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MONOTONIC PERMUTATIONS OF CHAINS

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An automorphism (opp) of a chain Ω is a permutation g of Ω which preserves order in the sense that $\omega < \tau$ iff $\omega g < \tau g$. An anti-automorphism (orp) is a permutation k of Ω which reverses order in the sense that $\omega < \tau$ iff $\omega k > \tau k$. A permutation which either preserves or reverses order is called *monotonic*, and the group of all monotonic permutations is denoted by $M(\Omega)$. $M(\Omega)$ is ordered pointwise, i.e., $g \leqq h$ iff $\omega g \leqq \omega h$ for all $\omega \in \Omega$. This yields a *po*-set but not a *po*-group. However the subgroup $A(\Omega)$ of all opps of Ω forms a lattice-ordered group (l-group). A subgroup K of $M(\Omega)$ is called *l-monotonic* if $K' = K \cap A(\Omega)$ is nonempty, i.e., if K contains an orp, and if $G(K) = K \cap A(\Omega)$ is a transitive l-subgroup of $A(\Omega)$. The group $M(\Omega)$ is l-monotonic iff Ω is homogeneous and admits an orp. The opp group $G(K)$ has index 2 in K and is o-isomorphic to K'. Thus K' is also a lattice and there exist orps k in K' such that $k^2 = 1$. The *stabilizer* of a point $\alpha \in \Omega$ is $M_\alpha = \{m \in M \mid \alpha m = \alpha\}$, and the *paired orbit* of Δ is $\Delta' = \{\alpha g \mid \alpha \in \Delta g$ for some $g \in G\}$. The Main Theorem 8 shows that a K_α-orbit is the union of a G_α-orbit and its paired G_α-orbit.

An l-subgroup H of $A(\Omega)$ is *extendable* if there exists an l-monotonic group (K, Ω) such that $G(K) = H$. Regular abelian opp groups and full periodically o-primitive groups are uniquely extendable. There exist both extendable and nonextendable o-2-transitive groups. A characterization of o-primitive l-monotonic groups is given.

The transitivity of $G(K)$ forces all $(G(K))_\alpha$'s to be conjugate in $G(K)$, and also forces all K_α's to be conjugate in $G(K)$, so that most statements about these stabilizer subgroups are independent of the choice of α. Transitive l-subgroups of $A(\Omega)$ have been studied extensively by Holland [3], [4], and [5]; Lloyd [6]; and McCleary [7], [8], [9], and [10]. Standard results about *po*-groups and l-groups can be found in [1], while standard results about permutation groups can be found in [12]. We make minimal use of these results since the main theme of this paper is the interplay between orps and opps.

2. Basic structure theory. Let Ω be a totally ordered set (chain) containing more than one point. Points of Ω will be denoted by lower case Greek letters; subsets, by upper case Greek letters; and permutations, by lower case Roman letters. The image of $\beta \in \Omega$ under the permutation f will be denoted by βf, so that if g is also a permutation, $\beta(fg) = (\beta f)g$.

Since Ω is totally ordered, a permutation g is automatically an opp (orp) provided only that $\omega < \tau$ implies $\omega g < \tau g (\omega g > \tau g)$ for all $\omega, \tau \in \Omega$. If k and m are orps of Ω and a and b are opps of Ω, the following facts are easily verified:

(1) ab, km, and a^{-1} are opps;

(2) ak, ka, and k^{-1} are orps.

It follows from these facts that $M(\Omega)$ is actually a group under composition, and that $A(\Omega)$ is a subgroup. It is well known that $A(\Omega)$ is a lattice-ordered group under the pointwise order, with $\beta(f \vee g) = \max\{\beta f, \beta g\}$ and $\beta(f \wedge g) = \min\{\beta f, \beta g\}$. A group G is called a *po-group* iff G is a *po*-set such that $a, b, c, d \in G$ with $b \leqq c$ implies $abd \leqq acd$. $M(\Omega)$ is not a *po-group*, for if $b \leqq c$ are opps of Ω and k is an orp of Ω, then $bk \geqq ck$.

If Ω is equipped with the order topology, then it is clear that the orps and opps of Ω are homeomorphisms of Ω. There exist orps of the integers without fixed points, but if k is an orp of a chain Ω such that $(\alpha, \alpha k)$ is connected, since the continuous image of connected set is connected, k has a fixed point in $(\alpha, \alpha k)$.

