PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 2, 1974

ON COMPLETENESS AND SEMICOMPLETENESS OF FIRST COUNTABLE SPACES

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In this paper, well known completeness conditions in Moore spaces are generalized to arbitrary first countable spaces. Relationships are established between these conditions and various other completeness concepts including Čech completeness, countable completeness, and countable subcompactness. Finally, conditions are given for embedding a given first countable space in a "first countable complete" space. As one application of the theory developed, a necessary and sufficient condition is obtained for the embedding of a Moore space in a semicomplete or "Rudin" complete Moore space.

There has been considerable work done concerning complete Moore spaces, i.e., spaces satisfying R. L. Moore's Axiom 1, and completable Moore spaces, i.e., spaces which are dense subspaces of complete Moore spaces. There has also been much interest in M. E. Estill Rudin's concept of semicomplete Moore spaces, i.e., spaces satisfying Axiom 1″. In this paper the author applies the concepts of completeness and semicompleteness to more general first countable spaces and establishes some theorems involving these concepts. Embedding theorems are also given. The last theorem of the paper answers a question discussed by Steve Armentrout at the Arizona State Topology Conference in 1967 by supplying a necessary and sufficient condition for a Moore space to be a dense subspace of a semicomplete Moore space.

The lower case letters m, n, i, j, and k will denote positive integers unless otherwise stated.

DEFINITION 1. The statement that the sequence $G_1, G_2, G_3, \cdots$ is an *f.c.* development for the space X means that for each n, $G_n = \{g_i(x) \mid i \geqq n, x \in X\}$, where for each $x \in X$, $g_1(x), g_2(x), g_3(x), \cdots$ is a sequence of open sets forming a local base at x.

DEFINITION 2. The *f.c.* development is complete (semicomplete) provided that if $M_1, M_2, M_3, \cdots$ is a sequence of sets such that for each n, M_n is a closed set contained in some element g_n of G_n and contains M_{n+1} (M_n is an element of G_n and contains $\overline{M_{n+1}}$) then $\bigcap_{i=1}^{\infty} M_i \neq \emptyset$. The first countable space is complete (semicomplete) if and only if it has an *f.c.* complete (semicomplete) development.

THEOREM 1. *A T_2 regular first countable semicomplete space X*

satisfies Baire's theorem (the intersection of countably many dense open sets is a dense subset of the space).

Proof. For each $x \in X$, let $g_1(x)$, $g_2(x)$, $\cdots$ denote a local base for x such that the corresponding *f.c.* development $G_1, G_2, \cdots$ is semi-complete. Let $D = d_1, d_2, \cdots$ denote a countable collection of open sets, each dense in X, and let R be an open set. R contains a point p_1 of d_1. Let $g_{n_1}(p_1) \in G_1$ such that $p_1 \in g_{n_1}(p_1)$ and $\overline{g_{n_1}(p_1)} \subseteqq R \cap d_1$. For $i > 1$, let $p_i \in g_{n_{i-1}}(p_{i-1}) \cap d_i$ and let $g_{n_i}(p_i) \in G_i$ such that $p_i \in g_{n_i}(p_i)$ and $\overline{g_{n_i}(p_i)} \subseteqq g_{n_{i-1}}(p_{i-1}) \cap d_i$. Notice that for each i, $g_{n_i}(p_i) \subseteqq d_i \cap R$ and $\overline{g_{n_i}(p_i)} \subseteqq g_{n_{i-1}}(p_{i-1})$. Thus $\bigcap_{i=1}^{\infty} g_{n_i}(p_i) \neq \emptyset$. Since $\bigcap_{i=1}^{\infty} g_{n_i}(p_i) \subseteqq (\bigcap_{i=1}^{\infty} d_i) \cap R$, the intersection of D is a dense subset of the space.

The reader should compare the following two theorems with those analogous theorems of Creede in [4].

THEOREM 2. *A completely regular f.c. complete space X with a G_δ diagonal is Čech complete.*

Proof. Let Y be a T_2 compact space such that X is a dense subspace of Y. For each $x \in X$, let $g_1(x)$, $g_2(x)$, $\cdots$ be a sequence of open sets of X forming a local base at x such that the corresponding *f.c.* development is complete, and $\{x, y\} \subseteqq \bigcap_{i=1}^{\infty} g_i(x_i)$ implies $x = y$. For each $x \in X$ and positive integer i, let $G_i(x)$ be an open set of Y such that $G_i(x) \cap X = g_i(x)$, and let $H_i(x)$ be an open set of Y containing x such that $\overline{H_i(x)} \subseteqq G_i(x)$ (in Y). Notice $\overline{H_i(x) \cap X} \cap X \subseteqq G_i(x) \cap X$. For each positive integer i, let $H_i = \bigcup \{H_i(x) \mid x \in X\}$. Thus H_1, H_2, H_3, $\cdots$ is a sequence of open sets in Y such that $X \subseteqq \bigcap_{i=1}^{\infty} H_i$.

