i} \subseteqq \bigcap_{s \in S}^{\mathscr{F}} \bigcup_{q=1}^{Q_s} F_{q,s} = \bigwedge_{\mathscr{F}} ,$$

a contradiction. Thus $\mathscr{F}$ is m-regular in $\mathscr{F}'$.

The next lemma assumes there is a Boolean algebra $\mathscr{A}$ such that an m-extension is not an m-completion. Sikorski [2] cites an example due to Katětov of such a Boolean algebra for the case $m = \sigma$. As Lemmas 3.5 and 3.6 imply, there is such an $\mathscr{A}$ for all infinite cardinal numbers m.

Assume for the moment that $\mathscr{A}$ is a Boolean algebra such that $\mathscr{K}$ contains more than one element and $\{i, \mathscr{B}\} \in \mathscr{K}$ is an m-extension that is not an m-completion. Thus there is a $B \in \mathscr{B}$ such that $i(A) \subseteqq B$, $A \in \mathscr{A}$, implies $A = \bigwedge_{\mathscr{A}}$. Let $\mathscr{F}'$ be the Boolean m-fold product of $\mathscr{B}$, h_0 an isomorphism of $\mathscr{B}$ onto the Stone space $\mathscr{F}$ of

$\mathscr{B}$, X the Cartesian product of $\mathscr{F}$ with itself m times and indexed by T, and

$$B_t = \varphi_t h_0(B) \quad \text{for all} \quad t \in T .$$

Let

$$B_0 = \bigcup_{t \in T'} B_t ,$$

where T' is a fixed, but arbitrary subset of T such that $\bar{\bar{T'}} \geqq \sigma$, and define

$$\mathscr{F}_0 = \langle \mathscr{F}', B_0 \rangle .$$

Since $\bar{\bar{T'}} \geqq \sigma$, $\mathscr{F}_0 \neq \mathscr{F}'$.

LEMMA 3.3. *If $\mathscr{F}$ is the Boolean m-fold product of $\mathscr{A}$ then $\mathscr{F}$ is isomorphic to an m-regular subalgebra of $\mathscr{F}_0$.*

Proof. It may be assumed, without loss of generality, that $\mathscr{A} \subseteqq \mathscr{B}$. Thus $\mathscr{F} \subseteqq \mathscr{F}_0$. Let $\mathscr{S}(\mathscr{S}')$ be a generating set for $\mathscr{F}(\mathscr{F}')$. Let

$$\mathscr{S}_0 = \mathscr{S}' \cup \{B_0\} ,$$

so $\mathscr{S}_0$ is a generating set for $\mathscr{F}_0$. As in the previous lemma, to prove $\mathscr{F}$ is m-regular in $\mathscr{F}_0$ it suffices to show that if

$$\bigcap_{n=1}^{N} F'_n \subseteqq \bigcup_{q=1}^{Q_s} F_{s,q}$$

for all $s \in S$, $\bar{\bar{S}} \leqq m$; and

$$\bigcap_{s \in S}^{\mathscr{F}} \bigcup_{q=1}^{Q_s} F_{s,q} = \Lambda_{\mathscr{F}} ;$$

$F_{s,q} \in \mathscr{S}$ for all $s \in S$ and $1 \leqq q \leqq Q_s$, $F'_n \in \mathscr{S}_0$, $1 \leqq n \leqq N$; then

$$\bigcap_{n=1}^{N} F'_n = \Lambda_{\mathscr{F}'} .$$

Since $F'_n \in \mathscr{S}_0$, there is an n, $1 \leqq n \leqq N$, such that $F'_n = B_0$ or $F'_n = -B_0$, otherwise there is nothing to prove. This may be reduced to two cases:

Case 1.

$$\bigcap_{n=1}^{N} F'_n \cap B_0 \subseteqq \bigcup_{q=1}^{Q_s} F_{s,q}$$

for all $s \in S$, where $F'_n \in \mathscr{S}'$ and $F_{s,q} \in \mathscr{S}$.

Case 2.

$$(-B_0) \cap \bigcap_{n=1}^{N} F'_n \subseteqq \bigcup_{q=1}^{Q_s} F_{s,q}$$

for all $s \in S$, where $F'_n \in \mathscr{S}'$ and $F_{s,q} \in \mathscr{S}$.