The *intial number* of a cardinal number is the smallest ordinal number of that cardinality. An ordinal number ω_β is *regular* if it is an initial number and all of its cofinal subsets have cardinality $\aleph_\beta$. Following [9] we say that a point $\alpha \in \Omega$ has *character* $c_{\beta\gamma}$ if ω_β is the unique regular ordinal which is o-isomorphic to a cofinal subset of $\{\sigma \in \Omega \mid \sigma < \alpha\}$ (or equivalently, if $\aleph_\beta$ is the smallest cardinality of any cofinal subset of $\{\sigma \in \Omega \mid \sigma < \alpha\}$), and dually for ω_γ. A chain is *homogeneous* if $A(\Omega)$ is transitive. A point of Ω has *symmetric* characters if its left character equals its right character. A necessary condition for a homogeneous chain Ω to admit an orp is that points of Ω have symmetric characters. Examples of chains with nonsymmetric characters are easy to produce, e.g., the semi-long line with points of character c_{01}.

In the sequel all chains will be homogeneous and will admit orps. A monotonic group (K, Ω) is *t-monotonic* if $G(K)$ is transitive on Ω.

THEOREM 1. *If* (K, Ω) *is monotonic, then* $(K\colon G(K)) = 2$, *so that* $G(K)$ *is normal in* K.

Proof. It follows from facts (1) and (2) that if $k, m \in K'$, km^{-1} is an opp. Since $G = G(K)$ is the group of all opps in K, km^{-1} is in G, so that $Gk = Gm$. Hence $(K\colon G) = 2$.

THEOREM 2. *If* (K, Ω) *is t-monotonic, then for any* $\alpha \in \Omega$, $(K_\alpha\colon (G(K))_\alpha) = 2$.

Proof. Since $G = G(K)$ is transitive and K contains at least one orp k, if $\alpha \in \Omega$, there exists $g \in G$ such that $\alpha k g = \alpha$. Thus K'_α is not empty. The result now follows from a proof analogous to the proof of Theorem 1.

An opp group (G, Ω) is called *regular* if G is transitive and $G_\alpha = \{1\}$ for one (and hence, every) $\alpha \in \Omega$.

COROLLARY 3. *If (K, Ω) is monotonic, $G(K)$ is regular and $\alpha \in \Omega$, then K'_α contains precisely one element.*

THEOREM 4. *If (K, Ω) is monotonic and $G = G(K)$, then left multiplication by a fixed orp $r \in K'$ provides an o-isomorphism* (order preserved both ways) *from G onto K'.*

Proof. If $k \leqq m \in M(\Omega)$ and p is *any* permutation of Ω, $\alpha p k \leqq \alpha p m$ and $\alpha p^{-1} k \leqq \alpha p^{-1} m$. Thus $pk \leqq pm$ and $p^{-1}k \leqq p^{-1}m$. It follows from Theorem 1 that if $r \in K'$, $rG = K'$ and $rK' = r^2 G = G$. Thus if $g, h \in G$, $g \leqq h$ iff $rg \leqq rh$; and similarly if $k, m \in K'$, $k \leqq m$ iff $rk \leqq rm$. Thus left multiplication by r is an o-isomorphism from G onto K'.

COROLLARY 5. *If (K, Ω) is monotonic and $G(K)$ is an l-subgroup of $A(\Omega)$, then K' is a lattice with $\alpha(k \wedge m) = \min\{\alpha k, \alpha m\}$, and dually for suprema.*

Proof. Since every o-isomorphism of a lattice is a lattice isomorphism, the first statement follows from Theorem 4. If $k, m \in K'$, by Theorem 1, $m = ka$ for some $a \in G(K)$. Since $1 \wedge a \in G(K)$, it follows from Theorem 4 that $k(1 \wedge a) = k \wedge m \in K'$. If $\alpha \in \Omega$, $\alpha(k \wedge m) = \alpha k(1 \wedge a)$, and since $\alpha k \in \Omega$ (and $1 \wedge a \in G(K)$ where infs are pointwise), $\alpha k(1 \wedge a) = \alpha k \wedge \alpha k a = \min\{\alpha k, \alpha m\}$. A dual argument shows that $\alpha(k \vee m) = \max\{\alpha k, \alpha m\}$.

COROLLARY 6. *When ordered pointwise, the orps of any chain* (homogeneous or not) *form a lattice.*

Proof. $A(\Omega)$ is an l-permutation group.

The following lemma uses the lattice properties of $M(\Omega)$ to establish the existence of orps which square to the identity. These orps will be very useful in §3, and in the upcoming example which shows that this nice behavior is not valid for t-monotonic groups.

LEMMA 7. *If k is an orp of any chain Ω, then $(k \wedge k^{-1})^2 = 1 =$*

$(k \vee k^{-1})^2$.

Proof. If k is an orp of Ω, $M(\Omega)$ is monotonic so that $k \wedge k^{-1} \in M'$ by Corollary 5. If $\beta \in \Omega$ and $\beta k^{-1} \leqq \beta k$, since k^{-1} is an orp, $\beta k^{-2} \geqq \beta$. Thus by Corollary 5, $\beta(k \wedge k^{-1})^2 = \beta k^{-1}(k \wedge k^{-1}) = \min \{\beta, \beta k^{-2}\} = \beta$. Similarly if $\beta k \leqq \beta k^{-1}$, we have $\beta(k \wedge k^{-1})^2 = \beta$ so that $(k \wedge k^{-1})^2 = 1$. The dual argument shows that $(k \vee k^{-1})\, 2 = 1$.

