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AN APPLICATION OF STEINBERG'S CONSTRUCTION OF TWISTED GROUPS

EIVIND STENSHOLT

The construction of the twisted groups uses automorphisms of certain Chevalley groups derived from symmetries of the Dynkin diagrams. This paper applies the same method to the symmetries of the extended Dynkin diagrams. The groups so constructed turn out to be other Chevalley groups exhibited in a way which let a particular subgroup structure appear.

Introduction. Let Φ be an n-dimensional indecomposable crystallographic root system of one rootlength. Steinberg's construction of the twisted Chevalley groups makes use of a nontrivial isometry of Φ which permutes the roots of a given fundamental system. Such isometries exist except for $\Phi = A_1; E_7; E_8$ and are defined by symmetries of the Dynkin diagrams.

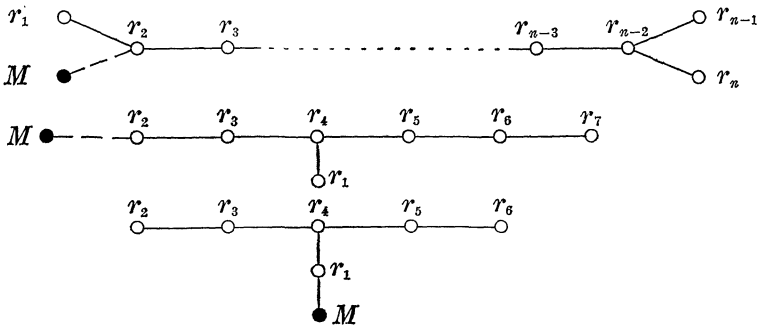
This paper applies the same method to another isometry of Φ which permutes the roots of a set $S = \{r_1, \cdots, r_n, M\}$ of $n+1$ roots where $\{r_1, \cdots, r_n\}$ is a fundamental system and S has the following properties in common with a fundamental system:

$$(M, r_i) \leqq 0 , \qquad 1 \leqq i \leqq n . \tag{0.1}$$

(0.2) The graph with the $n+1$ roots of S as nodes, r and s being connected if and only if $(r, s) < 0$, is a tree.

The condition (0.1) shows that $-M$ is in the fundamental chamber defined by $\{r_1, \cdots, r_n\}$, and because all roots in Φ are conjugates, M is uniquely determined as the lowest root. Therefore the graph is the extended Dynkin diagram as defined by Bourbaki [2, p. 198]. The cases that satisfy (0.2) are $\Phi = D_n; E_7; E_6$. The extended diagrams are:

(0.3)



The extended diagrams suggest the definition of an isometry η

of Φ. The following five special cases will be treated: $\Phi = D_{m+2}$; D_{2m}; D_{2m+1}; E_7; E_6 ($m \geqq 2$) and $\eta = \lambda$; μ; ν; φ; ψ respectively, where

$$\lambda: \quad r_i \longrightarrow r_i \text{ for } 2 \leqq i \leqq n-2,\ r_{n-1} \longleftrightarrow r_n,\ r_1 \longrightarrow M \qquad (n = m+2)$$

$$\mu, \nu: \quad r_i \longrightarrow r_{n-i} \text{ for } 1 \leqq i \leqq n-1,\ r_n \longrightarrow M \qquad (n = 2m, 2m+1) \tag{0.4}$$

$$\varphi: \quad r_1 \longrightarrow r_1,\ r_4 \longrightarrow r_4,\ r_3 \longleftrightarrow r_5,\ r_2 \longleftrightarrow r_6,\ r_7 \longrightarrow M$$

$$\psi: \quad r_4 \longrightarrow r_4,\ r_1 \longrightarrow r_3 \longrightarrow r_5 \longrightarrow r_1,\ r_2 \longrightarrow r_6 \longrightarrow M\,.$$

Because the root M is uniquely determined by $\{r_1, \cdots, r_n\}$, $\lambda(M) = r_1$, $\mu(M) = r_n$, $\nu(M) = r_n$, $\varphi(M) = r_7$, $\psi(M) = r_2$. Hence η is an isometry of order

$$|\eta| = 2; 2; 2; 2; 3 \text{ for } \eta = \lambda; \mu; \nu; \varphi; \psi \text{ respectively.} \tag{0.5}$$

About certain automorphisms of Chevalley groups we state a few facts that are easily checked:

Let G_1 and G_2 be Chevalley groups defined by the same indecomposable root system and the same finite field $GF(q)$, G_1 universal, and let γ be a homomorphism of G_1 onto G_2 with kernel in $Z(G_1)$. Further let $\alpha_1 \in \operatorname{Aut} G_1$ and $\alpha_2 \in \operatorname{Aut} G_2$ be either (a) field automorphisms or (b) products of field and graph automorphisms as used in Steinberg's twisting construction, and such that $\alpha_2\gamma = \gamma\alpha_1$.

Let $\tilde{G}_i$ be the fixpointgroup of α_i in G_i, $i = 1, 2$. Then $\tilde{G}_2$ is generated by $\gamma(\tilde{G}_1)$ and α_2-invariant elements of the diagonal subgroup in G_2, $\gamma(\tilde{G}_1) \lhd \tilde{G}_2$ and the index $[\tilde{G}_2 : \gamma(\tilde{G}_1)]$ is prime to q. The inclusion may be proper.

If G_2 is adjoint, $\gamma(\tilde{G}_1)$ is in case (b) a twisted group which is simple with a few exceptions. If G_2 is not adjoint, $\gamma(\tilde{G}_1)$ is in general a central extension of the mentioned twisted group. We will still call it a twisted group and denote it by the usual symbol, ${}^2A_n(q^2)$ etc.

The content of the paper is as follows.

Section 1 describes the η-orbits of the roots in Φ. Let Pr be the orthogonal projection of the vectorspace spanned by Φ onto the subspace of η-invariant points. $Pr^{-1}(0)$ contains a root system $\Phi^{\ker}$. It is shown that $Pr(\Phi) - \{0\}$ is a root system, and it will be denoted by Φ^{Pr}. Subsystems Φ_i^{sub}, $i = 1, \cdots, N$, that are conjugates under $W(\Phi^{\ker})$, the reflection group defined by $\Phi^{\ker}$, are defined such that

$$\eta(\Phi_i^{\text{sub}}) = \Phi_i^{\text{sub}}, \quad \Phi_i^{\text{sub}} \cap \Phi^{\ker} = \varnothing, \quad Pr(\Phi_i^{\text{sub}}) = \Phi^{Pr}\,. \tag{0.6}$$

Let $\mathscr{L}(\Phi)$ be the simple complex Lie algebra defined by Φ, and let $\mathscr{L}(\Phi_i^{\text{sub}})$, $\mathscr{L}(\Phi^{\ker})$ be the subalgebras supported by Φ_i^{sub}, $1 \leqq i \leqq N$,

and $\Phi^{\ker}$. Similarly, let $\Phi(q)$ be a Chevalley group defined by Φ and the field $GF(q)$, and let $\Phi_i^{\mathrm{sub}}(q)$, $\Phi^{\ker}(q)$ be subgroups. In § 2 automorphisms of $\mathscr{L}(\Phi)$ and $\Phi(q^{|\eta|})$ corresponding to η are defined. They are shown to be of order as in (0.5) and will also be denoted by $\eta = \lambda; \mu; \nu; \varphi; \psi$. The subalgebras and subgroups introduced above are η-invariant, and the group automorphism η is so defined that the restrictions $\eta \mid \Phi_i^{\mathrm{sub}}(q^{|\eta|})$ are the usual twisting automorphisms, defining subgroups of the types

$$\begin{gathered} {}^2D_{m+1}(q^2);\ {}^2A_{2m-1}(q^2);\ {}^2A_{2m}(q^2);\ {}^2E_6(q^2);\ {}^3D_4(q^3) \\ \text{for} \quad \eta = \lambda; \mu; \nu; \varphi; \psi \text{ respectively.} \end{gathered} \tag{0.7}$$

Then §§ 3 and 4 establish the following main result:

(0.8) THEOREM. *The $\Phi(q^{|\eta|})$-automorphism η is conjugate in* Aut $\Phi(q^{|\eta|})$ *to a field automorphism if $\eta \neq \nu$ and to the product of a graph- and a field automorphism if $\eta = \nu$.*

Hence the fixpointgroup of η is or contains (see the discussion above) a group of one of the types

$$\begin{gathered} D_{m+2}(q);\ D_{2m}(q);\ {}^2D_{2m+1}(q^2);\ E_7(q);\ E_6(q) \\ \text{for} \quad \eta = \lambda; \mu; \nu; \varphi; \psi \text{ respectively.} \end{gathered} \tag{0.9}$$

As a consequence of (0.9) and (0.7) one obtains the embeddings

$$\begin{gathered} {}^2D_{m+1}(q^2) \subset D_{m+2}(q) \\ {}^2A_{2m-1}(q^2) \subset D_{2m}(q)\ , \qquad {}^2A_{2m}(q^2) \subset {}^2D_{2m+1}(q^2) \\ {}^2E_6(q^2) \subset E_7(q)\ , \qquad {}^3D_4(q^3) \subset E_6(q)\ . \end{gathered} \tag{0.10}$$

In § 5 these embeddings are described somewhat closer. The paper aims at a unified presentation of the arguments, with special treatment of the individual cases $\eta = \lambda; \mu; \nu; \varphi; \psi$ only when technical reasons call for it.

1. The isometry of the root system. We first derive some information about the isometry η of Φ defined in (0.4). This information is given in the following table, using the notation of the introduction.

(1.1)

η	Φ	dim $Pr^{-1}(0)$	$\Phi^{\ker}$	Φ_i^{sub}	Φ^{Pr}
λ	D_{m+2}	2	A_1^2	D_{m+1}	B_m
μ	D_{2m}	m	A_1^m	A_{2m-1}	C_m
ν	D_{2m+1}	$m+1$	A_1^m	A_{2m}	BC_m
φ	E_7	3	A_1^3	E_6	F_4
ψ	E_6	4	A_2^2	D_4	G_2

This may be checked in each case by means of the standard models of the root systems $\Phi = D_n; E_7; E_6$, but for convenience we offer a reasonably unified argument.

Since M is a negative root, $M - \eta(M)$ for $\eta \neq \psi$ and $M - 1/3.(M + \eta(M) + \eta^2(M))$ for $\eta = \psi$ is not in the subspace spanned by the fundamental roots different from $\eta(M)$, $\eta^2(M)$. Hence the multiplicity of an eigenvalue different from 1 of η equals the number of η-orbits of length > 1 containing a fundamental root. Therefore dim $Pr^{-1}(0) = 2; m; m+1; 3; 4$ and dim $V = m; m; m; 4; 2$ where V is the space of η-invariant points.