Proof of Case 1. If for each $s \in S$ there is an n_s, $1 \leqq n_s \leqq N$, such that there is a q_s, $1 \leqq q_s \leqq Q_s$, with $F'_{n_s} \subseteqq F_{s,q_s}$, then

$$\bigcap_{n=1}^{N} F'_n \subseteqq \bigcup_{q=1}^{Q_s} F_{s,q}$$

for all $s \in S$, and

$$\bigcap_{n=1}^{N} F'_n \in \mathscr{F}'$$

implies

$$\bigcap_{n=1}^{N} F'_n = \bigwedge_{\mathscr{F}'} .$$

Thus it may be assumed there is an s_0 such that

$$\bigcap_{n=1}^{N} F'_n \not\subseteqq \bigcup_{q=1}^{Q_{s_0}} F_{s_0,q} .$$

Hence for all n, $F'_n \subseteqq F_{s_0,q}$ for some q, is false. If

$$\bigcap_{n=1}^{N} F'_n \cap B_0 \neq \bigwedge_{\mathscr{F}'} ,$$

let $x \in X$ be defined as follows. Let $t_1, \cdots, t_n \in T$ be such that $\lambda_{t_i}(F'_i) \neq \bigvee_{\mathscr{B}}$, $1 \leqq i \leqq N$. Choose an $x \in X$ such that it satisfies the following conditions:

(a)

$$\pi_i(x) \in \begin{cases} \lambda_{t_i}(F'_i) \text{ if } \lambda_{t_i}(F_{s_0,q}) = \bigvee_{\mathscr{B}} \text{ for all } q, 1 \leqq q \leqq Q_{s_0} \\ \lambda_{t_i}(F'_i) - \lambda_{t_i}(F_{s_0,q_0}) \text{ if } \lambda_{t_i}(F_{s_0,q_0}) \neq \bigvee_{\mathscr{B}} \end{cases}$$

for $1 \leqq i \leqq N$;

(b) $\pi_{t_q}(s) \in -\lambda_{t_q}(F_{s_0,q})$ for each $t_q \in T$ such that $\lambda_{t_q}(F_{s_0,q}) \neq \bigvee_{\mathscr{B}}$, $1 \leqq q \leqq Q_{s_0}$ and $t_q \neq t_i$, $1 \leqq i \leqq n$;

(c) $\pi_t(x) \in h_0(B)$ for all $t \neq t_q$; $1 \leqq i \leqq N$, $1 \leqq q \leqq Q_{s_0}$.

Now x is well defined,

$$x \in B_0 \quad \text{and} \quad x \in \bigcap_{n=1}^{N} F'_n ,$$

by its definition. But $x \notin F_{s_0,q}$ for all q, $1 \leqq q \leqq Q_{s_0}$, hence

$$x \notin \bigcup_{q=1}^{Q_{s_0}} F_{s_0,q} ,$$

a contradiction.

Proof of Case 2. If

$$-B_0 \cap \bigcap_{n=1}^{N} F'_n \neq \bigwedge_{\mathscr{F}'}$$

and $\lambda_{t_n}(F'_n) \neq \bigvee_{\mathscr{B}}$, $t_n \in T$, let $A_n = \varphi_{t_n}(-B_0)$, $1 \leqq n \leqq N$. Then

$$\bigcap_{n=1}^{N} F'_n \cap (-B_0) = \bigcap_{n=1}^{N} (F'_n \cap A_n) \cap (-B_0)$$

and

$$\bigcap_{n=1}^{N} (F'_n \cap A_n) \in \mathscr{F}' .$$

As before, an $s_0 \in S$ may be found such that

$$\bigcap_{n=1}^{N} (F'_n \cap A_n) \nsubseteq \bigcup_{q=1}^{Q_{s_0}} F_{s_0,q} .$$

Define $t_1, \cdots, t_N$ as before so that $\lambda_{t_i}(F'_i \cap A_i) \neq \bigvee_{\mathscr{B}}$, $1 \leqq i \leqq N$. Choose $x \in X$ satisfying the following conditions:

(a)

$$\pi_{t_i}(x) \in \begin{cases} \lambda_{t_i}(F'_i \cap A_i) \text{ if } \lambda_{t_i}(F_{s_0,q}) = \bigvee_{\mathscr{B}}, 1 \leqq q \leqq Q_{s_0} \\ \lambda_{t_i}(F'_i \cap A_i) - \lambda_{t_i}(F_{s_0,q}) \text{ if } \lambda_{t_i}(F_{s_0,q_0}) \neq \bigvee_{\mathscr{B}} \end{cases}$$

for $1 \leqq i \leqq N$.

(b) $\pi_{t_q}(x) \in -\lambda_{t_q}(F_{s_0,q})$ for each $t_q \in T$ such that $\lambda_{t_q}(F_{s_0,q}) \neq \bigvee_{\mathscr{B}}$; $1 \leqq q \leqq Q_{s_0}$, and $t_q \neq t_i$, $1 \leqq i \leqq N$.