If (G, Ω) is a transitive l-permutation group and $\delta \in \Omega$, the G_α-*orbit* containing δ is $\{\delta g \mid g \in G_\alpha\}$. It is easy to show [7, Proposition 1] that the orbits of G_α are convex. Thus the G_α-orbits partition Ω into convex subsets, and this set inherits the natural total order, i.e., if Δ and Λ are G_α-orbits, then $\Delta \leqq \Lambda$ iff $\delta \leqq \gamma$ for all $\delta \in \Delta$, $\gamma \in \Lambda$. Furthermore, this natural total order is independent of α [7, Theorem 9]. We define for each G_α-orbit Δ, a *paired orbit* $\Delta' = \{\alpha g \mid \alpha \in \Delta g\}$, and always use the notation Δ' to refer to pairings with respect to some distinguished point α. It is shown in [12, §16] and [7, Theorem 9] that Δ' is indeed a G_α-orbit, and in [7, Proposition 4] that the map $\Delta \rightarrow \Delta'$ is an o-anti-isomorphism of the set of G_α-orbits with the property that $\Delta'' = \Delta$ for any G_α-orbit Δ.

If $\beta \in \Omega$ and $\beta G_\alpha = \{\beta\}$, then β is called a *fixed point* of G_α. If $\beta G_\alpha \neq \{\beta\}$, $\{\beta G_\alpha\}$ is a *long* G_α-orbit which must necessarily be infinite. A G_α-orbit Δ is called *positive* (*negative*) iff $\delta > \alpha (\delta < \alpha)$ for each $\delta \in \Delta$, and $\{\alpha\}$ is called the *zero* G_α-orbit.

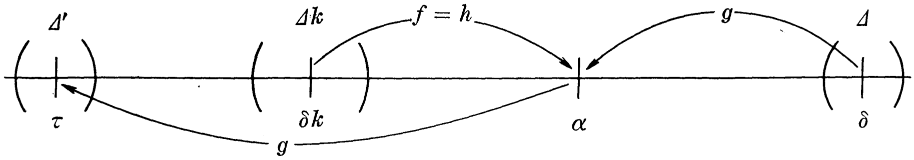
The following theorem describes the relationship between the $G(K)_\alpha$-orbits and the K_α-orbits of an l-monotonic group (K, Ω).

THEOREM 8 (Main Theorem). *If (K, Ω) is l-monotonic, $G = G(K)$, Δ is a G_α-orbit, and $k \in K'_\alpha$, then $\Delta k = \Delta'$. Thus a K_α-orbit is the union of a G_α-orbit and its paired G_α-orbit.*

Proof. If $k, m \in K'_\alpha$, since $km^{-1} \in G_\alpha$, $\Delta km^{-1} = \Delta$ and $\Delta k = \Delta m$. It follows that $\Delta K'_\alpha$ is the union of G_α-orbits, for suppose that Γ is a G_α-orbit not contained in Δk which meets Δk. Then if $\gamma \in \Gamma \backslash \Delta k$ and $\beta \in \Gamma \cap \Delta k$, there exists $g \in G_\alpha$ such that $\beta g = \gamma$. But then $\Delta kg \neq \Delta k$, which is a contradiction since $kg \in K'_\alpha$. Thus Δk is the union of G_α-orbits.

Suppose $\delta k, \tau k \in \Delta k$. Then since Δ is a G_α-orbit, there exists $g \in G_\alpha$ such that $\delta g = \tau$. Thus $\delta k(k^{-1}gk) = \delta gk = \tau k$, and since $k^{-1}gk \in G_\alpha$, δk and τk are in the same G_α-orbit. This shows that Δk is a G_α-orbit.

We note that both pairing and $k \in K'_\alpha$ provide an involution of the set of G_α-orbits, so that we may assume that Δ is a positive G_α-orbit. To show $\Delta k = \Delta'$, first suppose $\Delta' < \Delta k$.



If $\delta \in \varDelta$, there exists $g \in G$ such that $\delta g = \alpha$, and from the definition of pairing we have $\alpha g = \tau \in \varDelta'$ so that $\alpha g < \delta k$. But G is a transitive l-group, so there exists $h \in G$ such that $\delta k h = \alpha$. Now $g^{-1} \geqq g^{-1} \wedge h = f$ and $\alpha = \delta k h = \delta k f$; also $\alpha k g^{-1} k f = \alpha g^{-1} k f = \delta k f = \alpha$. But since $g^{-1} \geqq f$, $\delta k g^{-1} > \delta k f = \alpha$, so that $\delta k g^{-1} k f < \alpha f$. Since pairing is an orp of the G_α-orbits and $\varDelta k > \varDelta'$, $(\varDelta k)' < \varDelta'' = \varDelta$. Since $(\delta k)f = \alpha$, from the definition of pairing we have $\alpha f \in (\varDelta k)' < \varDelta$. Thus $\delta k g^{-1} k f < \alpha f \in (\varDelta k)' < \varDelta$, and this is a contradiction since $k g^{-1} k f \in G_\alpha$ and $\varDelta$ is a G_α-orbit.