(1.2) Let Φ_1^{sub} be the η-invariant subsystem of Φ of type $D_{m-1}; A_{2m-1}; A_{2m}; E_6; D_4$ generated by fundamental roots. See (0.3). Then the restriction $\eta \mid \Phi_1^{\text{sub}}$ is the usual twisting isometry, and so $Pr(\Phi_1^{\text{sub}})$ is a root system of type $B_m; C_m; BC_m; F_4; G_2$ which clearly spans V.

Let Φ_2^{sub} be another η-invariant subsystem of Φ of the same type as Φ_1^{sub} spanned by roots in the extended diagrams (0.3) if $\eta = \lambda; \mu; \nu$, by $r_1, M + r_2, r_3, r_4, r_5, r_6 + r_7$ if $\eta = \varphi$ and by $r_1 + M, r_2 + r_3, r_4, r_5 + r_6$ if $\eta = \psi$.

Now $Pr(\Phi_1^{\text{sub}})$ and $Pr(\Phi_2^{\text{sub}})$ are root systems of the same type, each of them spanning V. We will show that they coincide:

To any root $r \in Pr(\Phi_1^{\text{sub}}) \cup Pr(\Phi_2^{\text{sub}})$ the usual twisting procedure assigns an element $w_r \in W(\Phi)$, the reflection group defined by Φ, such that $w_r(V) = V$ and $w_r \mid V$ is the reflection along r. Consider the group generated by the elements w_r,

$$W' = \langle w_r; r \in Pr(\Phi_1^{\text{sub}}) \cup Pr(\Phi_2^{\text{sub}}) \rangle .$$

Since $W' \subset W(\Phi)$, W' and $W' \mid V$ are finite. Since $W' \mid V$ is a finite reflection group, $Pr(\Phi_1^{\text{sub}}) \cup Pr(\Phi_2^{\text{sub}})$ generate a root system Φ' in V.

If $Pr(\Phi_1^{\text{sub}}) \neq Pr(\Phi_2^{\text{sub}})$, there is a proper inclusion $Pr(\Phi_1^{\text{sub}}) \subset \Phi'$. But the only inclusions between indecomposable root systems we have to consider are $B_m \subset BC_m$, $C_m \subset BC_m$, $B_4 \subset F_4$, $C_4 \subset F_4$. The first two possibilities are excluded by the fact that $Pr(\Phi_1^{\text{sub}})$ and $Pr(\Phi_2^{\text{sub}})$ consist of the same types of roots. The two last possibilities ($\eta = \lambda; \mu$, $m = 4$) may then be excluded by regarding $\eta = \lambda; \mu$ as a restriction to an η-invariant $D_6; D_8$-subsystem of $\Phi = D_{m+2}; D_{2m}$ with $m > 4$. Hence $Pr(\Phi_1^{\text{sub}}) = Pr(\Phi_2^{\text{sub}})$.

All fundamental roots of Φ are contained in Φ_1^{sub}, Φ_2^{sub} or $\Phi_1^{\text{sub}} + \Phi_2^{\text{sub}}$. Therefore $\Phi' = Pr(\Phi) - \{0\}$, and so $\Phi^{Pr} = Pr(\Phi_i^{\text{sub}})$, $i = 1, 2$. This establishes (0.6) for $i = 1, 2$, and that Φ^{Pr} is of type $B_m; C_m; BC_m; F_4; G_2$ in the cases $\eta = \lambda; \mu; \nu; \varphi; \psi$ respectively.

The classification of the systems $\Phi^{\text{ker}} = \Phi \cap Pr^{-1}(0)$ may now be done by the following counting argument:

Φ_2^{sub} contains η-orbits of lengths 2 and 3 of the same types as

Φ_1^{sub} contains, that are not contained in Φ_1^{sub}. This is obvious when there is only one such type to consider, and for $\eta = \nu\,(M, r_{2m+1})$ and $(M + r_2 + \cdots + r_m, r_{m+1} + \cdots + r_{2m-1} + r_{2m+1})$ are examples, with reference to (0.3). Thus one, hence (by application of elements of W') all short roots and halves of long roots in Φ^{Pr} are projections of more than one η-orbit of roots in Φ.

Φ contains $2m^2 + 6m + 4$; $8m^2 - 4m$; $8m^2 + 4m$; 126; 72 roots. Φ^{Pr} contains $2m$; $2m^2-2m$; $2m^2-2m$; 24; 6 short and $2m^2-2m$; $2m$; $2m$; 24; 6 long or mixed roots. So far then, $Pr^{-1}(\Phi^{Pr})$ accounts for $2m\cdot 4 + (2m^2 - 2m)\cdot 1$; $(2m^2 - 2m)\cdot 4 + 2m\cdot 1$; $(2m^2 - 2m)\cdot 4 + 2m\cdot 5$; $24\cdot 4 + 24\cdot 1$; $6\cdot 6 + 6\cdot 1$ roots of Φ, and 4; $2m$; $2m$; 6; 30 remain to be placed.

But for $\eta = \psi$, all 30 roots cannot be in $Pr^{-1}(0)$, since no root system of dimension $\leqq 4$ has 30 roots. So in this case 3 η-orbits of length 3 have the same projection. Now the remaining 4; $2m$; $2m$; 6; 12 roots are so few that none of them can be in $Pr^{-1}(\Phi^{Pr})$. Hence they form a root system $\Phi^{\text{ker}} \subset Pr^{-1}(0)$.

$W(\Phi^{\text{ker}})$ permutes the roots inside each $Pr^{-1}(r)$, $r \in \Phi^{Pr}$. Since $\Phi \cap Pr^{-1}(\Phi^{Pr})$ generates Φ (it contains $\Phi_1^{\text{sub}} \cup \Phi_2^{\text{sub}}$), $W(\Phi^{\text{ker}})$ has a faithful permutation representation on $\Phi \cap Pr^{-1}(\Phi^{Pr})$. For

$$\eta \neq \psi\,, \qquad \eta \mid Pr^{-1}(0) = -1\,,$$

so η centralizes $W(\Phi^{\text{ker}})$ and $W(\Phi^{\text{ker}})$ permutes the η-orbits inside each $Pr^{-1}(r) \cap \Phi$, $r \in \Phi^{Pr}$. Hence the elements of $W(\Phi^{\text{ker}})$ have orders 1, 2 or (conceivably) 4 if $\eta \neq \psi$. For $\eta = \psi$ we remark that the only one-rootlength systems with 12 roots and dimension $\leqq 4$ are $A_2 \times A_2$ and A_3. But if $\Phi^{\text{ker}} = A_3$, the restriction of ψ to the subspace spanned by Φ^{ker} must have an eigenvalue equal to 1. Hence $\Phi^{\text{ker}} = A_2 \times A_2$. Consequently the only possibilities are $\Phi^{\text{ker}} = A_1^2$; A_1^m; A_1^m; A_1^3; A_2^2 in the five cases $\eta = \lambda$; μ, ν; φ; ψ respectively.

This establishes (1.1).

(1.3) *In the sequel, Ω will denote an arbitrary component of Φ^{ker}.* By (1.1) Ω is of type A_1 if $\eta \neq \psi$ and of type A_2 if $\eta = \psi$.

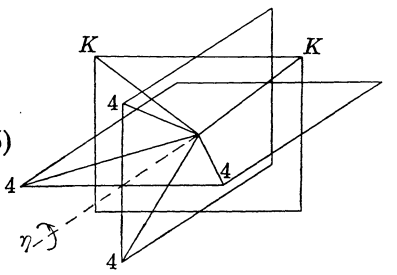
The discussion above shows that any root in Φ belongs to one of five classes: K, L, 4, 5, 9 *defined as follows*:

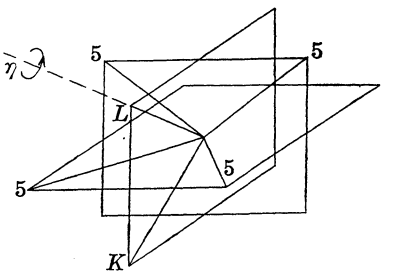
(1.4)

- K: a root in Φ^{ker}.
- L: an η-invariant root, i.e., a long root in Φ^{Pr}.
- 4: a root in an η-orbit of two orthogonal roots, i.e., belonging to a set $Pr^{-1}(r) \cap \Phi$ of 4 roots, $r \in \Phi^{Pr}$ a short root, $\eta \neq \psi$.
- 5: a root in an η-orbit of two roots making a 120°-angle, i.e., in a set $Pr^{-1}(\{s, 2s\}) \cap \Phi$ of 5 roots, $s \in \Phi^{Pr}$ a half-root, $\eta = \nu$.
- 9: a root in an η-orbit of three orthogonal roots, i.e., in a set $Pr^{-1}(r) \cap \Phi$ of 9 roots, $r \in \Phi^{Pr}$ a short root, $\eta = \psi$.

Let $\eta \neq \psi$ and let $r, s \in \Phi^{Pr}$, r a short root, s a half-root. Then $Pr^{-1}(\{r, -r\}) \cap \Phi$ and $Pr^{-1}(\{2s, s, -s, -2s\}) \cap \Phi$ generate root systems X and Y of dimensions $\leqq 3$ with $\geqq 8$ and $\geqq 10$ roots respectively. Hence Y is an A_3-system. X must be an $A_2 \times A_1$- or an A_3-system, but the first is impossible, since the 8 roots would have to form 4 η-orbits of orthogonal roots. X contains four K-roots and Y contains two. It is useful to show $\eta \mid X$ and $\eta \mid Y$ in two figures. Only a positive subsystem is shown.

(1.5)





We now introduce a new ordering of the roots in Φ, with the property that if $Pr(r) = Pr(s) \neq 0$, $r, s \in \Phi$ and r is positive in the new ordering, then also s is positive. This is done by selecting positive systems Φ^{Pr+} in Φ^{Pr} and $\Phi^{\ker+}$ in $\Phi^{\ker}$ (i.e., *in each component Ω of $\Phi^{\ker}$*), *and defining*

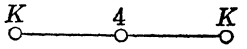
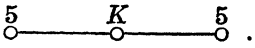
$$\Phi^{+} = (Pr^{-1}(\Phi^{Pr+}) \cap \Phi) \cup \Phi^{\ker+} . \tag{1.6}$$

Φ^{+} spans a convex cone containing one half of Φ, hence Φ^{+} is a positive system in Φ, different from the one defining the original fundamental roots in (0.3).