(c) $\pi_t(x) \in \lambda_t(-B_0)$ if $t \neq t_i, t_q$; $1 \leqq i \leqq n$, $1 \leqq q \leqq Q_{s_0}$.

Now x is well defined and

$$x \in (-B_0) \cap \bigcap_{n=1}^{N} (F'_n \cap A_n) = -B_0 \cap \bigcap_{n=1}^{N} F'_n ,$$

so

$$x \notin \bigcup_{q=1}^{Q_{s_0}} F_{s,q} ,$$

a contradiction.

Consequently, in either case

$$\bigcap_{n=1}^{N} F'_n = \bigwedge_{\mathscr{F}'} .$$

LEMMA 3.4. *If j is the identity isomorphism of $\mathscr{F}$ into $\mathscr{F}_0$ and $\{i, \mathscr{C}\}$ is an m-completion of $\mathscr{F}_0$, then $\{ij, \mathscr{C}\}$ is an m-extension of $\mathscr{F}$.*

Proof. All that needs to be shown is that $ij(\mathscr{F})$ m-generates $\mathscr{C}$. But this follows immediately from the fact that $\mathscr{A}$ m-generates $\mathscr{B}$ and the definition of $\mathscr{F}$ and $\mathscr{F}_0$.

THEOREM 3.1. *If $\mathscr{A}$ m-generates $\mathscr{B}$ then $\mathscr{K}(\mathscr{F})$ does not contain a smallest element.*

Proof. $F \in \mathscr{F}$ and $F \supseteqq B_0$ then $F = \bigvee_{\mathscr{F}_0}$, by definition of B_0. Thus if j and $\{i, \mathscr{C}\}$ are defined as in Lemma 3.4, $\{ij, \mathscr{C}\}$ is an m-extension of $\mathscr{F}$ and $ij(B_0) \in K(\mathscr{C})$. By Proposition 3.1, $\mathscr{K}(\mathscr{F})$ does not contain a smallest element.

The results of this theorem may be generalized as follows. Let $\{\mathscr{A}_t\}_{t \in T}$ be an infinite indexed set of Boolean algebras and $\{\{i_t\}_{t \in T}, \mathscr{B}\}$ be the Boolean product of $\{\mathscr{A}_t\}_{t \in T}$. Let T' be the set of all $t \in T$ such that $\mathscr{K}(\mathscr{A}_t)$ contains more than one element.

THEOREM 3.2. *The class of m-extensions $\mathscr{K}(\mathscr{B})$ does not contain a smallest element if $\bar{\bar{T}}' \geqq \sigma$.*

Proof. Define $\mathscr{F}'$ to be the Boolean product of $\{\{j_t, \mathscr{B}_t\}\}_{t \in T}$, where $\{j_t, \mathscr{B}_t\} \in \mathscr{K}(\mathscr{A}_t)$ for all $t \in T$ and $\{j_t, \mathscr{B}_t\}$ is not an m-completion of $\mathscr{A}_t$ for all $t \in T'$. For each $\mathscr{B}_t$, $t \in T'$, there is a $B_t \in \mathscr{B}_t$ such that $j_t(A) \subseteqq B_t$, $A \in \mathscr{A}_t$, implies $A = \bigwedge_{\mathscr{A}_t}$. Let φ_t map $\mathscr{B}_t$ into $\mathscr{B}$ and set

$$B_0 = \overset{\mathscr{B}}{\bigcup_{t \in T'}} \varphi_t(B_t)$$

and

$$\mathscr{F}_0 = \langle \mathscr{F}', B_0 \rangle .$$

Then by an argument similar to the proofs of Lemmas 3.2, 3.3, and 3.4, and Theorem 3.1, $\mathscr{K}(\mathscr{B})$ does not contain a smallest element.

COROLLARY 3.1. *If $\mathscr{A}_t = \mathscr{A}_{t'}$ for all $t, t' \in T$ then $\mathscr{K}(\mathscr{B})$ contains a smallest element if, and only if, an m-extension of $\mathscr{B}$ is an m-completion.*

Proof. If $\mathscr{K}(\mathscr{B})$ contains an m-extension which is not an m-completion, let $\mathscr{B}$ play the role of $\mathscr{A}$ in Lemmas 3.2, 3.3, and 3.4. By Theorem 3.1, $\mathscr{K}(\mathscr{F})$ does not contain a smallest element. As the m-fold product $\mathscr{F}$ of $\mathscr{B}$ is isomorphic to $\mathscr{B}$, $\mathscr{K}(\mathscr{B})$ does not contain a smallest element. The converse is clear.