If $\varDelta k < \varDelta'$ a dual argument leads to a contradiction. This completes the proof of Theorem 8.

COROLLARY 9. *If (K, Ω) is l-monotonic and $G = G(K)$, the paired G_α-orbits are o-anti-isomorphic.*

Proof. The o-anti-isomorphism is achieved by means of any $k \in K'_\alpha$.

Transitive l-subgroups G of $A(\Omega)$ such that fixed points of G_α are never paired with long G_α-orbits were called *balanced* in [7]. Examples of unbalanced l-permutation groups can be constructed, but it follows from Corollary 9 that if (K, Ω) is l-monotonic, then $G(K)$ is balanced.

A monotonic group (K, Ω) is called *c-monotonic* if $G(K)$ is *coherent*, i.e., $\alpha < \beta \in \Omega$ implies that there exists $1 \leqq g \in G(K)$ such that $\alpha g = \beta$. The chain of implications l-monotonic $\Rightarrow$ c-monotonic $\Rightarrow$ t-monotonic $\Rightarrow$ monotonic is easy to verify. The following is an example of a c-monotonic group (K, Ω) which not only has no orp k such that $k^2 = 1$ but also does not have the orbit pairing property of Theorem 8.

Suppose H is the subgroup of the linear group of the reals which consists of the elements $\{\alpha x + \beta \mid \alpha \text{ is a positive rational and } \beta \text{ is any real number}\}$. An element g of H is positive iff $\alpha = 1$ and $\beta > 0$. Then H is a coherent opp group, but not an l-permutation group. Let k be the orp of the reals which sends each real number α to $-\sqrt{2}\alpha$. Since $\alpha k^2 = 2\alpha$, $k^2 \in H$. If $h \in H$ there is a positive rational number τ and a real number β such that $\alpha h = \tau\alpha + \beta$ for each real number α. Hence $\alpha khk = 2\tau\alpha - \sqrt{2}\beta$, so that $khk \in H$ for each $h \in H$. The element g of H defined by $\tau g = \tau/2$ has the property that $k^{-1} = kg$, so that $khk^{-1} \in H$ for each $h \in H$. Since $k^2 \in H$ if $K = \langle k, H \rangle$, it is easy to show that $G(K) = H$. (In fact this result follows from Theorem 10.) Thus K is c-monotonic, and if $m \in K'_0$, there exists a positive rational number γ such that $\alpha m = -\sqrt{2}\gamma\alpha$ for each α. Since $\alpha m^2 = 2\gamma^2\alpha$ and $\sqrt{2}$ is irrational, $m^2 \neq 1$.

If $r \in K'$ there is a positive rational number η and a real number λ such that $\alpha r = -\sqrt{2}\eta\alpha + \lambda$ for each α. But then if $r^2 = 1$, since $0 = 0r^2 = \lambda(1 - \sqrt{2}\eta)$, either $\lambda = 0$ or $\sqrt{2}$ is rational, and because of the above, both of these statements lead to contradictions. Thus no orp in K squares to the identity.

Since the positive rationals are an orbit of H_0, the orbits of H_0 are not convex; and furthermore, since the positive rationals are paired (in H_0) with the negative rationals, K clearly does not have the orbit pairing property of Main Theorem 8.

3. Extendable *l*-permutation groups. If H is an l-subgroup of $A(\Omega)$, an orp of k of Ω will be said to *extend* H iff $G(\langle k, H\rangle) = H$. If k extends H, we note that for any $a \in H$, ka also extends H. If (K, Ω) is l-monotonic and $G(K) = H$, K will be called an *extension* of H, and H will be called *extendable*. The following theorem provides a computational necessary and sufficient condition for extendability.

THEOREM 10. *An orp m extends an l-subgroup H of $A(\Omega)$ iff m normalizes H and $m^2 \in H$.*

Proof. If H is extendable, any $m \in K\backslash H$ normalizes H, and clearly, $m^2 \in H$.

Conversely, if such an orp m exists, $mgm^{-1} \in H$ for each $g \in H$, so that since $m^2 \in H$, $mgm = mgm^{-1}m^2 \in H$. Similarly $m^{-1}gm^{-1} \in H$. Since only words which contain an even number of m's or m^{-1}'s are opps, it follows that m extends H.