Assume $p \in \bigcap_{i=1}^{\infty} H_i$. For each i, let $H_i(x_i)$ contain p. Let $A_i = \overline{H_i(x_i) \cap X} \cap X$. Since X is dense in Y, $A_1, A_2, A_3, \cdots$ is a sequence of nonempty closed sets of X with the finite intersection property such that for each i, $A_i \subseteqq g_i(x_i)$. Thus there is a point x of X such that $x \in \bigcap_{i=1}^{\infty} A_i$. Assume $x \neq p$. For each i, let R_i be an open set of Y containing p such that $\overline{R_i}$ (in Y) does not contain x and $\overline{R_i} \subseteqq H_i(x_i)$. Let $B_i = \overline{R_i \cap X} \cap X$. As before, $\bigcap_{i=1}^{\infty} B_i$ contains some point k of X. But $\{x, k\} \subseteqq \bigcap_{i=1}^{\infty} g_i(x_i)$, so $x = k$. However, $x \neq k$ since $x \notin \overline{R_i}$ and $k \in B_i \subseteqq \overline{R_i}$. Thus it must be that $x = p$ and $p \in X$. Hence $\bigcap_{i=1}^{\infty} H_i = X$.

THEOREM 3. *A Čech complete first countable space X is f.c. complete.*

Proof. Let Y be a T_2 compact space such that X is a G_δ set in Y. Let $P_1, P_2, \cdots$ be a sequence of open sets of Y such that $X = \bigcap_{i=1}^{\infty} P_i$. For each $x \in X$, let $g_1(x)$, $g_2(x)$, $\cdots$ be a sequence of open sets of X forming a local base at x, such that for each i, $\overline{g_{i+1}(x)} \subseteqq g_i(x)$. For each $x \in X$ and positive integer i, let $G_i(x)$ be an open set in Y such that $g_i(x) = G_i(x) \cap X$, and let $H_i(x)$ be an open set of Y containing x such that $\overline{H_i(x)} \subseteqq G_i(x) \cap P_i$ and $\overline{H_{i+1}(x)} \subseteqq H_i(x)$. Thus $h_1(x), h_2(x), \cdots$ is a local base at x in X where for each i, $h_i(x) = H_i(x) \cap X$. For each positive integer i, let $H_i = \{h_n(x) \mid n \geqq i, x \in X\}$. Thus $H_1, H_2, H_3, \cdots$ is an *f.c.* development for X.

Let $A_1, A_2, A_3, \cdots$ be a monotonically decreasing sequence of closed sets of X such that for each i, $A_i \subseteqq h_{n_i}(x_i) \in H_i$. The sequence $Cl\ Y A_1$, $Cl\ Y A_2$, $Cl\ Y A_3, \cdots$, where $Cl\ Y A_i$ denotes the closure in Y of A_i, is a sequence of closed sets in Y with the finite intersection property; hence, there is a point y of Y such that $y \in \bigcap_{i=1}^{\infty} Cl\ Y A_i$. For each i, $Cl\ Y A_i \subseteqq Cl\ Y(H_{n_i}(x_i) \cap X) \subseteqq P_i$. Thus $y \in \bigcap_{i=1}^{\infty} P_i$ and hence $y \in X$. Thus $H_1, H_2, H_3, \cdots$ is a complete development for X.

In first countable spaces, the concepts of *f.c.* completeness and *f.c.* semicompleteness are related to Frolic's concept of countable completeness [6] and de Groot's concept of countable subcompactness [7]. To avoid confusion the term "countable Čech-completeness", coined by Lutzer and Aarts [1] will be used for the term "countable completeness".