Now to prove the assumption on which these results are based.

LEMMA 3.5. *For each infinite cardinal number m there is a Boolean algebra $\mathscr{A}$ such that an m-completion $\{i, \mathscr{B}\}$ of $\mathscr{A}$ contains an element B with*

$$B \neq \bigcup_{u \in U}^{\mathscr{B}} \bigcap_{v \in V}^{\mathscr{B}} A_{u,v} ,$$

for all m-indexed sets $\{A_{u,v}\}_{u \in U, v \in V}$ in $\mathscr{A}$.

Proof. The proof will be by constructing such an $\mathscr{A}$ for each m. Let S be an indexing set of cardinality m. Let $\mathscr{D}_m$ be the Cartesian product of S with itself m times and indexed by T. Define

$$D_{t,s} = \{d \in \mathscr{D}_m : \pi_t(d) = s\} .$$

Fix $s_1', s_2' \in S$, $s_1' \neq s_2'$, and set $S' = S - \{s_1', s_2'\}$. Let $D = \bigcup_{t \in T}(D_{t,s_1'} \cup D_{t,s_2'})$. Thus $\overline{\overline{D}} = 2^m$ and $d \in \mathscr{D}_m - D$ implies $\pi_t(d) \neq s_k'$, $k = 1, 2$, for all $t \in T$.

Let

$$\mathscr{S} = \{\{d\} : d \in \mathscr{D}_m\} \cup \{D_{t,s} : t \in T,\ s \in S'\} .$$

Let $\mathscr{A}$ be generated by $\mathscr{S}$ in $\mathscr{D}_m$ and let $\mathscr{B}$ be the m-field of sets m-generated by $\mathscr{S}$ in $\mathscr{D}_m$. Then $\mathscr{A}$ is dense in $\mathscr{B}$ and m-generates $\mathscr{B}$, so if i is the identity map of $\mathscr{A}$ into $\mathscr{B}$, $\{i, \mathscr{B}\}$ is an m-completion of $\mathscr{A}$.

Let

$$B = \mathscr{D}_m - D .$$

Suppose

$$B = \bigcup_{u \in U} \bigcap_{v \in V} A_{u,v} ,$$

$\{A_{u,v}\}_{u \in U, v \in V}$ an m-indexed set in $\mathscr{A}$. This can be written in the form

$$\bigcup_{u \in U} \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} ;$$

$$A_{u,v,m} \text{ or } -A_{u,v,m} \in \mathscr{S}, \quad \overline{\overline{M_{u,v}}} < \sigma .$$

Let $B' = \{d \in \mathscr{D}_m : \{d\} = A_{u,v,m}$ for some $u \in U$, $v \in V$, and $m \in M_{u,v}\}$. Then $\overline{\overline{B}}' \leqq m$, so if

$M_{u,v}' = \{m \in M_{u,v} : A_{u,v,m}$ is not of the form $\{d\}$, $d \in \mathscr{D}_m\}$, it follows that

$$\overline{\overline{B - \bigcup_{u \in U} \bigcap_{v \in V} \bigcup_{m \in M'_{u,v}} A_{u,v,m}}} \leqq m \ .$$

It will now be shown that in fact

$$\overline{\overline{B - \bigcup_{u \in U} \bigcap_{v \in V} \bigcup_{m \in M'_{u,v}} A_{u,v,m}}} > m \ ,$$

a contradiction. Hence it may be assumed that $A_{u,v,m}$ is not of the form $\{d\}$, $d \in \mathscr{D}_m$, for all $u \in U$, $v \in V$, and $m \in M_{u,v}$.

If $A_{u,v,m} = -\{d\}$, $d \in \mathscr{D}_m$, for some $m \in M_{u,v}$, then either

$$(1) \qquad \bigcup_{m \in M_{u,v}} A_{u,v,m} = -\{d\}$$

or

$$(2) \qquad \bigcup_{m \in M_{u,v}} A_{u,v,m} = \mathsf{V} \ .$$

If (1) occurs, it may be assumed that $M_{u,v} = \{1\}$ and $A_{u,v,1} = -\{d\}$. If (2) occurs, the term $\bigcup_{m \in M_{u,v}} A_{u,v,m}$ may be dropped. Thus for all $u \in U$, V may be written as $V_u \cup V'_u$, where (1) $V_u \cap V'_u = \varnothing$; (2) $A_{u,v,m} = -\{d_{u,v}\}$, $d_{u,v} \in \mathscr{D}_m$, for all $v \in V_u$; and (3) $A_{u,v,m}$ is either of the form $-D_{t,s}$ or $D_{t,s}$ for all $v \in V'_u$. Consequently, for all $u \in U$,

$$\bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} = \bigcap_{v \in V_u} - \{d_{u,v}\} \cap \bigcap_{v \in V'_u} \bigcup_{m \in M_{u\ v}} A_{u,v,m} \ .$$