THEOREM 11. *Suppose that (H, Ω) is regular. Then H is extendable iff H is abelian; and then H uniquely determines its extension.*

Proof. If K is an extension of H and $H = G(K)$ is regular, we know by Corollary 3 that there is precisely one $k \in K'_\alpha$. Since each $\beta \in \Omega$ is an H_α-orbit, we know by Main Theorem 8 and the definition of pairing that $\beta k = \beta' = \alpha g$, where g is the unique element of H such that $\beta g = \alpha$. We next show that this orp k extends H iff H is abelian.

Since k fixes α, k^2 fixes α so that by regularity, $k^2 = 1$ and $k = k^{-1}$. It suffices (by Theorem 10) to show that $kgk \in H$ for each $g \in H$.

From the definition of k we have $\alpha kgk = \alpha gk = (\alpha g)' = \alpha g^{-1}$ for any g in H. If $\beta \in \Omega$, there is a unique h in H such that $\beta h = \alpha$.

Thus $\beta kgk = \alpha hgk = (\alpha hg)' = \alpha g^{-1}h^{-1}$. But since H is abelian $\alpha g^{-1}h^{-1} = \alpha h^{-1}g^{-1} = \beta g^{-1}$. Thus for each $g \in H$, we have $kgk = g^{-1}$ so that k extends H. Also H uniquely determines its extension, since k must belong to any extension of H, and thus, all extensions are simply extensions by k.

If H is not abelian, by regularity there exist $c, d \in H$ such that $\gamma c^{-1}d^{-1} \neq \gamma d^{-1}c^{-1}$ for any $\gamma \in \Omega$. Then picking γ such that $\gamma c = \alpha$, we have $\alpha kdk = \alpha dk = (\alpha d)' = \alpha d^{-1}$; but since $\gamma c = \alpha$, $\gamma kdk = \alpha cdk = (\alpha cd)' = \alpha d^{-1}c^{-1} \neq \alpha c^{-1}d^{-1} = \gamma d^{-1}$. Thus kdk agrees with d^{-1} at α but not at γ; so by regularity, $kdk \notin H$. Thus since k must belong to any extension of H, H is not extendable.

COROLLARY 12. *Suppose that (K, Ω) is montonic and $G = G(K)$ is regular. Then for any $m \in K'$ and $g \in G$, $m^2 = 1$ and $mg = g^{-1}m$.*

Proof. In the proof of Theorem 11 we actually showed that for any $h \in G$, $khk = h^{-1}$ where k is the only orp in K'_α. Thus $(kh)^2 = 1$ and since $k = k^{-1}$, $kh = h^{-1}k$. If $m \in K'$, by Theorem 1 $m = kf$ for some $f \in G$, so that $m^2 = (kf)^2 = 1$. Now if $g \in G$, $mg = (kf)g$ so that since $fg \in G$, $(mg)^2 = (k(fg))^2 = 1$, and since $m^2 = 1$ we have $mg = g^{-1}m$.

If F is any group of permutations on Ω, then a *convex F-congruence* on Ω is an equivalence relation Q on Ω such that each Q-class is convex, and such that if $\alpha Q \beta$ then $\alpha f Q \beta f$ for each $f \in F$. If L is any t-monotonic subgroup of $M(\Omega)$, it follows from the transitivity of $G(L)$ that all convex L-congruence classes for any one convex L-congruence are o-isomorphic. If Q is a convex L-congruence, we call each Q-class an *o-block* of L; thus an o-block of L is a nonempty convex subset Δ of Ω such that $\Delta m = \Delta$ or $\Delta m \cap \Delta = \{\ \}$ for each $m \in L$. L is called *o-primitive* iff the only convex L-congruences are trivial ones.

A subgroup G of $A(\Omega)$ is *o-2-transitive* iff whenever $\alpha < \beta$ and $\gamma < \delta$, there exists $g \in G$ such that $\alpha g = \gamma$ and $\beta g = \delta$. It is clear that if (H, Ω) is an o–2–transitive opp group, then H is o-primitive. Proposition 24 [7] states that a regular opp group (H, Ω) is o-primitive iff H is isomorphic as an ordered group to a subgroup of the additive reals.

The easiest example of a transitive l-permutation group which is neither o-2-transitive nor regular is the group (G, Ω), where Ω is the reals and $G = \{g \in A(\Omega) \mid \alpha g + 1 = (\alpha + 1)g \text{ for all } \alpha \in \Omega\}$. Some comments on this group will facilitate the understanding of the next few theorems. The long orbits of any G_α form a chain o-isomorphic to the integers (in fact, the long G_0-orbits are the intervals $(n, n+1)$ where n is an integer). The o-permutation z defined by $\alpha z = \alpha + 1$ generates (as a group) the centralizer $Z_{A(\Omega)}G$, and z is called the *Ω-period* of G.