The phrase "C is a centered system on X" means "C is a collection of subsets of X such that for any finite $C_0 \subseteqq C$, $\bigcap C_0 \neq \varnothing$". A collection F of nonempty subsets of X is called a regular filterbase [7] if whenever $F_1, F_2 \in F$, some $F_3 \in F$ has $\bar{F}_3 \subseteqq F_1 \cap F_2$. A regular space X is countably Čech-complete [6] if there is a sequence $\{\hat{B}_n\}$ of open bases for X such that if $n_1 < n_2 < n_3 < \cdots$ and if the sequence $\{B_{n_k}\}$, where $B_{n_k} \in \hat{B}_{n_k}$ forms a centered system, then $\bigcap\{\bar{B}_{n_k}\} = \varnothing$. A regular space is countably subcompact [7] with respect to a base B of open sets provided that any countable regular filterbase $F \subseteqq B$ has $\bigcap F \neq \varnothing$.

THEOREM 4. *In regular first countable spaces, the following implications hold:*

$$(1)\ f.c.\ completeness \Longrightarrow (2)\ countable\ Čech\text{-}completeness \Longrightarrow (3)\ f.c.\ semicompleteness\ .$$

None of the above implications are reversible. In Moore spaces, countable Čech-completeness $\Leftrightarrow$ More completeness $\Leftrightarrow$ *f.c.* completeness [1], and Rudin completeness $\Leftrightarrow$ *f.c.* semicompleteness. M. E. Rudin's example [5] of a Rudin complete space that is not Moore complete shows (3) $\nRightarrow$ (2). The following example shows (2) $\nRightarrow$ (1).

EXAMPLE A. The Sorgenfrey line is the topological space S of real numbers topologized by taking sets of the form $[a, b)$ to be basic open sets. Let $D_1, D_2, D_3, \cdots$ be a monotonic sequence of open sets on the line such that $\bigcap_{i=1}^{\infty} D_i = I$, where I is the set of irrationals. Let $r_1, r_2, r_3, \cdots$ denote the rationals. For each $x \in I$, let $\{[x, x_i)\}$ be a sequence of open sets closing on x such that for each i, x_i is a rational not in $\{r_1, r_2, \cdots, r_i\}$. For each rational x, let $\{[x, x_i)\}$ be a sequence of open sets closing on x such that for each i, x_i is a rational, and $[x, x_i) \cap (\{r_1, r_2, \cdots, r_i\} - \{x\}) = \varnothing$.

For each positive integer n, let $B_n = \{[x, x_i)/x \in S, i \geqq n\}$. Notice that if r is a rational there exists an n such that for $i \geqq n$, $r \in g \in B_i$ implies $g = [r, x_i)$ for some x_i. Let $\{[p_i, q_i)\}$ be a centered sequence such that for each i, $[p_i, q_i) \in B_i$. Let $x \in \bigcap \{[p_i, q_i]\}$. Due to the construction of $\{B_n\}$, $x \in \bigcap \{[p_i, q_i)\}$. Thus S is a first countable, countably Čech-complete space.

Now let $G_1, G_2, G_3, \cdots$ represent any *f.c.* development for S. There must exist a monotonically decreasing sequence having the following properties. (1) the nth term of S belongs to G_n and contains its left endpoint, p_n, and (2) $p_1, p_2, p_3, \cdots$ is an increasing sequence converging to a number x on the line. The sequence $\{[p_i, x)\}$ is a monotonically decreasing sequence of closed sets such that the nth term is a subset of some $g \in G_n$, but has no common part. Thus S is not *f.c.* complete.

The following theorems follow readily.

THEOREM 5. *Any open subspace of an f.c. complete* (*f.c.* semicomplete) *is f.c. complete* (*f.c.* semicomplete).

THEOREM 6. *A G_δ subspace of a regular f.c. complete space is f.c. complete.*

In Moore spaces, Rudin completeness $\Leftrightarrow$ countable subcompactness. In regular first countable spaces, countable subcompactness implies *f.c.* semicompleteness. In [1], Lutzer and Aarts use the term "completeness property" for any property implying the Baire property. Of the several completeness properties they examine, all but countable subcompactness and subcompactness are such that a space may have the property locally but fail to have it globally. The proof they give showing that if a regular space is locally countably subcompact then it is countably subcompact may be slightly altered to yield this theorem.

THEOREM 7. *If a regular first countable space is locally f.c. complete, then it is f.c. semicomplete.*

It would be interesting to know if in first countable spaces, *f.c.*

semicompleteness implies countable subcompactness.

Some technical definitions are needed for the following theorems.