The fundamental roots in Φ^{+} span the extremelines of the cone. Therefore the numbers of fundamental roots of types K, L, 4, 5, 9 may be read off from (1.1). They are listed in Table (1.7):

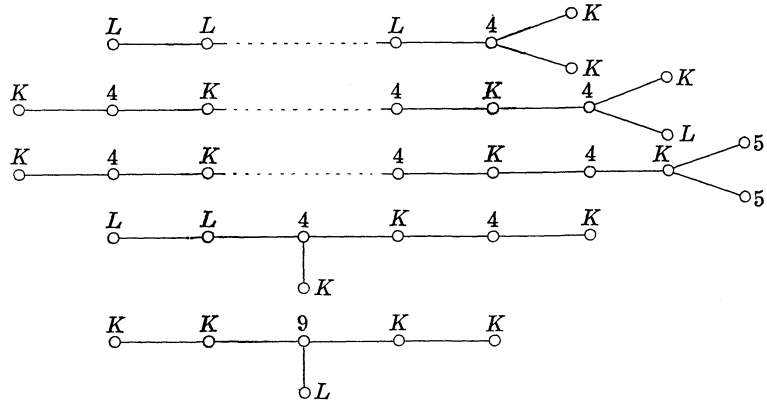
(1.7)

η	K	L	4	5	9
λ	2	$m-1$	1	0	0
μ	m	1	$m-1$	0	0
ν	m	0	$m-1$	2	0
φ	3	2	4	0	0
ψ	4	1	0	0	1

From (1.5) we see that in the Dynkin diagram with the fundamental roots of Φ^{+} as nodes, a 4-root or a 5-root always occur in a subdiagram  or  . The K-roots and any L-root form a subdiagram of type $\Phi^{\ker} \times A_1$.

These remarks together with (1.7) *suffice to determine the new Dynkin diagrams by simple conbinatorial arguments. They are*:

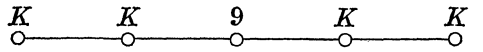
(1.8)



Now we make the following observations:

(1.9) If $\eta \neq \psi$, a reflection along a K-root in (1.5) interchanges the two η-orbits with the same projection.

(1.10) If $\eta = \psi$, (1.8) shows that the 18 roots in $Pr^{-1}(\{r, -r\}) \cap \Phi$, $r \in \Phi^{Pr}$ a short root, generate an ψ-invariant A_5-system



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ψ rotates the two A_2-components of Φ^{ker} 120 degrees. The centralizer of ψ in $W(\Phi^{\text{ker}})$ is isomorphic to $Z_3 \times Z_3$, and it is easily checked that a subgroup permutes the three ψ-orbits in $Pr^{-1}(r) \cap \Phi$ cyclically.

Any subsystem $\Phi_i^{\text{sub}} \subset \Phi$ which satisfies (0.6) is specified by the selection of one η-orbit inside each $Pr^{-1}(r) \cap \Phi$ where $r \in \Phi^{Pr+}$ is a fundamental short root or half-root. The distribution of K-roots in (1.8) together with (1.9) and (1.10) now show that there are exactly N such subsystems, with

(1.11) $N = 2; 2^{m-1}; 2^m; 4; 3$ *for* $\eta = \lambda; \mu; \nu; \varphi; \psi$ *respectively.*

In the sequel η-invariant subsystems of the types shown in (1.5) and described in (1.10) play an important role. We therefore introduce the following notation:

(1.12) *The η-invariant A_3-system with four K-roots and eight 4-roots, $\eta \neq \psi$, will be called a K4K-system. The η-invariant A_3-system with two K-roots, two L-roots, and eight 5-roots, $\eta = \nu$, will be called a 5K5-system. The η-invariant A_5-system of* (1.10) *with twelve K-roots and eighteen 9-roots, $\eta = \psi$, will be called a KK9KK-system.*

Clearly there are one-to-one correspondences between $K4K$-systems and pairs $\{r, -r\}$, r short, $\eta \neq \psi$, between $5K5$-systems and pairs $\{s, -s\}$, s a half-root, $\eta = \nu$, between $KK9KK$-systems and pairs

$\{r, -r\}$, r short, $\eta = \psi$. See (1.5) and (1.10).

2. **The Lie algebras and the Chevalley groups.** In the complex simple Lie algebra $\mathscr{L}(\Phi)$ defined by the root system Φ, let

$$\{X_r; r \in \Phi\} \cup \{H_{r_i}; r_i \text{ fundamental in } \Phi\} \tag{2.1}$$

be a Chevalley basis, $r_1, r_2, \cdots$ being the original fundamental roots of (0.3).

From the theory of Lie algebras we recall a few facts:

(2.2) *The selection of a Cartan algebra in $\mathscr{L}(\Phi)$ determines the rootspaces, and inside these the selection of*

$$\{X_{r_i}; r_i \textit{ fundamental in } \Phi\}$$

is arbitrary and determines the Chevalley basis (2.1) *except for signs. For every $r \in \Phi$, X_r determines X_{-r} through the equation*

$$[[X_r, X_{-r}], X_r] = 2X_r .$$

We introduce some subalgebras of $\mathscr{L}(\Phi)$:

$$\mathscr{L}(\Phi_i^{\text{sub}}) = \langle X_r; r \in \Phi_i^{\text{sub}} \rangle, \quad 1 \leqq i \leqq N$$

$$\mathscr{L}(\Phi^{\text{ker}}) = \langle X_r; r \in \Phi^{\text{ker}} \rangle$$

(2.3) $\mathscr{L}(\Omega)$, $\mathscr{L}(K4K)$, $\mathscr{L}(5K5)$, $\mathscr{L}(KK9KK)$ denote four types of subalgebras generated by the X_r-elements with r in an Ω-, $K4K$-, $5K5$- or $KK9KK$-system. See (1.3) and (1.12).

(2.4) LEMMA. *For any i, $1 \leqq i \leqq N$, $\langle \mathscr{L}(\Phi_i^{\text{sub}}), \mathscr{L}(\Phi^{\text{ker}}) \rangle = \mathscr{L}(\Phi)$.*

Proof. Observe that for any i, $1 \leqq i \leqq N$, and any given $K4K$-, $5K5$- or $KK9KK$-system and with Φ^+ as in (1.6),

$\Phi^+ \cap \Phi_i^{\text{sub}} \cap K4K$ consists of one η-orbit of two orthogonal roots,

$\Phi^+ \cap \Phi_i^{\text{sub}} \cap 5K5$ consists of three roots, r, $\eta(r)$, $r + \eta(r)$,

$\Phi^+ \cap \Phi_i^{\text{sub}} \cap KK9KK$ consists of one η-orbit of three orthogonal roots.

This follows from (0.6) and the Definition (1.12). Taking Lie-products inside each $\mathscr{L}(K4K)$, $\mathscr{L}(5K5)$, and $\mathscr{L}(KK9KK)$ one obtains $\pm X_r$ for all $r \notin \Phi^{\text{ker}} \cup \Phi_i^{\text{sub}}$.

(2.5) DEFINITION. Let $\eta(=\lambda; \mu; \nu; \varphi; \psi)$ denote an $\mathscr{L}(\Phi)$-automorphism such that

$$\eta\colon X_{r_i} \longrightarrow X_{\eta(r_i)}, \qquad X_{-r_i} \longrightarrow X_{-\eta(r_i)}$$

where $r_1, r_2, \cdots$ are the original fundamental roots from (0.3).

This defines an $\mathscr{L}(\Phi)$-automorphism by the isomorphism theorem for Lie algebras, but because of the facts (2.2) it is necessary to make the following remark:

(2.6) *The Definition* (2.5) *is ambiguous, given the elements* $X_{r_1}, X_{r_2}, \cdots$, *pending the choice between two possibilities for* X_M *in the Chevalley basis,* M *being as in* (0.3).

(2.7) LEMMA. *All the subalgebras of* (2.3) *are* η*-invariant.*

Proof. This follows from (2.5) and the fact that all the root subsystems $\Phi_i^{\text{sub}}(1 \leqq i \leqq N)$, Φ^{ker}, $\Omega \subset \Phi^{\text{ker}}$, $K4K$, $5K5$, $KK9KK$ of Φ are η-invariant (for the isometry η).

(2.8) LEMMA. η *is expressed by an integral matrix with respect to the Chevalley basis* (2.1).

Proof. The η-images of the Chevallay basis elements different from $H_{\eta^{-1}(M)}$ of (2.1) are clearly in the Z-span of that basis. But also $\eta(H_{\eta^{-1}(M)}) = \eta[X_{\eta^{-1}(M)}, X_{-\eta^{-1}(M)}] = [X_M, X_{-M}]$ belongs to this Z-span, by a fundamental property of a Chevalley basis.

Clearly, for every $r \in \Phi$ either $\eta(X_r)$ or $-\eta(X_r)$ belongs to the Chevalley basis. Hence the $\mathscr{L}(\Phi)$-automorphism η has order $|\eta| = 2; 2; 2; 2; 3$ if and only if $\eta(X_M) = X_{\eta(M)}$. The lemmas (2.10)–(2.13) show that this is actually so, and also settles the question of how the choice of X_M affects the $\mathscr{L}(\Phi)$-automorphism η. See (2.6).

(2.9) Two $\mathscr{L}(\Phi)$-automorphisms α and β will be called equivalent if there exists an $\mathscr{L}(\Phi)$-automorphism γ such that

$$\gamma(X_r) = \pm X_r \quad \text{for all } r \in \Phi \text{ and } \alpha = \gamma \cdot \beta \cdot \gamma^{-1} .$$

(2.10) LEMMA. *In case* $\eta = \psi$, *the two choices of* X_M *give two equivalent* $\mathscr{L}(E_6)$*-automorphisms.*

Proof. Call the two $\mathscr{L}(E_6)$-automorphisms ψ_1 and ψ_2, so that

$$\psi_1(X_{r_6}) = -\psi_2(X_{r_6}) . \qquad \text{See (0.4)} .$$

Define γ by

$$\gamma(X_{r_4}) = -X_{r_4} , \quad \gamma(X_{r_i}) = X_{r_i} \quad \text{for } i \neq 4 .$$

The highest root in E_6 with respect to the ordering of (0.3) is

$$-M = \sum k_i \cdot r_i = 2r_1 + r_2 + 2r_3 + 3r_4 + 2r_5 + r_6 ,$$

so, in particular, k_4 is odd. Expressing X_{-M} as a repeated Lie-product of the X_{r_i}-elements we see that

$$\gamma(X_{-M}) = -X_{-M} , \quad \text{hence by (2.2)} \quad \gamma(X_M) = -X_M .$$

It is now easily checked that $\gamma \cdot \psi_1 \cdot \gamma^{-1} = \psi_2$.