Because of this periodicity, the action of $g \in G$ on any long G_α-orbit determines its action on all of Ω. The long G_α-orbit Δ_{j+1} is "one period up" from Δ_j in the sense that $\Delta_j z = \Delta_{j+1}$.

McCleary's Theorem 40 [7] states that any transitive o-primitive l-permutation group which is neither o-2-transitive nor regular looks strikingly like (G, Ω). These groups were called *periodically o-primitive* in [7]. Here, more precisely, is what Theorem 40 says.

Let (G, Ω) be an o-primitive transitive l-permutation group which is neither o-2-transitive nor regular, and let $\alpha \in \Omega$. Then the long orbits of G_α form a chain o-isomorphic to the integers. Suppose $\Delta_1 = (\Delta_1)_\alpha$ is the first positive long orbit of G_α, Δ_{j+1} is the first long orbit greater than Δ_j, and $\bar{\omega}_j$ is the sup of $\bar{\Delta}_j$. Either there is a positive integer n such that sup $\Delta_j = \bar{\omega} \in \Omega$ iff $j \equiv 0 \pmod{n}$, and we say that G has *Config*(n); or sup $\Delta_j = \omega_j \in \Omega$ only when $j = 0$, and we say that G has *Config*(∞). The o-permutation $\bar{z}$ of $\bar{\Omega}$, $\bar{\Omega}$ the Dedekind completion (without end points) of Ω, such that $\alpha\bar{z} = \sup(\Delta_1)_\alpha = \bar{\omega}_1$ for each $\alpha \in \Omega$ is called the *$\bar{\Omega}$-period* of G in the sense that it generates (as a group) the centralizer $Z_{A(\bar{\Omega})}G$; so that $(\bar{\beta}\bar{z})g = (\bar{\beta}g)\bar{z}$ for all $\bar{\beta} \in \bar{\Omega}$, $g \in G$. If G has Config(n), $z = \bar{z}^n$ is called the *Ω-period* of G. G is called *full* if G is the entire centralizer $Z_{A(\bar{\Omega})}\bar{z}$.

$$
\text{——}\,|\,(\underset{\omega_{-1}}{\text{——}})(\overset{\Delta_0}{\text{——}})\,|\,(\underset{\alpha=\omega_0}{\overset{\Delta_1}{\text{——}}})(\underset{\omega_1=\alpha\bar{z}}{\overset{\Delta_2}{\text{——}}})\,|\,(\underset{\omega_2}{\text{——}}
$$

CONFIG (2)

LEMMA 13. *Suppose that (F, Ω) is a periodically o-primitive l-permutation group and t is either the Ω-period or the $\bar{\Omega}$-period of F. Then if an orp k extends F, $tk = kt^{-1}$, i.e., if β is one period up from γ, βk is one period down from γk. Conversely if (H, Ω) is full periodically o-primitive with t either period of H, and k is an orp of Ω such that $tk = kt^{-1}$, then k extends H.*

Proof. Suppose t is the $\bar{\Omega}$-period of F. If k extends F, then for some $a \in F$, $m = ka$ fixes α. Since $\bar{\omega}_n$ is fixed by F_α for each integer n, it follows from Theorem 8 that $\bar{\omega}_n m = \bar{\omega}_{-n}$. Thus $\alpha tm = \bar{\omega}_1 m = \bar{\omega}_{-1} = \alpha t^{-1} = \alpha m t^{-1}$. If $\beta \in \Omega$, since F is transitive, $\beta f = \alpha$ for some $f \in F$. Then $\beta f m t^{-1} = \alpha m t^{-1} = \alpha t m = \beta f t m = \beta t f m$ since t centralizes F. Since k (and hence m) extends F, there exists $c \in F$ such that $fm = mc$. But then $\beta f m t^{-1} = \beta m c t^{-1} = \beta m t^{-1} c$, and since also $\beta f m t^{-1} = \beta t f m = \beta t m c$, we have $\beta t m = \beta m t^{-1}$. Thus $tm = mt^{-1}$ and since $m = ka$, we have $tka = kat^{-1} = kt^{-1}a$ so that $tk = kt^{-1}$ as desired.

Conversely suppose (H, Ω) is full and t is the $\bar{\Omega}$-period of H. If k is an orp such that $tk = kt^{-1}$, it follows that $t^{-1}k = kt$ and $t^{-1}k^{-1} = k^{-1}t$.

Since H is full, using Theorem 10, k extends H iff for each g in H, kgk^{-1}, $k^{-1}gk$, and k^2 all commute with t. If $g \in H$, $kgk^{-1}t = kgt^{-1}k^{-1} = tkgk^{-1}$. Thus kgk^{-1} (and similarly $k^{-1}gk$ and k^2) commutes with t. Thus k extends H. The proof for the corresponding Ω-period is similar.