DEFINITIONS. (a) If $G = G_1, G_2, G_3, \cdots$ is a sequence of open sets, the sequence $g_1, g_2, \cdots$ is a nested sequence (g-sequence) wrt G if and only if for each i, $g_i \in G_i$ and contains $\overline{g_{i+1}}$ ($g_i \in G_i$ and contains g_{i+1}). (b) The set sequence $g_1, g_2, \cdots$ is adjacent to the set M if and only if for each i, $g_i \cap M \neq \emptyset$. (c) The set sequences $g_1, g_2, g_3, \cdots$ and $k_1, k_2, k_3, \cdots$ are mutually separated if and only if for some i, $g_i \cap k_i = \emptyset$. (d) The open set D covers the set sequence $g_1, g_2, g_3, \cdots$ if and only if for some i, $g_i \subseteq D$.

THEOREM 8. *Each T_2 first countable space is a dense subspace of an f.c. complete space.*

Proof. Let X be a first countable space. For each $x \in X$, let $g_1(x), g_2(x), \cdots$ be a monotonically decreasing sequence of open sets forming a local base at x, and let $G = G_1, G_2, G_3, \cdots$ be the corresponding *f.c.* development. The statement that $g_1, g_2, \cdots$ is an f-sequence means for each n, $g_n = \bigcap_{i=1}^{n} g'_i$ for some sequence $g'_1, g'_2, g'_3, \cdots$ where for each i, $g'_i \in G_i$. If each of $g = g_1, g_2, g_3, \cdots$ and $k = k_1, k_2, k_3, \cdots$ is an f-sequence then we will say $g \sim k$ if and only if for each positive integer n there exist positive integers i and j such that $k_i \subseteq g_n$ and $g_j \subseteq k_n$. If g is an f-sequence, let g' denote $\{k \mid g \sim k\}$. Let $X' = \{g' \mid g$ is an f-sequence such that each term of g is non-empty$\}$. For each open set D, let $D' = \{g' \mid D$ covers $g\}$. Let $0 = \{D' \mid D$ is open in $X\}$ be a basis for a topology Ω on X.

The ordered pair (X', Ω) is a topological space. X is a dense subspace of X' because X is homeomorphic to the subspace $\{I(x)' \mid x \in X, I(x) = g_1(x), g_2(x), \cdots\}$. X' is first countable. For each $I' \in X'$, where $I = g_1, g_2, \cdots$, and for each positive integer n, let $D_n(I') = g_n$. Thus $D_1(I')', D_2(I')', D_3(I')', \cdots$ forms a local base at I' in X'.

X' is *f.c.* complete. For each positive integer i, let $G'_i = \{D_n(I')' \mid n \geqq i, I' \in X'\}$. Let $M_1, M_2, M_3, \cdots$ be a monotonically decreasing sequence of closed sets of X' such that for each i, $M_i \subseteq D_{n_i}(y'_i)'$ for some $D_{n_i}(y'_i)' \in G'_i$. Since $\bigcap_{i=1}^{m} D_{n_i}(y'_i)' \neq \emptyset$ for each m, then $\bigcap_{i=1}^{m} D_{n_i}(y'_i) \neq \emptyset$. Now for each i, $D_{n_i}(y'_i) = \bigcap_{k=1}^{n_i} g_k(y_i)$ for some sequence $g_1(y_i), g_2(y_i), g_3(y_i), \cdots$ where $g_k(y_i) \in G_k$. Thus $D_{n_i}(y'_i) \subseteq g_i(y_i)$ since $n_i \geqq i$. Since $\bigcap_{i=1}^{m} D_{n_i}(y_i) \neq \emptyset$ for each m, then $\bigcap_{i=1}^{m} g_i(y_i) \neq \emptyset$. Observe that $g_i(y_i) \in G_i$, so $\bigcap_{i=1}^{1} g_i(y_i), \bigcap_{i=1}^{2} g_i(y_i), \bigcap_{i=1}^{3} g_i(y_i), \cdots$ is an f-sequence. Let J denote this f-sequence and examine any open set D' of X' containing J'. D contains $\bigcap_{i=1}^{k} g_i(y_i)$ for some k. So D contains $\bigcap_{i=1}^{k} D_{n_i}(y'_i)$, and thus D' contains $\bigcap_{i=1}^{k} D_{n_i}(y'_i)'$ and M_k. J' is thus a point or limit point of M_n for each n, and hence $J' \in \bigcap_{i=1}^{\infty} M_i$.

Thus, $G'_1, G'_2, G'_3, \cdots$ is an *f.c.* development for X'.