Because of (2.10) it is not necessary to distinguish between the two cases, and ψ will mean any of the two $\mathscr{L}(E_6)$-automorphisms of (2.5).

(2.11) LEMMA. *$\psi^3 = 1$ as an $\mathscr{L}(E_6)$-automorphism.*

Proof. As remarked above it suffices to show that $\psi(X_M) = X_{r_2}$. Suppose contrariwise $\psi(X_M) = -X_{r_2}$. Then $\psi^3(X_{r_i}) = X_{r_i}$ for $i = 1, 3, 5, 4$ and $\psi^3(X_{r_i}) = -X_{r_i}$ for $i = 2, 6$ and $\psi^3(X_M) = -X_M$. Express X_{-M} as in the preceding proof. Since $k_2 + k_6$ is even,

$$\psi^3(X_{-M}) = 1(X_{-M}) = X_{-M} , \quad \text{hence by (2.2)} \quad \psi^3(X_M) = X_M ,$$

which is a contradiction. Consequently $\psi^3 = 1$.

(2.12) LEMMA. *For $\eta \neq \psi$, $\eta^2 = 1$ as an $\mathscr{L}(\Phi)$-automorphism, for both choices of X_M.*

Proof. Let Φ_1^{sub} be as in (1.2). Clearly $\eta^2 \mid \mathscr{L}(\Phi_1^{\text{sub}}) = 1$, and so by (2.4) it suffices to prove that $\eta^2 \mid \mathscr{L}(\Phi^{\text{ker}}) = 1$.

For a given K-root $r \in \Phi$, there exist $a, b = \pm 1$ such that $\eta(X_r) = aX_{-r}$ and $\eta(X_{-r}) = bX_r$. The equation in (2.2) becomes $[[aX_{-r}, bX_r], aX_{-r}] = 2aX_{-r}$, which shows that $ab = 1$, $a = b$ and consequently $\eta^2(X_r) = X_r$.

(2.13) LEMMA. *For $\eta \neq \psi$, the two choices of X_M give two inequivalent $\mathscr{L}(\Phi)$-automorphisms η^+ and η^- (see (2.9)) where*

$$\eta^+(X_r) = X_{-r} \text{ and } \eta^-(X_r) = -X_{-r} \text{ for all } K\text{-roots } r \in \Phi .$$

Proof. Let Φ_1^{sub} be as in (1.2). Since $\eta \mid \mathscr{L}(\Phi_1^{\text{sub}})$ is independent of the choice of X_M, the two choices must give different restrictions $\eta \mid \mathscr{L}(\Phi^{\text{ker}})$ because of (2.4).

Let r, s, t be fundamental roots of a $K4K$-system, r and t being K-roots and s a 4-root. See (1.4), (1.5), (1.12). In the standard model for $\mathscr{L}(K4K)$ of 4×4-matrices, let E_{12}, E_{23}, E_{34} represent X_r, X_s, X_t respectively. Identifying $\mathscr{L}(K4K)$ with this matrix algebra we have

$$\eta\colon\ E_{12}, E_{34}, E_{13}, E_{31}, E_{14}, E_{41} \longrightarrow aE_{21}, bE_{43}, cE_{24}, cE_{42}, dE_{23}, dE_{32}$$

for certain $a, b, c, d = \pm 1$. By (2.12), $\eta^2 = 1$ and so

$$cE_{24} = \eta(E_{13}) = \eta([E_{12}, E_{23}]) = [aE_{21}, dE_{14}] = adE_{24}$$
$$dE_{41} = \eta(E_{32}) = \eta([E_{34}, E_{42}]) = [bE_{43}, cE_{31}] = bcE_{41}\ .$$

Hence $cd = abcd$, $ab = 1$, $a = b$ and so $\eta(X_r) = aX_{-r}$, $\eta(X_t) = aX_{-t}$. Applying this result to all $K4K$-subsystems defined by the Dynkin diagrams of (1.8), we find that $\eta(X_r) = aX_{-r}$ for all K-roots $r \in \Phi$ with $a = \pm 1$ independent of r.

The next result shows that the restrictions $\eta \mid \mathscr{L}(\Phi_i^{\text{sub}})$, $1 \leqq i \leqq N$, are related to each other in a sense similar to (2.9):

(2.14) LEMMA. *For every pair (i, j) such that $1 \leqq i < j \leqq N$, there is an isometry β_{ij} of Φ which maps Φ_i^{sub} onto Φ_j^{sub} and a corresponding isomorphism (also denoted β_{ij}) of $\mathscr{L}(\Phi_i^{\text{sub}})$ onto $\mathscr{L}(\Phi_j^{\text{sub}})$*, i.e., *$\beta_{ij}(X_r) = \pm X_{\beta_{ij}(r)}$ for every $r \in \Phi_j^{\text{sub}}$, with the additional property:*

$$\beta_{ij} \cdot \eta \mid \mathscr{L}(\Phi_i^{\text{sub}}) \cdot \beta_{ij}^{-1} = \eta \mid \mathscr{L}(\Phi_j^{\text{sub}})\ .$$

Proof. The subsystems Φ_i^{sub}, $1 \leqq i \leqq N$ are conjugates under $W(\Phi^{\text{ker}})$ (see the remarks in connection with (1.11)). Let $\gamma \in W(\Phi^{\text{ker}})$ be such that $\gamma(\Phi_i^{\text{sub}}) = \Phi_j^{\text{sub}}$. Let γ also denote a corresponding $\mathscr{L}(\Phi)$-automorphism, such that $\gamma(X_r) = \pm X_{\gamma(r)}$ for all $r \in \Phi$. Then

$$\gamma \cdot \eta \mid \mathscr{L}(\Phi_i^{\text{sub}}) \cdot \gamma^{-1} = \eta \mid \mathscr{L}(\Phi_j^{\text{sub}}) \cdot \beta \tag{2.15}$$

where β is an $\mathscr{L}(\Phi_j^{\text{sub}})$-automorphism such that $\beta(X_r) = \pm X_r$ for $r \in \Phi_j^{\text{sub}}$, since $\gamma\eta\gamma^{-1} = \eta$ as isometries.

Now, if $\eta(r) = r$, i.e., if r is an L-root, then $r \in \Phi_i^{\text{sub}} \cap \Phi_j^{\text{sub}}$ and $\gamma(r) = r$. Hence by (2.15) $\eta(X_r) = \gamma \cdot \eta \cdot \gamma^{-1}(X_r) = \eta \cdot \beta(X_r)$ which implies

$$\beta(X_r) = X_r \quad \text{if } r \text{ is an } L\text{-root.} \tag{2.16}$$

All mappings $\alpha\colon X_r \to \pm X_r$ where r belongs to the fundamental system in $\Phi^+ \cap \Phi_j^{\text{sub}}$ extend to $\mathscr{L}(\Phi_j^{\text{sub}})$-automorphisms. Hence for a suitable $\mathscr{L}(\Phi_j^{\text{sub}})$-automorphism α, because of (2.16),

$$\alpha \cdot (\eta \mid \mathscr{L}(\Phi_j^{\text{sub}}) \cdot \beta) \cdot \alpha^{-1} = \eta \mid \mathscr{L}(\Phi_j^{\text{sub}})\ .$$

Now $\beta_{ij} = \alpha\gamma$ meets the requirement of (2.14).

(2.17) Let $GF(q)$ be the finite field with q elements, and let θ be the automorphism $t \to t^q$ of $GF(q^{|\eta|})$ over $GF(q)$.

(2.18) *Let $\mathscr{L}(\Phi, q)$ be the Lie algebra $\mathscr{L}(\Phi)_Z \otimes GF(q)$ over the field $GF(q)$, $\mathscr{L}(\Phi)_Z$ being the Z-span of the Chevalley basis* (2.1). *$X_r \otimes 1$ is for short denoted X_r, $r \in \Phi$. Let $\Phi(q)$ be the Chevalley group with generators*

$$x_r(t) = \exp(t \cdot \rho(X_r)) , \quad t \in GF(q) , \quad r \in \Phi ,$$

ρ being a representation of $\mathscr{L}(\Phi, q)$.

We introduce some subgroups of $\Phi(q)$. $\mathscr{X}_r$ is the rootgroup defined by $r \in \Phi$, and:

(2.19)
$\Phi_i^{\mathrm{sub}}(q) = \langle \mathscr{X}_r ; r \in \Phi_i^{\mathrm{sub}} \rangle, \quad 1 \leqq i \leqq N$
$\Phi^{\mathrm{ker}}(q) = \langle \mathscr{X}_r ; r \in \Phi^{\mathrm{ker}} \rangle$.
$\Omega(q)$, $K4K(q)$, $5K5(q)$, $KK9KK(q)$ denote four types of subgroups generated by the rootgroups $\mathscr{X}_r$ with r in an Ω-, $K4K$-, $5K5$- or $KK9KK$-system. See (1.3) and (1.12).
$\mathscr{Y}_r = \langle \mathscr{X}_s ; s \in Pr^{-1}(r) \cap \Phi \rangle, \quad r \in \Phi^{Pr}$.

The structure of $\mathscr{Y}_r$ depends on r as follows:

(2.20)
r a long root: $r = s$, $\mathscr{Y}_r = \mathscr{X}_s$, s an L-root in Φ.
r a short root: $\mathscr{Y}_r$ is a direct product of $|\eta|^2$ rootgroups $\mathscr{X}_s$ defined by $|\eta|^2$-roots $s \in Pr^{-1}(r) \cap \Phi$.
r a half-root: Here $\eta = \nu$ and $\mathscr{Y}_r$ is a nondirect product of 4 rootgroups $\mathscr{X}_s$ defined by 5-roots $s \in Pr^{-1}(r) \cap \Phi$ and $\mathscr{X}_{2r}$, $2r \in \Phi \cap \Phi^{Pr}$ being an L-root.

(2.21) LEMMA. *The subgroup $\langle \mathscr{Y}_r, \mathscr{Y}_{-r} \rangle$, $r \in \Phi^{Pr}$, is of the type $A_1(q)$; $K4K(q)$; $5K5(q)$; $KK9KK(q)$ for r long; r short and $\eta \neq \psi$; r a half-root and $\eta = \nu$; r short and $\eta = \psi$ respectively.*

Proof. With Chevalley's commutator formulas one easily obtains a set of generators for these groups.