Let

$$C_u = \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} \ .$$

Suppose U is the set of all ordinals $u < \alpha$, where $\alpha = \overline{\overline{U}}$. Let $D_1 = \{d \in \mathscr{D}_m \colon \pi_t(d) = s'_1, s'_2\}$. Now $\overline{\overline{D}}_1 = 2^m$ implies there is a $d_1 \in D$ such that

$$d_1 \in \bigcap_{v \in V_1} - \{d_{1,v}\} \ .$$

Since $d_1 \notin B$, this implies

$$d_1 \notin \bigcap_{v \in V'_1} \bigcup_{m \in M_{1,v}} A_{1,v,m} \ ,$$

hence for some $v_1 \in V'_1$,

$$d_1 \notin \bigcup_{m \in M_{1,v_1}} A_{1,v_1,m} \ .$$

Also, $D_1 \subseteqq -D_{t,s}$ for all $t \in T$ and $s \in S'$, hence

$$A_{1,v_1,m} = D_{t_{1,m}, s_{t_{1,m}}}$$

for some $t_{1,m} \in T$ and $s_{t_{1,m}} \in S'$, for all $m \in M_{1,v_1}$. Let $T_1 = \{t_{1,m} \colon m \in M_{1,v_1}\}$

and pick $s_1 \in S'$ such that $s_1 \neq s_{t_{1,m}}$ for all $m \in M_{1,v_1}$. Define

$$\varphi(t) = s_1$$

for all $t \in T_1$. Let $B_1 = \varnothing$ and define $B_2 = \{d \in \mathscr{D}_m\colon \pi_t(d) = \varphi(t)$ for all $t \in T_1\}$.

Note that $B_2 \cap C_1 = \varnothing$.

Suppose $i > 1$ and a finite set $T_{i'}$ has been defined for each $i' < i$ so that $T_{i'} \cap T_{i''} = \varnothing$ if $i', i'' < i$, $i' \neq i''$; $s_{i'} \in S'$ has been chosen; φ has been defined on each $T_{i'}$, $i' < i$, so that $\varphi(t) = s_{i'}$ for all $t \in T_{i'}$; and if

$$B_i = \{d \in \mathscr{D}_m\colon \pi_t(d) = \varphi(t) \text{ for all } t \in \bigcup_{i'<i} T_{i'}\}$$

then

$$B_i \cap \bigcup_{i'<i} C_{i'} = \varnothing\ .$$

Let

$$\hat{T}_i = \bigcup_{i'<i} T_{i'}$$

and note that $\overline{\overline{\hat{T}_i}} < m$. Let

$$\begin{aligned} D_i = \{d \in \mathscr{D}_m\colon \pi_t(d) = \varphi(t) &\text{ for all } t \in \hat{T}_i \\ &\text{and } \pi_t(d) = s'_k,\ k = 1, 2, \text{ if } t \in T - \hat{T}_i\}\ . \end{aligned}$$

Then $D_i \subseteqq D$ and $\overline{\overline{D_i}} = 2^m$, hence there is a $d_i \in D_i$ such that

$$d_i \in \bigcap_{v \in V_i} - \{d_{i,v}\}\ .$$

Since $d_i \notin B$, this implies

$$d_i \notin \bigcap_{v \in V'_i} \bigcup_{m \in M_{i,v}} A_{i,v,m}\ ,$$

hence for some $v_i \in V'_i$,

$$d_i \notin \bigcup_{m \in M_{i,v_i}} A_{i,v_i,m}\ .$$

If $B_i \cap C_i = \varnothing$ set $T_i = \varnothing$. If not, there is a $d'_i \in B_i$ such that $d'_i \in C_i$, so

$$d'_i \in \bigcup_{m \in M_{i,v_i}} A_{i,v_i,m}\ .$$

Note that $\pi_t(d'_i) = \pi_t(d_i)$ for all $t \in \hat{T}_i$.