THEOREM 9. *Each regular T_2 space X is a dense subspace of a T_2 f.c. semicomplete space.*

Proof. For each $x \in X$, let $I(x) = g_1(x), g_2(x), \cdots$ be a nested sequence of open sets forming a local base at x. Let $G = G_1, G_2, G_3, \cdots$ be the corresponding *f.c.* development for X.

If I is a nested sequence wrt G, let $I' = \{K \mid K$ is a nested sequence wrt G and $K \sim I\}$ (refer to the technical definitions and the proof of Theorem 8). Notice that $x \neq y$ implies $I(x)$ and $I(y)$ are mutually separated.

Let K denote a maximal collection of mutually separated nested sequences wrt G such that for each x, $I(x) \in K$. *Let $K' = \{k' \mid k \in K\}$. If D is an open set, let $D' = \{k' \mid D$ covers k and $k \in K\}$. Let $B = \{D' \mid D$ is open in $X\}$ be a basis for a topology Ω on K'. X is a dense subspace of the T_2 space (K', Ω).

K' is first countable. For each $I' \in K'$, where $I = g_1, g_2, g_3, \cdots$, and each positive integer n, let $D_n(I') = g_n$. Thus $D_1(')', D_2(I')', \cdots$ forms a local base at I' in K'.

K' is *f.c.* semicomplete. Examine the *f.c.* development for K', $G' = G'_1, G'_2, \cdots$, where for each i, $G'_i = \{D_n(I')' \mid n \geqq i, I' \in K'\}$. Let $M = D_{n_1}(I'_1)', D_{n_2}(I'_2)', \cdots$ be a nested sequence wrt G', i.e., for each i, $\overline{D_{n_{i+1}}(I'_{i+1})'} \subseteqq C_{n_i}(I'_i)'$ and $D_{n_i}(I'_i)' \in G'_i$. Now for each i, $D_{n_i}(I'_i)' = g_{m_i}(x_i)'$ for some $g_{m_i}(x_i) \in G_i$. Since $\overline{g_{m_{i+1}}(x_{i+1})'} \subseteqq g_{m_i}(x_i)'$ in K', then $\overline{g_{m_{i+1}}(x_{i+1})} \subseteqq g_{m_i}(x_i)$ in X. Hence, $g_{m_1}(x_1), g_{m_2}(x_2), g_{m_3}(x_3), \cdots$ is a nested sequence wrt G in X. Since K is maximal, there is an element $I = g_1, g_2, g_3, \cdots$ of K such that for each k, $g_{m_k}(x_k) \cap g_k \neq \emptyset$. Let D' be an open set of K' containing I'. D contains g_i for some i, and in fact D contains g_k for $k \geqq i$. Thus for all k, $D \cap g_{m_k}(x_k) \neq \emptyset$ and $D' \cap g_{m_k}(x'_k)' = D' \cap D_{n_k}(I')' \neq \emptyset$. So $I' \in \overline{D_{n_k}(I'_k)}'$ for each positive integer k. Thus $I' \in \bigcap_{i=1}^{\infty} D_{n_i}(I'_i)'$. Hence, G' is an *f.c.* semicomplete development for K'.

DEFINITION 3. Property R. A T_2 regular first countable space has property R provided that there exists an *f.c.* development $G = G_1, G_2, G_3, \cdots$ and a collection K of mutually separated nested sequences wrt G such that: (1) for each $x \in X$, some element of K forms a local base at x, (2) if g is a nested sequence wrt G, there is an element k of K such that g and k are not mutually separated, and (3) if the open set D covers the sequence $g_1, g_2, g_3, \cdots$ of K, there is an integer n such that D covers any element k of K adjacent to g_n.

THEOREM 10. *A T_2 regular first countable space X with property*

R is a dense subspace of a T_2 regular f.c. semicomplete space.

Proof. Let G and K respectively be the *f.c.* development and collection as assured by property R. That X is a dense subspace of a T_2 *f.c.* semicomplete space can be seen by applying the part of the proof of Theorem 9 following *, using K as defined here.

We will now see that the space K' is regular. Let D' be an open set containing I' of K', where $I = g_1, g_2, g_3, \cdots$. Thus D covers I, and there is an integer n such that if $k = k_1, k_2, \cdots$ of K is adjacent to g_n, then D covers k. Let k' be a limit point of $D_n(I')' = g'_n$. Thus $D_1(k')', D_2(k')', \cdots$ is adjacent to $D_n(I')'$ and so $k_1, k_2, k_3, \cdots$ is adjacent to g_n. Thus D covers k, and $k' \in D'$. So $\overline{D_n(I')'} \subseteqq D'$.