Because of (2.8) η may be regarded as an $\mathscr{L}(\Phi, q)$-automorphism and we proceed to define a corresponding $\Phi(q^{|\eta|})$-automorphism:

(2.22) DEFINITION. η will also denote the $\Phi(q^{|\eta|})$-automorphism given by

$$\eta: x_r(t) \longrightarrow \exp(\theta(t) \cdot \rho(\eta(X_r))) = \exp(t^q \cdot \rho(\pm X_{\eta(r)})) = x_{\eta(r)}(\pm t^q) .$$

The special cases are denoted $\eta = \lambda; \mu; \nu; \varphi; \psi$, occasionally subdivided into $\eta^+; \eta^- = \lambda^+; \lambda^-; \mu^+; \mu^-; \nu^+; \nu^-; \varphi^+; \varphi^-$. See (2.13).

(2.23) LEMMA. *All the subgroups of* (2.19) *are η-invariant.*

Proof. This is so because the sets of roots which support these groups are η-invariant.

(2.24) LEMMA. *The $\Phi(q^{|\eta|})$-automorphism η has order $|\eta| = 2; 2; 2; 2; 3$ in the cases $\eta = \lambda; \mu; \nu; \varphi; \psi$ respectively, independently of whether $\eta = \eta^+$ or $\eta = \eta^-$.*

Proof. By (2.5) the sign $\pm$ in (2.22) is $a +$ if r or $-r$ is one of the original fundamental roots $r_1, r_2, \cdots$ in (0.3). Also

$$\eta(x_M(t)) = \exp(\theta(t)\cdot\rho(\eta(X_M))) = \exp(t^q\cdot\rho(X_{\eta(M)})) = x_{\eta(M)}(t^q)$$

because of (2.11) and (2.12). Similarly $\eta(x_{-M}(t)) = x_{-\eta(M)}(t^q)$. This implies (2.24).

(2.25) LEMMA. *In $\Phi_i^{\mathrm{sub}}(q^{|\eta|})$ there is a subgroup, denoted ${}^{|\eta|}\Phi_i^{\mathrm{sub}}(q^{|\eta|})$, of η-invariant elements which is isomorphic to the groups in* (0.7) *for $\eta = \lambda; \mu; \nu; \varphi; \psi$ respectively, independently of i, $1 \leqq i \leqq N$, and of whether $\eta = \eta^+$ or $\eta = \eta^-$.*

Proof. Because of (2.14), the fixpointgroups of the N restrictions $\eta \mid \Phi_i^{\mathrm{sub}}(q^{|\eta|})$, $1 \leqq i \leqq N$, are all isomorphic. Hence it suffices to consider $\eta \mid \Phi_1^{\mathrm{sub}}(q^{|\eta|})$, Φ_1^{sub} being as in (1.2). But this restriction is just the automorphism which is the basis for Steinberg's twisting procedure.

3. **The restriction $\eta \mid \Omega(q^{|\eta|})$, $\Omega \subset \Phi^{\ker}$.** Let $M(n, q)$ be the algebra of $n \times n$-matrices over $GF(q)$. Let

$$R^+ \subset M(2, q^2)\ ; \quad R^- \subset M(2, q^2)\ ; \quad S \subset M(3, q^3)$$

respectively consist of the matrices

$$(3.1) \qquad \begin{pmatrix} v^q & u^q \\ u & v \end{pmatrix}; \quad \begin{pmatrix} v^q & -u^q \\ u & v \end{pmatrix}; \quad \begin{pmatrix} z^{q^2} & bx^{q^2} & ay^{q^2} \\ cy^q & z^q & ax^q \\ x & y & z \end{pmatrix}$$

where $u, v \in GF(q^2)$, $x, y, z \in GF(q^3)$ and $a, b, c = \pm 1$ are constants such that $abc = 1$.

(3.2) LEMMA. *There exist similarity transformations $\alpha^+; \alpha^-; \beta$ of $M(2, q^2)$; $M(2, q^2)$; $M(3, q^3)$ respectively such that*

$$\alpha^+(M(2, q)) = R^+\ ; \quad \alpha^-(M(2, q)) = R^-\ ; \quad \beta(M(3, q)) = S\ .$$

Proof. Let $K = GF(q^2); GF(q^2); GF(q^3)$ and let U be the K-vectorspace $GF(q^2)^2; GF(q^2)^2; GF(q^3)^3$ in the three respective cases. Let V be the $GF(q)$-subspace of U consisting of the vectors

$$(v^q, v)\ ; \quad (Zv^q, v)\ ; \quad (cw^{q^2}, bw^q, w)$$

where $v \in GF(q^2)$, $w \in GF(q^3)$, b, c are as in (3.1) and $Z \in GF(q^2) - GF(q)$ is such that $Z^q = -Z^{-1}$. This means that

$$Z = k^{(t+1/2)(q-1)} \quad \text{for } q \text{ odd and } Z = k^{t(q-1)} \text{ for } q \text{ even,}$$

t being an integer and $\langle k \rangle = GF(q^2) - \{0\}$.

The K-span of V is U since the vander Monde determinant $|k^{i \cdot q^j}| \neq 0$, $i, j = 0, 1;\ 0, 1;\ 0, 1, 2$, in the three cases and $\langle k \rangle = K - \{0\}$.

Clearly R^+; R^-; S have the same number of elements as $M(2, q)$; $M(2, q)$; $M(3, q)$ and it is easily checked that R^+; R^-; S act $GF(q)$-linearly on V by right multiplication.

Now replace the standard K-basis of U with a $GF(q)$-basis of V. This basis-change defines α^+; α^-; β.

Since $\Phi^{\ker}(q^{|\gamma|})$ is a direct product of groups $\Omega(q^{|\gamma|})$ (see (1.3)) and since each $\Omega(q^{|\gamma|})$ is η-invariant (see (2.23)), the study of $\eta \mid \Phi^{\ker}(q^{|\gamma|})$ reduces to the study of $\eta \mid \Omega(q^{|\gamma|})$. Two cases must be distinguished: $\eta \neq \psi$ with Ω of type A_1 and $\eta = \psi$ with Ω of type A_2.

(3.3) $\eta \neq \psi$, $\Omega = \{r, -r\}$, r a K-root in Φ.

Let $\gamma\colon SL(2, q^2) \longrightarrow \Omega(q^2)$ be a homomorphism such that

$$\gamma\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_r(t)\,, \qquad \gamma\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_{-r}(t)\,.$$

Then, by (2.22)

$$\gamma\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \xrightarrow{\eta} \gamma\begin{pmatrix} 1 & 0 \\ abt^q & 1 \end{pmatrix}, \qquad \gamma\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \xrightarrow{\eta} \gamma\begin{pmatrix} 1 & abt^q \\ 0 & 1 \end{pmatrix}$$

with $a, b = \pm 1$, i.e., in general

$$\eta\gamma\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \gamma\left(\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\begin{pmatrix} x^q & y^q \\ u^q & v^q \end{pmatrix}\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right) = \gamma\begin{pmatrix} v^q & abu^q \\ aby^q & x^q \end{pmatrix}.$$

Hence we get a set of fixpoints of $\eta \mid \Omega(q^2)$ (all the fixpoints if γ is an isomorphism):

$$\gamma\begin{pmatrix} v^q & abu^q \\ u & v \end{pmatrix}, \qquad u, v \in GF(q^2)\,.$$

By (3.2) there exists an $\alpha \in \operatorname{Aut} SL(2, q^2)$ such that

(3.4) *$\gamma\alpha SL(2, q)$ is contained in the fixpointgroup of $\eta \mid \Omega(q^2)$.*

Note that $ab = 1$ for $\eta = \eta^+$ and $ab = -1$ for $\eta = \eta^-$.

(3.5) $\eta = \psi$, $\Omega = \{r, \psi(r), r + \psi(r), -r, -\psi(r), -r - \psi(r)\}$, r a K-root in Φ.

Let $\gamma: SL(3, q^3) \to \Omega(q^3)$ be a homomorphism such that

$$\gamma\begin{pmatrix}1&0&0\\0&1&t\\0&0&1\end{pmatrix} = x_r(t)\,, \quad \gamma\begin{pmatrix}1&t&0\\0&1&0\\0&0&1\end{pmatrix} = x_{\psi(r)}(t)\,, \quad \gamma\begin{pmatrix}1&0&t\\0&1&0\\0&0&1\end{pmatrix} = x_{r+\psi(r)}(t)$$

$$\gamma\begin{pmatrix}1&0&0\\0&1&0\\0&t&1\end{pmatrix} = x_{-r}(t)\,, \quad \gamma\begin{pmatrix}1&0&0\\t&1&0\\0&0&1\end{pmatrix} = x_{-\psi(r)}(t)\,, \quad \gamma\begin{pmatrix}1&0&0\\0&1&0\\t&0&1\end{pmatrix} = x_{-r-\psi(r)}(t)\,.$$

Then, by (2.22)

$$\gamma\begin{pmatrix}1&0&0\\0&1&t\\0&0&1\end{pmatrix} \xrightarrow{\psi} \gamma\begin{pmatrix}1&abt^q&0\\0&1&0\\0&0&1\end{pmatrix} \xrightarrow{\psi} \gamma\begin{pmatrix}1&0&0\\0&1&0\\bct^{q^2}&0&1\end{pmatrix}$$

and similarly for the transposed matrices, with $a, b, c = \pm 1$, i.e., in general

$$\psi\gamma\begin{pmatrix}f&g&h\\u&v&w\\x&y&z\end{pmatrix} = \gamma\left(\begin{pmatrix}0&a&0\\0&0&b\\c&0&0\end{pmatrix}\begin{pmatrix}f^q&g^q&h^q\\u^q&v^q&w^q\\x^q&y^q&z^q\end{pmatrix}\begin{pmatrix}0&0&c\\a&0&0\\0&b&0\end{pmatrix}\right)$$

$$= \gamma\begin{pmatrix}v^q&abw^q&acu^q\\aby^q&z^q&bcx^q\\acg^q&bch^q&f^q\end{pmatrix}.$$

Hence we get a set of fixpoints of $\psi \mid \Omega(q^3)$ (all the fixpoints if γ is an isomorphism):

$$\gamma\begin{pmatrix}z^{q^2}&acx^{q^2}&bcy^{q^2}\\aby^q&z^q&bcx^q\\x&y&z\end{pmatrix}, \quad x, y, z \in GF(q^3)\,. \tag{3.6}$$

By (3.2) there exists a $\beta \in \operatorname{Aut} SL(3, q^3)$ such that

(3.7) *$\gamma\beta SL(3, q)$ is contained in the fixpointgroup of $\psi \mid \Omega(q^3)$.*

(3.8) Clearly, the restrictions of $\eta \neq \psi$ to subgroups of types $K4K(q^2)$, $5K5(q^2)$ and of ψ to subgroups of type $KK9KK(q^3)$ can be treated the same way by means of homomorphisms

$$SL(4, q^2) \longrightarrow K4K(q^2)\,, \quad SL(4, q^2) \longrightarrow 5K5(q^3)\,,$$
$$SL(6, q^3) \longrightarrow KK9KK(q^3)\,.$$

The matrices are conveniently subdivided into 2×2-blocks or 3×3-blocks. Thus the fixpointgroup of $\eta \mid K4K(q^2)$ contains the image

of the matrices (equals in the case of isomorphism):

$$\begin{pmatrix} g^q & abf^q & bt^q & ah^q \\ f & g & h & t \\ bv^q & au^q & y^q & abx^q \\ u & v & x & y \end{pmatrix} \quad \text{with determinant 1,}$$
$$f, g, h, t, u, v, x, y \in GF(q^2) .$$

Similarly 6×6-matrices with blocks like (3.6) give fixpoints of $\psi \mid KK9KK(q^3)$.