It immediately follows that if

$$d'_i \in \bigcup_{m \in M_{i,v_i}} A_{i,v_i,m}$$

then

$$A_{i,v_i,m} = D_{t_{i,m},s_{t_{i,m}}} ,$$

where $t_{i,m} \notin \hat{T}_i$ and

$$\pi_{t_{i,m}}(d'_i) = s_{t_{i,m}} ,$$

for some $m \in M_{i,v_i}$.

Let

$$T_i = \{t_{i,m} \in T - \hat{T}_i \colon A_{i,v_{i,m}} = D_{t_{i,m},s_{t_{i,m}}} \text{ for some } m \in M_{i,v_i}\}$$

and pick $s_i \in S'$ such that if $t_{i,m} \in T_i$ then

$$s_i \neq S_{t_{i,m}} ,$$

for all $m \in M_{i,v_i}$. Now define

$$\varphi(t) = s_i \quad \text{for all} \quad t \in T_i .$$

Thus $T_i \cap \hat{T}_i = \varnothing$ which implies $T_i \cap T_{i'} = \varnothing$ for all $i' < i$. If

$$B_{i+1} = \{d \in \mathscr{D}_m \colon \pi_t(d) = \varphi(t) \text{ for all } t \in T_i \cup \hat{T}_i\}$$

then it is clear that

$$B_{i+1} \cap \bigcup_{i'<i} C_i = \varnothing .$$

Now let $\hat{T} = \bigcup_{i<\alpha} T_i$ and set

$$\begin{aligned}\hat{B} = \{d \in \mathscr{D}_m \colon \pi_t(d) = \varphi(t) \text{ for all } t \in \hat{T} \\ \text{and } \pi_t(d) \neq s'_1, s'_2 \text{ if } t \in T - \hat{T}\} .\end{aligned}$$

Then $\hat{B} \neq \varnothing$ and $\hat{B} \subseteqq B$. But $\hat{B} \cap \bigcup_{u \in U} C_u = \varnothing$ which implies

$$B - \bigcup_{u \in U} C_u \neq \varnothing .$$

If $B' = B - \bigcup_{u \in U} C_u$ then for each $b \in B'$,

$$b = \bigcap_{t \in T} D_{t,s_{t,b}} ,$$

for some m-indexed set $\{s_{t,b}\}_{t \in T}$ in S'. Thus

$$B = \bigcup_{u \in U} \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} \cup \bigcup_{b \in B'} \bigcap_{t \in T} D_{t,s_{t,b}} ,$$

but the above construction shows that

$$B - (\bigcup_{u \in U} \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} \cup \bigcup_{b \in B'} \bigcap_{t \in T} D_{t,s_{t,b}}) \neq \emptyset$$

if $\bar{\bar{B}}' \leqq m$. Hence

$$\overline{\overline{B - \bigcup_{u \in U} C_u}} > m \; .$$

LEMMA 3.6. *If $\{i, \mathscr{B}\}$ is an m-completion of the Boolean algebra $\mathscr{A}$ and there is a $B \in \mathscr{B}$ such that*

$$B \neq \overset{\mathscr{B}}{\bigcup_{t \in T}} \overset{\mathscr{B}}{\bigcap_{s \in S}} i(A_{t,s})$$

for all m-indexed sets $\{A_{t,s}\}_{t \in T, s \in S}$ in $\mathscr{A}$, then there is an m-ideal Δ in $\mathscr{B}$ such that $\{j, \mathscr{B}_\Delta\}$ is an m-extension of $i_\Delta(\mathscr{A})$ but not an m-completion, where $i_\Delta(A) = [i(A)]_\Delta$ for all $A \in \mathscr{A}$, $\mathscr{B}_\Delta = \mathscr{B}/\Delta$ and j is the identity map of $i_\Delta(\mathscr{A})$ into $\mathscr{B}_\Delta$.

Proof. Let

$$\Delta' = \{B' \in \mathscr{B} : B' \subseteqq B \text{ and } B' = \overset{\mathscr{B}}{\bigcap_{t \in T}} i(A_t) \; , \quad \text{for some } m\text{-indexed set } \{A_t\}_{t \in T} \text{ in } \mathscr{A}\}$$

and let $\Delta = \langle \Delta' \rangle_m$. Then if $\delta \in \Delta$, $\delta \subseteqq B$, so $B \notin \Delta$. If $A \in \mathscr{A}$ and $[i(A)]_\Delta \subseteqq [B]_\Delta$ then $i(A) - B \in \Delta$ so $i(A) - B \subseteqq B$ which implies $i(A) \subseteqq B$, hence $i(A) \in \Delta$ and $[i(A)]_\Delta = \bigwedge_{\mathscr{B}_\Delta}$, implying $i_\Delta(\mathscr{A})$ is not dense in $\mathscr{B}_\Delta$.