LEMMA 14. *Suppose (H, Ω) is periodically o-primitive with finite* Config (n), *Δ_i is the* ith *positive H_α-orbit, and $\Psi = \Delta_1 \cup \cdots \cup \Delta_n$. Then Ω has an orp iff Ψ has an orp.*

Proof. Since H has Config(n) if z is the Ω-period of H, $\alpha z \in \Omega$ so that if m is an orp of Ω, $\alpha < \alpha z$ and $\alpha zm < \alpha m$. Since H is periodically o-primitive, $A(\Omega)$ is o-primitive. Since $A(\Omega)$ is not periodic [6], it must be o-2-transitive. Thus there is a g in $A(\Omega)$ such that $\alpha zmg = \alpha$ and $\alpha mg = \alpha z$. Thus mg induces an orp on Ψ.

Since H has Config (n), z is actually in $A(\Omega)$. Thus if m is an orp of Ψ, we define a function k by

$$\beta k = \begin{cases} ((\beta z^{-t})m)z^{-t-1} \text{ if } \beta \in \Psi_t = \Psi z^t \\ \bar{\omega}_{-t} = \bar{\omega}_{t'} \text{ if } \beta = \bar{\omega}_t \in \bar{\Omega} \end{cases} .$$

Since the long orbits and fixed points of H_α partition Ω, and m is an orp of Ψ, k is an orp of Ω which is essentially the "period extension" of m to Ω.

THEOREM 15. *If (H, Ω) is full periodically o-primitive with finite* Config (n), *and Ω has an orp, then H is uniquely extendable.*

Proof. If $\alpha \in \Omega$ and Ψ is as in Lemma 14, then by Lemma 14 Ψ has an orp m which we periodically extend to the orp k of Ω as in Lemma 14. To show that H is extendable it suffices by Lemma 13 to show that if z is the Ω-period of H, then $zk = kz^{-1}$. If $\beta \in \Delta_t$, $t = an + b$, $0 \leqq b < n$, then $\beta z \in \Psi_{a+1} = \Psi z^{a+1}$ so that by the definition of k, $\beta zk = (\beta z)z^{-a-1}mz^{-a-2} = (\beta(z^{-a}mz^{-a-1}))z^{-1} = \beta kz^{-1}$. Similarly if $\beta = \omega_{an} \in \Omega$, $\beta zk = \beta kz^{-1}$ so that $zk = kz^{-1}$, and thus k extends H by Lemma 13.

If k and r both extend H, it follows from Lemma 13 that $zk = kz^{-1}$ and $zr = rz^{-1}$. By Theorem 1, $r = ak$ for some $a \in A(\Omega)$, so that $zak = zr = rz^{-1} = akz^{-1} = azk$, i.e., $za = az$. Thus $a \in H$ since H is full, and it follows that H is uniquely extendable.

If Δ and Γ are subsets of a chain Ω, we write $\Delta < \Gamma$ iff $\delta < \gamma$ for all $\delta \in \Delta$, $\gamma \in \Gamma$. Let α be an ordinal number. An *α-set* is a chain

Ω of cardinality $\aleph_\alpha$ in which for any two (possibly empty) subsets $\Delta < \Gamma$ of cardinality less that $\aleph_\alpha$, there exists $w \in \Omega$ such that $\Delta < \omega < \Gamma$. If ω_α is a regular ordinal, then (assuming the generalized continuum hypothesis) there exists an α-set, and it is unique up to o-isomorphism [2, pp. 179–181]. Reversing the ordering on an α-set yields an α-set, so by the uniqueness of α-sets, every α-set possesses an orp.

It is shown in [8, Lemma 22] that if H is a periodically o-primitive l-subgroup of $A(\Omega)$ and $\Delta_1 = (\Delta_1)_\beta$ is an α-set, then all long H_β-orbits Δ_i are α-sets. Theorem 24 [8] states that if $n = 1, 2, \cdots$, or ∞, and Δ is an α-set (where ω_α is a regular ordinal number) then there exists a unique (up to o-permutation group isomorphism) full periodically o-primitive group (H, Ω) having Δ as the first positive orbit of a stabilizer subgroup G_β and having Config(n). We have

THEOREM 16. *Let $n = 1, 2, \cdots$, or ∞, let ω_α be a regular ordinal number, and let Δ_1 be an α-set. Then the unique full periodically o-primitive l-permutation group (H, Ω) having Δ_1 as the first positive orbit of a stabilizer subgroup H_β and having* Config(n) *is uniquely extendable.*

Proof. Suppose that n is finite, Δ_i be the ith positive orbit of H_β and $\Psi = \Delta_1 \cup \cdots \cup \Delta_n$. Since each Δ_i is an α-set, it has an orp and furthermore, Δ_i is also o-isomorphic to Δ_j for any integer j; therefore Ψ has an orp. Thus Ω has an orp by Lemma 14 and H is uniquely extendable by Theorem 15.