(3.9) LEMMA. *Let $r \in \Phi^{Pr}$ be a short root. Then the fixpointgroup of $\eta \mid \mathscr{Y}_r$ has order $q^{|\eta|^2}$.*

Proof. There are $|\eta|$ η-orbits of roots in $Pr^{-1}(r) \cap \Phi$. With one such orbit we associate as in the usual twisting procedure $q^{|\eta|}$ fixpoints $x_r(t)x_{\eta(r)}(\pm t^q)$ if $\eta \neq \psi$ and $x_r(t)x_{\eta(r)}(\pm t^q)x_{\eta^2(r)}(\pm t^{q^2})$ if $\eta = \psi$.

For later reference we need a result which is derived by means of a matrix model for the $5K5(q^2)$-groups. We consider the following situation:

(3.10) $\eta = \nu$, $r \in \Phi^{Pr}$ a half-root, $\langle \mathscr{Y}_r, \mathscr{Y}_{-r} \rangle$ a $5K5(q^2)$-group.

Let $\gamma: SL(4, q^2) \to \langle \mathscr{Y}_r, \mathscr{Y}_{-r} \rangle$ be a homomorphism such that $\gamma(I + E_{ij}t) = x_s(t)$ where

s is a K-root in Φ^+ if $(i, j) = (2, 3)$,
an L-root in Φ^+ if $(i, j) = (1, 4)$,
a 5-root in Φ^+ if $(i, j) = (1, 2), (1, 3), (2, 4), (3, 4)$.

Then

$$\gamma(I + E_{12}t) \xrightarrow{\nu} \gamma(I + E_{24}at^q)$$
$$\gamma(I + E_{34}t) \xrightarrow{\nu} \gamma(I + E_{13}bt^q)$$
$$\gamma(I + E_{21}t) \xrightarrow{\nu} \gamma(I + E_{42}at^q)$$
$$\gamma(I + E_{43}t) \xrightarrow{\nu} \gamma(I + E_{31}bt^q) , \quad a, b = \pm 1 .$$

This determines η in the $5K5(q^2)$-group. One may easily verify that for $A \in SL(4, q^2)$,

$$\nu\gamma(A) = \gamma(X \cdot \theta(A^T)^{-1} \cdot X^{-1})$$

where

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -b & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$\theta(A)$ is obtained by applying the field automorphism of (2.17) on each entry of A, and T denotes transposing. Hence we have:

(3.11) *The fixpointgroup of* $\eta \mid \langle \mathscr{Y}_r, \mathscr{Y}_{-r} \rangle$ *contains a homomorphic image of* $SU(4, q^2)$.

Here $ab = 1$ for $\nu = \nu^-$ and $ab = -1$ for $\nu = \nu^+$.

(3.12) LEMMA. *Let* $r \in \Phi^{Pr}$ *be a half-root,* $\eta = \nu$. *Then the fixpointgroup of* $\eta \mid \mathscr{Y}_r$ *has order* q^5.

Proof. Clearly $\eta \mid \mathscr{Y}_r$ is a one-to-one mapping. It then suffices and is easy to verify that the fixpoints of $\eta \mid \mathscr{Y}_r$ are the elements

$$\gamma \begin{pmatrix} 1 & x & by^q & z \\ 0 & 1 & 0 & ax^q \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } x, y, z \in GF(q^2) \text{ and } z + z^q = axx^q + byy^q .$$

4. The proof of Theorem (0.8).

(4.1) LEMMA. *Let* $\sigma: t \to x \cdot t^q$ *be an additive automorphism of* $GF(q^n)$, $x \neq 0$ *and let* $\langle k \rangle = GF(q^n) - \{0\}$. *Then* σ *has fixpoints different from* 0 *if and only if* $x \in \langle k^{q-1} \rangle$. *The set of fixpoints is then* $t_0 \cdot GF(q)$, $t_0 \neq 0$ *being an arbitrary fixpoint.*

Proof. Let $x = k^a$, $t = k^b$. Then $t = \sigma(t)$ is equivalent to the congruence $(q-1)b + a \equiv 0 \pmod{(q^n - 1)}$, which implies $a = (q-1)c$ for some c and $b + c \equiv 0 \pmod{(q^n - 1)/(q-1)}$. Hence

$$t = k^{-c + d(q^n - 1)/(q-1)}, \qquad d = 0, 1, \cdots, q-2$$

where $\langle k^{(q^n-1)/(q-1)} \rangle = GF(q) - \{0\}$.

(4.2) LEMMA. *Let* γ *be a homomorphism of* $SL(m, q^n)$ *onto a group* H, *and let* $\mathscr{X}_{ij}(q^a) = \gamma\{I + E_{ij}t; t \in GF(q^a)\}$, $i \neq j$ *and* $a = 1$ *or* $a = n$. *Suppose* $\alpha \in \operatorname{Aut} H$ *has the property* $\alpha \mathscr{X}_{ij}(q) = \mathscr{X}_{ij}(q)$ *for all* (i, j). *Then also* $\alpha \mathscr{X}_{ij}(q^n) = \mathscr{X}_{ij}(q^n)$ *for all* (i, j).

Proof. Let $K_{ij} = \{(a, b); a \neq b \text{ and } [\mathscr{X}_{ij}(q^n), \mathscr{X}_{ab}(q)] = 1\}$. Then it is easily verified that $\mathscr{X}_{ij}(q^n)$ is the only p-Sylowgroup ($p = \operatorname{char} GF(q)$) in the centralizer of $\langle \mathscr{X}_{ab}(q); (a, b) \in K_{ij} \rangle$. This proves the lemma.

In (4.3)-(4.5) we list some facts about automorphisms of Chevalley groups over the field $GF(q)$.

(4.3) *A Chevalley group acts by conjugation transitively on the set of ordered pairs of disjoint p-Sylowgroups,* $p = \text{char } GF(q)$.

(4.4) *An automorphism which normalizes U and V (the products of the rootgroups* $\mathscr{X}_r$ *for* $r > 0$, *resp.* $r < 0$) *permutes the rootgroups and can be expressed by diagonal, field and graph automorphisms.*

(4.5) *Any set* $\{d_r \in GF(q); d_r \neq 0$ *and* r *is a fundamental root}* *determines a diagonal automorphism* d *where*

$$d: x_r(t) \longrightarrow x_r(d_r t) , \quad x_{-r}(t) \longrightarrow x_{-r}(d_r^{-1} t) .$$

See Steinberg [4, p. 158].

(4.6) LEMMA. $\Phi^{\ker}(q^{|\eta|})$ *normalizes* $\mathscr{Y}_r$ *for all* $r \in \Phi^{Pr}$.

Proof. This is an immediate consequence of Chevalley's commutator formulas (one-rootlength case) since $s \in \Phi^{\ker}$ and $t \in Pr^{-1}(r) \cap \Phi$ implies $t + s \in Pr^{-1}(r)$.

Proof of (0.8). Consider first a component Ω of $\Phi^{\ker}$. By (3.4) and (3.7) there exists a homomorphism

$$\gamma_\Omega: SL(|\eta|, q^{|\eta|}) \longrightarrow \Omega(q^{|\eta|})$$

such that $\gamma_\Omega SL(|\eta|, q)$ is contained in the fixpointgroup of $\eta \mid \Omega(q^{|\eta|})$.

An application of (4.3) and (4.4) to each component Ω of $\Phi^{\ker}$ shows that there exists an inner automorphism $\omega: x \to gxg^{-1}$ of $\Phi(q^{|\eta|})$ defined by an element $g \in \Phi^{\ker}(q^{|\eta|})$ such that

$$\omega\gamma_\Omega\{I + E_{ij}t; t \in GF(q^{|\eta|})\}$$

is a rootgroup in $\Phi^{\ker}(q^{|\eta|})$ for all rootgroups $\{I + E_{ij}t\}$ of $SL(|\eta|, q^{|\eta|})$. Then, for $t \in GF(q)$,

$$\omega\eta\omega^{-1} \cdot \omega\gamma_\Omega(I + E_{ij}t) = \omega\gamma_\Omega(I + E_{ij}t) \tag{4.7}$$

and so, because of Lemma (4.2)

$$\omega\eta\omega^{-1} \cdot \omega\gamma_\Omega\{I + E_{ij}t; t \in GF(q^{|\eta|})\} = \omega\gamma_\Omega\{I + E_{ij}t; t \in GF(q^{|\eta|})\}$$

i.e.,

$$\omega\eta\omega^{-1}\mathscr{X}_r = \mathscr{X}_r \quad \text{for all } K\text{-roots } r \in \Phi . \tag{4.8}$$

By (4.6) and (2.23)

$$\omega\eta\omega^{-1}\mathscr{Y}_r = \mathscr{Y}_r \quad \text{for all } r \in \Phi^{Pr} . \tag{4.9}$$

In particular, when r is a long root, this by (2.20) becomes

$$\omega\eta\omega^{-1}\mathscr{X}_r = \mathscr{X}_r \quad \text{for all } L\text{-roots } r \in \Phi . \tag{4.10}$$

From (4.8) and (4.9) it follows that $\omega\eta\omega^{-1}$ normalizes the subgroups U and V where

$$\begin{aligned} U &= \prod_{r \in \Phi^{\ker +}} \mathscr{X}_r \cdot \prod_{r \in Pr+} \mathscr{Y}_r \\ V &= \prod_{-r \in \Phi^{\ker +}} \mathscr{X}_r \cdot \prod_{-r \in \Phi Pr+} \mathscr{Y}_r . \end{aligned} \tag{4.11}$$

Hence, by (4.4), $\omega\eta\omega^{-1}$ either normalizes all rootgroups $\mathscr{X}_r$, $r \in \Phi$, or permutes them according to a graph automorphism. By (1.1) the K-roots and L-roots span the space of Φ if and only if $\eta \neq \nu$. Thus we have, by (4.8) and (4.10),

$$\omega\eta\omega^{-1}\mathscr{X}_r = \mathscr{X}_r \quad \text{for all } r \in \Phi \text{ if } \eta \neq \nu . \tag{4.12}$$

$$\omega\eta\omega^{-1}\mathscr{X}_r = \mathscr{X}_{\alpha^a(r)} \quad \text{for all } r \in \Phi \text{ if } \eta = \nu,\ a = 0 \text{ or } a = 1 \tag{4.13}$$

where α is the graph automorphism of $\Phi = D_{2m+1}$, a to be determined later. By (4.7) it is clear that a field automorphism is needed to express $\omega\eta\omega^{-1}$ as in (4.4). Hence by (4.12) and (4.13)

$$\omega\eta\omega^{-1}x_r(t) = x_s(b_r \cdot t^q) \quad \text{for all } r \in \Phi \tag{4.14}$$

with $b_r \in GF(q^{|\eta|})$, $s = r$ for $\eta \neq \nu$, $s = \alpha^a(r)$ if $\eta = \nu$.