It only remains to show that $i_\Delta(\mathscr{A})$ is m-regular in $\mathscr{B}_\Delta$. If

$$\overset{i_\Delta(\mathscr{A})}{\bigcap_{t \in T}} [i(A_t)]_\Delta = \bigwedge_{\mathscr{B}_\Delta}$$

then $i(A) \subseteqq i(A_t)$ for all $t \in T$ implies $i(A) \in \Delta$, so $i(A) \subseteqq B$. If

$$\overset{\mathscr{B}}{\bigcap_{t \in T}} i(A_t) \not\subseteqq B \; ,$$

then there is an $A \neq \bigwedge_{\mathscr{A}}$ in $\mathscr{A}$ such that

$$i(A) \subseteqq \overset{\mathscr{B}}{\bigcap_{t \in T}} i(A_t) - B \; ,$$

contradicting the above statement. Thus

$$\overset{\mathscr{B}}{\bigcap_{t \in T}} i(A_t) \subseteqq B$$

so

$$\overset{\mathscr{B}}{\bigcap_{t \in T}} i(A_t) \in \Delta$$

and

$$\bigwedge_{\mathscr{B}_\Delta} = [\bigcap_{t\in T}^{\mathscr{B}} i(A_t)]_\Delta = \bigcap_{t\in T}^{\mathscr{B}_\Delta} [i(A_t)]_\Delta .$$

Thus if $\mathscr{A}$ is the Boolean algebra constructed in Lemua 3.5, $i_\Delta(\mathscr{A})$ is a Boolean algebra such that $\mathscr{K}(i_\Delta(\mathscr{A}))$ contains more than one element. Hence it is justified to assume that for each infinite cardinal m there is a Boolean algebra $\mathscr{A}$ such that $\mathscr{A}$ has an m-extension which is not an m-completion.

4. Let $\{\mathscr{A}_t\}_{t\in T}$ be a (fixed) indexed set of Boolean algebras. Let h_t be an isomorphism of $\mathscr{A}_t$ onto the field $\mathscr{F}_t$ of all open-closed subsets of the Stone space X_t of $\mathscr{A}_t$. Let X denote the Cartesian product of all the spaces X_t. Let π_t be the projection of X onto $\mathscr{F}_t$ and define

$$\varphi_t: \mathscr{F}_t \longrightarrow X$$

by:

$$\text{if } F\in\mathscr{F}_t \text{ then } \varphi_t(F) = \{x\in X: \pi_t(x)\in F\} .$$

Let $\mathscr{F}$ be the Boolean product of $\{\mathscr{A}_t\}_{t\in T}$. Define $h_t^* = \varphi_t h_t$ and let $\mathscr{S}$ be the set of all sets $\bigcap_{t\in T'} h_t^*(A_t)$; $A_t\in\mathscr{A}_t$, $T'\subseteqq T'$, $\bar{\bar{T'}}\leqq n$. Define $\hat{\mathscr{F}}$ to be the field of sets generated by $\mathscr{S}$. Let J be the set of all sets $S\subseteqq\hat{\mathscr{F}}$ such that

1. $\bar{\bar{S}}\leqq m$;
2. there is a $t\in T$ such that $S\subseteqq h_t^*(\mathscr{A}_t)$;
3. the join $\bigcup_{A\in S}^{\hat{\mathscr{F}}} A$ exists.

Let M' be the set of all sets $S\subseteqq\hat{T}$ such that

1. $\bar{\bar{S}}\leqq m$;
2. there is a $t\in T$ such that $S\subseteqq h_t^*(\mathscr{A}_t)$;
3. the meet $\bigcap_{A\in S}^{\hat{\mathscr{F}}} A$ exists.

Let M'' be the set of all sets $S\subseteqq\hat{T}$ such that

1. $\bar{\bar{S}}\leqq n$;
2. if $A\in S$ then $A\in h_t^*(\mathscr{A}_t)$ for some $t\in T$;
3. if $A, B\in S$, $A\neq B$, then $A\in h_t^*(\mathscr{A}_t)$ implies $B\notin h_t^*(\mathscr{A}_t)$. Let $M = M'\cup M''$.

The following lemma is due to La Grange [1] and will be given without proof.