If $n = \infty$, one can use the $\bar{\Omega}$-period $\bar{z}$ of H and a special property of α-sets (namely Lemma 23 [8]) to show that H is uniquely extendable by a proof similar to the proof of Theorem 15.

A chain Ω is *o-2-homogeneous* iff $A(\Omega)$ is o-2-transitive. The *support* of $m \in M(\Omega)$ is $\{\beta \in \Omega \mid \beta m \neq \beta\}$. An *l-ideal* of an l-group G is a convex normal l-subgroup of G. We make the following definitions:

$$B(\Omega) = \{g \in A(\Omega) \mid g \text{ has bounded support}\}$$
$$BA(\Omega) = \{g \in A(\Omega) \mid g \text{ has support bounded above}\}$$
$$BB(\Omega) = \{g \in A(\Omega) \mid g \text{ has support bounded below}\}.$$

It is shown in [3, Theorem 6] that when Ω is o-2-homogeneous, B, BA, and BB are all o-2-transitive l-ideals of $A(\Omega)$. We have the following theorem.

THEOREM 17. *Suppose that Ω is o-2-homogeneous and has an orp. Then $B(\Omega)$ is extendable, but in general, not uniquely extendable. Furthermore $BA(\Omega)$ and $BB(\Omega)$ are not extendable.*

Proof. If m is any orp of Ω, it is straightforward to show that m fixes γ iff $m^{-1}hm$ fixes γm. Thus conjugation by any orp m fixes $B(\Omega)$ and interchanges $BA(\Omega)$ and $BB(\Omega)$, so that $BA(\Omega)$ and $BB(\Omega)$ are never extendable. If m is an orp of Ω which squares to 1 (such an orp exists by Lemma 7 since $M(\Omega)$ is l-monotonic), since $m^{-1}B(\Omega)m = B(\Omega)$ and $m^2 = 1 \in B(\Omega)$, m extends $B(\Omega)$ by Theorem 10.

If Ω is the reals, and the orps k, n of the reals are defined by: $\alpha k = -\alpha$ for each α; and $\alpha n = -2\alpha$ if $\alpha \geqq 0$, $\alpha n = -\alpha/2$ otherwise, then both k and n extend $B(\Omega)$. The extensions $K = \langle k, B\rangle$ and $N = \langle n, B\rangle$ are definitely not identical however, for n is clearly not in kB.

It follows from Corollary 9 that a necessary condition for (H, Ω) to be extendable is that the paired H_α-orbits be o-anti-isomorphic which implies that H must be balanced. Balanced is not sufficient for extendability since BA and BB are both balanced whenever Ω is o-2-homogeneous.

In [11] the generalized monotonic wreath product is constructed (along the same lines as the generalized ordered wreath product constructed in [5] but different in one crucial way), and it is shown that an l-monotonic group can be "nicely" embedded in the generalized monotonic wreath product of its "o-primitive components". Thus a study of o-primitive l-monotonic groups is called for.

If (K, Ω) is an o-primitive l-monotonic group, $G(K)$ is either o-2-transitive, the regular representation of a subgroup of the reals, or periodically o-primitive. If $G(K)$ is o-2-transitive, then K is actually 2-*transitive*, i.e., if $\alpha, \beta, \gamma, \delta \in \Omega$, there exists $k \in K$ such that $\alpha k = \gamma$ and $\beta k = \delta$. If $\alpha < \beta$ and $\gamma > \delta$ and k is an orp, then $\alpha k > \beta k$ so there exists $g \in G(K)$ such that $\alpha k g = \gamma$ and $\beta k g = \delta$. The other cases are similar, and it follows that K is 2-transitive. It is shown in [8] that when $A(\Omega)$ is o-2-transitive, it is actually o-n-transitive for $n \geqq 3$. Since an orp can have at most one fixed point, $M(\Omega)$ is not 3-transitive.

If $G = G(K)$ is the regular representation of a subgroup of the reals, Corollary 12 shows that if $k \in K'$, $k^2 = 1$ and $kg = g^{-1}k$ for any $g \in G$. If $G = G(K)$ is periodically o-primitive with $\bar{\Omega}$-period $\bar{z}$, and $k \in K'$, since k extends G, $\bar{z}k = k\bar{z}^{-1}$ by Lemma 13. We summarize these results in

THEOREM 18. *If (K, Ω) is an o-primitive l-monotonic group and $G = G(K)$, then either*:

(1) *G is the regular representation of a subgroup of the reals, and if $k \in K'$, $g \in G$, $k^2 = 1$ and $kg = g^{-1}k$; or*

(2) *G is o-2-transitive, and K is 2-transitive; or*

(3) *G is periodically o-primitive with $\bar{\Omega}$-period $\bar{z}$, and $\bar{z}k = k\bar{z}^{-1}$ for any $k \in K'$.*

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