Now design a diagonal automorphism d of $\Phi(q^{|\eta|})$ such that

(i) If $r \in \Phi^+$ is fundamental and $\omega\eta\omega^{-1}\mathscr{X}_r = \mathscr{X}_r$, then $\{d(x_r(t)); t \in GF(q)\}$ are the $\omega\eta\omega^{-1}$-invariant elements in $\mathscr{X}_r$. This is possible by (4.1) and (4.5).

(ii) If $r \in \Phi^+$ is fundamental and $\omega\eta\omega^{-1}\mathscr{X}_r = \mathscr{X}_s \neq \mathscr{X}_r$, then $d(x_s(t^q)) = \omega\eta\omega^{-1}d(x_r(t))$. s is fundamental in Φ^+ because the rootgroups are permuted according to a graph automorphism, so this may be achieved by adjusting d_s in (4.5).

Now it is easily verified that (4.14) is changed into

$$d^{-1}\omega\eta\omega^{-1}d(x_r(t)) = x_s(t^q), \quad \text{for } r \text{ or } -r \text{ fundamental.} \tag{4.15}$$

It remains to determine whether $a = 0$ or $a = 1$ in (4.13). If $a = 0$ however, the fixpointgroup of ν, hence of $\omega\nu\omega^{-1}$, in an arbitrary $5K5(q^2)$-group would contain an image of $SL(4, q)$, contradicting (3.11). Hence $a = 1$. This proves (0.8).

We remark that similar twists may be performed on the Chevalley groups $F_4(q^2)$ and $F_4(q^3)$. The long roots in F_4 form a D_4-system supporting a subgroup of type $D_4(q^2)$ or $D_4(q^3)$. Defining η as the isometry used for one of the twists of D_4, the short roots of F_4 form η-orbits of roots making 60°-angles and a kernel system of type A_1 or A_2. One may check that corresponding $\mathscr{L}(F_4)$-automorphisms of order 2 and 3 exist and proceed as above. The group automorphism

η is conjugate in Aut $F_4(q^{|\eta|})$ to a field automorphism, and so one obtains embeddings ${}^2D_4(q^2) \subset F_4(q)$ and ${}^3D_4(q^3) \subset F_4(q)$.

5. A property of the embeddings (0.10). Let G be the subgroup of the fixpointgroup of η in $\Phi(q^{|\eta|})$, of the type (0.9). G contains the groups ${}^{|\eta|}\Phi_i^{\mathrm{sub}}(q^{|\eta|})$ given in (0.7), because of (2.25).

G may be described in two ways:

First description. We first remark that

(5.1) the $\Phi(q^{|\eta|})$-automorphism $\omega^{-1}d$ of (4.15) normalizes $\mathscr{Y}_r$ for all $r \in \Phi^{Pr}$.

This follows from (4.6) and the definitions of ω and d.

We now introduce new rootgroups $\tilde{\mathscr{X}}_r$ of $\Phi(q^{|\eta|})$ with elements $\tilde{x}_r(t)$, $t \in GF(q^{|\eta|})$ by defining

$$\tilde{\mathscr{X}}_r = \omega^{-1}d\mathscr{X}_r\,, \quad \tilde{x}_r(t) = \omega^{-1}d(x_r(t))\,. \tag{5.2}$$

Then, by (4.15)

$$\eta\tilde{x}_r(t) = \tilde{x}_s(t^q) \quad \text{for } r \text{ or } -r \text{ fundamental in } \Phi^+, \tag{5.3}$$

$s = r$ if $\eta \neq \nu$ and $s = \alpha(r)$ if $\eta = \nu$, α being the graph symmetry of $\Phi = D_{2m+1}$.

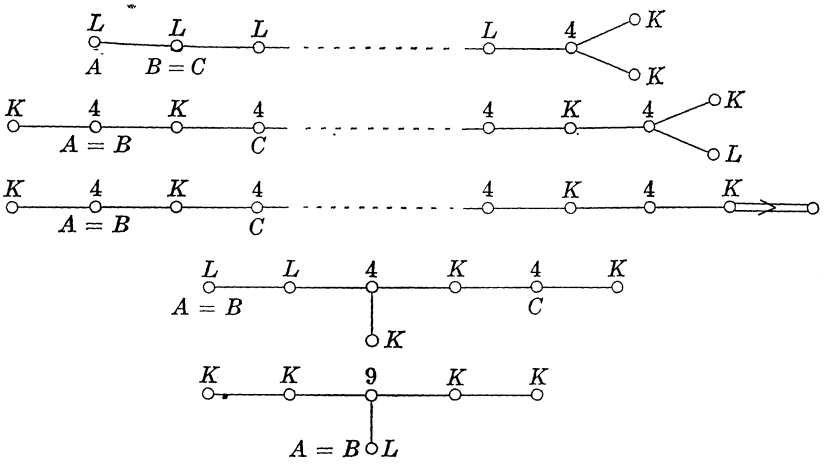
This yields a description of G as a group with a BN-pair with root system Φ^{new} where

$$\begin{aligned} &\Phi^{\mathrm{new}} = D_{m+2}\,;\ D_{2m}\,;\ B_{2m}\,;\ E_7\,;\ E_6 \quad \text{and} \\ &G \cong D_{m+2}(q)\,;\ D_{2m}(q)\,;\ {}^2D_{2m+1}(q^2)\,;\ E_7(q)\,;\ E_6(q) \end{aligned} \tag{5.4}$$

in the cases $\eta = \lambda; \mu; \nu; \varphi; \psi$ respectively.

Thus only the third Dynkin diagram of (1.5) must be changed when we move our attention from $\Phi(q^{|\eta|})$ to G. We repeat the other four diagrams however, indicating by A, B, C some fundamental roots for later reference:

(5.5)



The rootgroups of G will be called $\mathscr{X}_r^{\text{new}}$, $r \in \Phi^{\text{new}}$, with elements $x_r^{\text{new}}(t)$. If $\eta \neq \nu$ or $\eta = \nu$ and r is a long root in Φ^{new} (i.e., $r \in \Phi \cap \Phi^{\text{new}}$), then $\mathscr{X}_r^{\text{new}} = \widetilde{\mathscr{X}}_r \cap G$ and $x_r^{\text{new}}(t) = \tilde{x}_r(t)$, $t \in GF(q)$. If $\eta = \nu$ and r is a short root in Φ^{new}, then $\mathscr{X}_r^{\text{new}} = \widetilde{\mathscr{X}}_s \cdot \widetilde{\mathscr{X}}_{\alpha(s)} \cap G$ with $s \in \Phi$, $r = 1/2(s + \alpha(s))$ and $x_r^{\text{new}}(t) = \tilde{x}_s(t) \cdot \tilde{x}_{\alpha(s)}(t^q)$, $t \in GF(q^2)$. See (5.3) and (5.4).

Second description. G is described in terms of

(i) the subgroups $\Omega(q^{|\eta|}) \cap G$, Ω being a component of Φ^{ker}. These groups are homomorphic images of $SL(|\eta|, q)$. See (3.4), (3.7).

(ii) the rootgroups of the twisted subgroups ${}^{|\eta|}\Phi_i^{\text{sub}}(q^{|\eta|})$, i.e., the groups

$$\mathscr{X}_{r,i}^{\text{sub}} = \mathscr{Y}_r \cap \Phi_i^{\text{sub}}(q^{|\eta|}) \cap G\,, \quad r \in \Phi^{Pr}\,, \quad 1 \leqq i \leqq N$$

with elements $x_{r,i}^{\text{sub}}(t)$ or $x_{r,i}^{\text{sub}}(t, u)$. The parametrization is as follows.

r a long root: $r \in \Phi^{Pr} \cap \Phi$, $x_{r,i}^{\text{sub}}(t) = x_r(t)$, $t = t^q$ if $\eta \neq \nu$ and $t = -t^q$ if $\eta = \nu$. (See the proof of (3.12).)

r a short root, $\eta \neq \psi$: $x_{r,i}^{\text{sub}}(t) = x_s(t) \cdot x_{\eta(s)}(\pm t^q)$ with $s, \eta(s) \in \Phi_i^{\text{sub}} \cap Pr^{-1}(r)$.

r a half-root, $\eta = \nu$: $x_{r,i}^{\text{sub}}(t, u) = x_s(t) \cdot x_{\nu(s)}(\pm t^q) \cdot x_{s+\nu}(s)(u)$ with $u + u^q = \pm tt^q$. (See the proof of (3.12).)

r a short root, $\eta = \psi$: $x_{r,i}^{\text{sub}}(t) = x_s(t) \cdot x_{\psi(s)}(\pm t^q) \cdot x_{\psi^2(s)}(\pm t^{q^2})$ with $s, \psi(s), \psi^2(s) \in \Phi_i^{\text{sub}} \cap Pr^{-1}(r)$.

If r is a long root in Φ^{Pr}, the groups $\mathscr{X}_{r,i}^{\text{sub}}$ coincide for $i = 1, \cdots, N$. If not, there are $|\eta|$ different groups for $i = 1, \cdots, N$.