LEMMA 4.1. *If* $\{\{i_t\}_{t\in T}, \mathscr{B}\}\in\mathscr{P}_n$ *then there is one and only one* (J, M, m)*-isomorphism* h *mapping* $\hat{\mathscr{F}}$ *into* $\mathscr{B}$ *such that*

$$hh_t^* = i_t \quad \textit{for all} \quad t\in T .$$

THEOREM 4.1. *If* $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \mathscr{P}_n$ *then there is a mapping* h *of* $\hat{\mathscr{F}}$ *into* $\mathscr{B}$ *such that* $\{h, \mathscr{B}\}$ *is a* (J, M, m)*-extension of* $\hat{\mathscr{F}}$. *If* $\{h, \mathscr{B}\}$ *is a* (J, M, m)*-extension of* $\hat{\mathscr{F}}$ *then the ordered pair* $\{\{hh_t^*\}_{t\in T}, \mathscr{B}\} \in \mathscr{P}_n$.

Proof. Let h be the (J, M, m)-isomorphism from $\hat{\mathscr{F}}$ into $\mathscr{B}$ such that $hh_t^* = i_t$ for all $t \in T$. Then $\{h, \mathscr{B}\}$ is a (J, M, m)-extension of $\hat{\mathscr{F}}$.

Conversely, if $\{h, \mathscr{B}\}$ is a (J, M, m)-extension of $\hat{\mathscr{F}}$, it follows immediately that $\{\{hh_t^*\}_{t\in T'}, \mathscr{B}\}$ is an (m, n)-product of $\{\mathscr{A}_t\}_{t\in T}$.

THEOREM 4.2. *If* $\{\{i_t\}_{t\in T}, \mathscr{B}\}$, $\{\{i'_t\}_{t\in T}, \mathscr{B}'\}$ *are two* (m, n)*-products of* $\{\mathscr{A}_t\}_{t\in T}$ *then*

$$\{\{i_t\}_{t\in T}, \mathscr{B}\} \leqq \{\{i'_t\}_{t\in T}, \mathscr{B}'\}$$

if, and only if,

$$\{i, \mathscr{B}\} \leqq \{i', \mathscr{B}'\}$$

where $\{i, \mathscr{B}\}$ *and* $\{i', \mathscr{B}'\}$ *are the* (J, M, m)*-extensions of* $\hat{\mathscr{F}}$ *induced by the* (J, M, m)*-isomorphisms* i' *and* i *of* $\hat{\mathscr{F}}$ *into* $\mathscr{B}'$ *and* $\mathscr{B}$, *respectively, given by Lemma* 4.1.

Proof. Now

$$\{\{i_t\}_{t\in T}, \mathscr{B}\} \leqq \{\{i'_t\}_{t\in T}, \mathscr{B}'\}$$

if, and only if, there is an m-homomorphism h such that

$$h\colon \mathscr{B}' \longrightarrow \mathscr{B}$$

and $hi'_t = i_t$ for all $t \in T$. Similarly,

$$\{i, \mathscr{B}\} \leqq \{i', \mathscr{B}'\}$$

if, and only if, there is an m-homomorphism

$$h\colon \mathscr{B}' \longrightarrow \mathscr{B}$$

such that $h'i' = i$. Thus it suffices to show that $hi' = i$, if, and only if, $hi'_t = i_t$. Let h_t^* be defined as above. Then $ih_t^* = i_t$ and $i'h_t^* = i'_t$, so if $hi' = i$,

$$hi'_t = hi'h_t^* = ih_t^* = i_t\ ,$$

and if $hi'_t = i_t$, then

$$hi' = hi'_t h_t^{*-1} = i_t h_t^{*-1} = i\ .$$

La Grange [1] has given an example of an $(m, 0)$-product for which $\mathscr{P}$ does not contain a smallest element and an example of an (m, n)-product for which $\mathscr{P}_n$ does not contain a smallest element. Theorem 4.2 extends this result by showing that the question whether $\mathscr{P}$ or $\mathscr{P}_n$ contains a smallest element reduces to asking whether the class of all (J, M, m)-extensions of $\mathscr{A}_0$ or $\hat{\mathscr{F}}$ contains a smallest element for J and M defined appropriately in each case, where $\mathscr{A}_0$ and $\hat{\mathscr{F}}$ are defined as above. Now the class of all (J, M, m)-extensions of $\mathscr{A}_0$ contains a smallest element only if the class of all m-extensions of $\mathscr{A}$ contains a smallest element and Theorem 3.2 shows that the class of all m-extensions of $\mathscr{A}_0$ need not contain a smallest element, which implies the same is true for $\mathscr{P}$. Since Theorem 3.2 may be extended to Boolean algebras of the form $\hat{\mathscr{F}}$, it follows that $\mathscr{P}_n$ need not contain a smallest element.

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Received November 27, 1972 and in revised form March 28, 1973.

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