Between the elements of $\mathscr{X}_{r,i}^{\text{sub}}$ and $\mathscr{X}_{s,i}^{\text{sub}}$, $r, s \in \Phi^{Pr}$, there are commutator relations as described in Steinberg [4, p. 181]. In the case $\eta = \psi$ there is another nontrivial commutator relation of the type $[\mathscr{X}_{r,i}^{\text{sub}}, \mathscr{X}_{s,j}^{\text{sub}}] \subseteqq \mathscr{X}_{r+s,k}^{\text{sub}}$ where $r, s, r + s$ are short roots in $\Phi^{Pr} = G_2$ and $i \neq j \neq k \neq i$.

(5.6) LEMMA. *Let Ω be a component of Φ^{ker}. The elements in $\Omega(q^{|\eta|}) \cap G$ which for every $r \in \Phi^{Pr}$ permute the rootgroups $\mathscr{X}_{r,i}^{\text{sub}}$, $i = 1, \cdots, N$ are, with notation from* (3.3) *and* (3.5):

$$\left\{\gamma\begin{pmatrix} v^q & 0 \\ 0 & v \end{pmatrix}\right\} \cup \left\{\gamma\begin{pmatrix} 0 & u^q \\ u & 0 \end{pmatrix}\right\} \quad \textit{if } \eta \neq \psi \textit{ and}$$

$$\left\{\gamma\begin{pmatrix} z^{q^2} & 0 & 0 \\ 0 & z^q & 0 \\ 0 & 0 & z \end{pmatrix}\right\} \cup \left\{\gamma\begin{pmatrix} 0 & acx^{q^2} & 0 \\ 0 & 0 & bcx^q \\ x & 0 & 0 \end{pmatrix}\right\} \cup \left\{\gamma\begin{pmatrix} 0 & 0 & bcy^{q^2} \\ aby^q & 0 & 0 \\ 0 & y & 0 \end{pmatrix}\right\} \textit{ if } \eta = \psi\,.$$

Proof. This is checked by means of matrix models of the $K4K(q^2)$-groups and $KK9KK(q^3)$-groups as indicated in (3.8) or the model of the $5K5(q^2)$-groups in (3.10). Note that these elements are η-invariant

elements of the monomial group of $\Phi(q^{|\eta|})$.

(5.7) Let U_i^{sub}, $1 \leqq i \leqq N$, be the p-Sylowgroup ($p = \text{char } GF(q)$) of the twisted group ${}^{|\eta|}\Phi_i^{\text{sub}}(q^{|\eta|})$ defined by $U_i^{\text{sub}} = U \cap \Phi_i^{\text{sub}}(q^{|\eta|}) \cap G$ with U as in (4.11), i.e.,

$$U_i^{\text{sub}} = \prod_{r \in \Phi^{Pr+}} \mathscr{X}_{r,i}^{\text{sub}} .$$

(5.8) Let $U^{Pr+} = \langle U_i^{\text{sub}}; 1 \leqq i \leqq N \rangle = \prod_{r \in \Phi^{Pr+}} (\mathscr{Y}_r \cap G)$.

Expressing $\mathscr{Y}_r \cap G$ in each of the two descriptions of G we get because of (5.1)

$$(5.9) \qquad U^{Pr+} = \prod_{r \in \Phi^{Pr+},\ 1 \leqq i \leqq N} \mathscr{X}_{r,i}^{\text{sub}} = \prod_{r \in \Phi^{\text{new}+},\, r \text{ a non-}K\text{-root}} \mathscr{X}_r^{\text{new}} .$$

(5.10) LEMMA. *Let H be the highest root in Φ^{Pr+}. Then $H \in \Phi^{Pr} \cap \Phi \cap \Phi^{\text{new}}$ and $\mathscr{X}_H \cap G = \mathscr{X}_H^{\text{new}} = Z(U^{Pr+}) = \mathscr{X}_{H,i}^{\text{sub}} = Z(U_i^{\text{sub}})$, $1 \leqq i \leqq N$.*

Proof. The highest root in Φ^{Pr+} is long, hence H is an L-root in Φ, and $H \in \Phi^{Pr} \cap \Phi \cap \Phi^{\text{new}}$. The rest is easily checked with commutator formulas in G using the first description above and in the twisted subgroups using the second description.

(5.11) Let R be the set of fundamental roots in (5.5). Let $J \subset R$. We recall that the parabolic subgroup P_J defined by J has a normal series

$$P_J \rhd P_J^* \rhd O_p(P_J) , \quad p = \text{char } GF(q)$$

where P_J^* is generated by $O_p(P_J)$ and the rootgroups $\mathscr{X}_r^{\text{new}}$ with r or $-r$ expressible by the roots in J. P_J is generated by P_J^* and the diagonal subgroup of G, so the index $[P_J : P_J^*]$ is prime to p. $P_J^*/O_p(P_J)$ is isomorphic to the direct product of the Chevalley groups (twisted or not) defined by the connected components of J (regarded as a subset of the Dynkin diagram).

(5.12) The maximal parabolic subgroup $P_{R-\{r\}}$ will be denoted simpler by P_{R-r} and the parabolic subgroup defined by all K-roots in (5.5) will be denoted P_K.

The main result of this section is

(5.13) THEOREM. $N_G(U_i^{\text{sub}}) \subset P_K$, $1 \leqq i \leqq N$.

Proof. Let B be as in (5.5) and H as in (5.10). Then $(B, H) > 0$ and $(r, H) = 0$ for $r \neq B$, r fundamental in (5.5). Hence, using (5.10),

$$N_G(U_i^{\text{sub}}) \subseteqq N_G(Z(U_i^{\text{sub}})) = N_G(\mathscr{X}_H^{\text{new}}) = P_{R-B} .$$

We want to show that $N_G(U_i^{\text{sub}}) \subseteqq P_{R-A}$ with A as in (5.5). This is clear if $\eta \neq \lambda$, because then $A = B$.

(5.14) If $\eta = \lambda$, we argue as follows: Suppose $y \in N_G(U_i^{\text{sub}})$, and $y = u'hn_w u$ in the Bruhat decomposition of G associated with the first description above. Since $y \in P_{R-B}$, $w = w_A^a \cdot w'$ where w_A is the reflection along A and w' is expressible by reflections along fundamental roots difierent from A and B. Clearly $a = 0$ or $a = 1$, and we will show that $a = 0$. Since A is an L-root, $\mathscr{X}_A \cap G = \mathscr{X}_{A,i}^{\text{sub}} = \mathscr{X}_A^{\text{new}} \subset U_i^{\text{sub}} \subset U^{Pr+}$. Let $1 \neq x_A^{\text{new}}(t)$. Then, using commutator formulas, $u \cdot x_A^{\text{new}}(t) \cdot u^{-1} = x_A^{\text{new}}(t) \cdot \prod_{s>A} x_s^{\text{new}}(t_s)$ for certain t_s. Hence if $a = 1$, $n_w u \cdot x_A^{\text{new}}(t) \cdot u^{-1} n_w^{-1} \notin U^{Pr+}$ and consequently $y \cdot x_A^{\text{new}}(t) \cdot y^{-1} \notin U^{Pr+}$. This implies $y \notin N_G(U_i^{\text{sub}})$. Hence $a = 0$ and $y \in P_{R-A}$.

It is now enough to compute modulo $O_p(P_{R-A})$. The non-K fundamental roots in (5.5) are removed one by one. The detached K's define direct factors as described in (5.11), so the situation repeats itself.

If $\eta = \psi$, the 9-root in (5.5) can be removed by an easy calculation with 6×6-matrices. If $\eta \neq \psi$, the non-K fundamental roots are removed by repeating the argument, starting with C. If $\eta \neq \nu$, the last two steps are removal of a 4-root and an L-root as in the argument (5.14). If $\eta = \nu$, the last step is removal of the short root in the subsystem K $\circ\!\!=\!\!=\!\!\circ$, an easy verification.

(5.15) COROLLARY. *Let* $y \in N_G(U_i^{\text{sub}})$. *Then*

$$y = h \cdot \Big(\prod_{\Omega \subset \Phi^{\text{ker}}} y_\Omega \Big) \cdot y^+$$

with h *in the diagonal group of* G, $y_\Omega \in \Omega(q^{|\eta|}) \cap G$, $y^+ \in U^{Pr+}$ *and* y_Ω *as in Lemma* (5.6).

Proof. The decomposition of y follows from Theorem (5.13) and the description (5.11) of parabolic groups. Note that $O_p(P_K) = U^{Pr+}$. Because of (4.6) y_Ω normalizes $\mathscr{Y}_r \cap G$ for all $r \in \Phi^{Pr}$. It is checked with the models of (3.8) and (3.10) that y_Ω is as in (5.6).

(5.16) Let E be a finite group, D a subgroup and p a prime. We will say that D is p-maximal in E if for a subgroup F with greater p-part than D, $D \subset F \subseteqq E$ implies $F = E$.

(5.17) COROLLARY. *If* $\eta \neq \psi$ *and* $p \neq 2$ *or if* $\eta = \nu$, *then* ${}^{|\eta|}\Phi_i^{\text{sub}}(q^{|\eta|})$ *is* p-*maximal in* G, $p = \text{char}\, GF(q)$.

Proof in outline. If not, we may suppose that $y \in N_G(U_i^{\mathrm{sub}}) - U_i^{\mathrm{sub}}$ is a p-element. By (5.15) and the argument of (1.11), $y \in U^{Pr+}$. Then we may assume that

$$1 \neq y = \prod_{r \in \Phi^{Pr+}} x_{r,j(r)}^{\mathrm{sub}}(t_r) \quad \text{with } \mathscr{X}_{r,j(r)}^{\mathrm{sub}} \not\subset U_i^{\mathrm{sub}} .$$

Taking commutators between y and elements from U_i^{sub} we may reduce to the case with just one term in the product. Conjugation by elements from the monomial group of ${}^{|\eta|}\Phi_i^{\mathrm{sub}}(q^{|\eta|})$ then yields sufficiently many elements to show that $\langle y, {}^{|\eta|}\Phi_i^{\mathrm{sub}}(q^{|\eta|})\rangle = G$.

In particular this shows that if $p \neq 2$ the embedding ${}^2D_{m+1}(q^2) \subset D_{m+2}(q)$ obtained for $\eta = \lambda$, cannot be refined to the sequence

$${}^2D_{m+1}(q^2) \subset B_{m+1}(q) \subset D_{m+2}(q)$$

which may be obtained in orthogonal geometry by choosing appropriate nonisotropic vectors and their orthogonal complements. See e.g. Artin [1, p. 147].

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The Norwegian School of Economics and Business Administration