THEOREM 11. *Each T_2 regular f.c. semicomplete space X has property R.*

Proof. Let $G = G_1, G_2, \cdots$ be an *f.c.* semicomplete development for X. For each $x \in X$, let $I(x) = g_1(x), g_2(x), \cdots$ where for each i, $g_i(x) \in G_i$, and $I(x)$ forms a local base at x. Let $K = \{I(x) \mid x \in X\}$.

Let $g = g_1, g_2, \cdots$ be a nested sequence wrt G. Thus for some x, $x \in \bigcap_{i=1}^{\infty} g_i$. So $I(x) \in K$ and g and $I(x)$ are not mutually separated.

Let D be an open set covering $I(x)$ of K. Let n be an integer such that $\overline{g_n(x)} \subseteqq D$. Thus if $I(y)$ is adjacent to $g_n(x)$, $y \in \overline{g_n(x)}$ and thus, $y \in D$. Thus D covers $I(y)$. This completes the proof that X satisfies property R.

DEFINITION. If X is a topological space, the statement that G is a nested development for X means that (1) for each positive integer n, G_n is an open cover of X containing G_{n+1} as a subcover and (2) if U is an open set and p is any point of U there is an integer n such that $p \in g \in G_n$ implies $g \subseteqq U$.

DEFINITION. The topological space X is a Moore space if and only if it is a regular T_2 space with a nested development.

DEFINITION. A complete (semicomplete) Moore space is a Moore space with a nested development G having the property that if $M_1, M_2, M_3, \cdots$ is a sequence of sets such that for each n, M_n is a closed set containing M_{n+1} and M_n is a subset of some element g_n of G_n (M_n is an element of G_n and contains $\overline{M_{n+1}}$), then $\bigcap_{i=1}^{\infty} M_i \neq \varnothing$. Such a development is called a complete (semicomplete) nested development.

Every complete Moore space is semicomplete, but the converse is not true [5]. Not every Moore space is a dense subspace of a complete Moore space [5]. In [9], Whipple provides a necessary and sufficient condition for a Moore space to be completable. Not every Moore space is a dense subspace of a semicomplete Moore space [5]. In his thesis for the University of Iowa, Alzoobaee provides a sufficient condition but it is not known if this condition is necessary for a Moore space to be semicompletable. The following definition and theorem provide a necessary and sufficient condition.

DEFINITION 4. The Moore space X satisfies Axiom K provided that there exists a nested development Q and a collection K of mutually separated g-sequences wrt Q such that: (1) for each $x \in X$, $x \in \cap g$ for some g of K, (2) if d is a nested sequence wrt Q, there is an element g of K such that d and g are not mutually separated, and (3) if the open set D covers the sequence d of K, there is an integer n such that if $g_1, g_2, g_3, \cdots$ is an element of K and g_n covers d, then D covers any element k of K adjacent to g_n. (The reader should refer to definitions a, b, and c for explanation of above terms.)

THEOREM 12. *A Moore space X satisfying Axiom K is a dense subspace of a semicomplete Moore space.*

Proof. Let X be a Moore space with $Q = Q_1, Q_2, Q_3, \cdots$ and K defined as in Definition 4. Form the topological space (K', Ω) as in the paragraph following * in the proof of Theorem 5, using Q for G.

For each $k' \in K'$, where $k = k_1, k_2, k_3, \cdots$, let $g_n(k') = k_n$. As before, $g_1(k')', g_2(k')', \cdots$ forms a local base at k' in K'. For each positive integer n, let $Q'_n = \{g_i(k')' \mid i \geqq n, k' \in K'\}$. $Q' = Q'_1, Q'_2, Q'_3, \cdots$ is a nested development for K'. Let D' be an open set containing k' of K'. Thus D covers $k_1, k_2, k_3, \cdots$ of K. There is an integer n such that if $g_1, g_2, g_3, \cdots$ is an element of K and g_n covers $k_1, k_2, k_3, \cdots$, then D covers any element $m_1, m_2, m_3, \cdots$ of K adjacent to g_n. Let d' be an element of Q'_n containing k'. Now $d = g_t(j')$ for some positive integer $t \geqq n$ and $j \in K$. Also $g_t(j') = j_t$ for some $j_1, j_2, j_3, \cdots$ K. Thus j_t covers k. Assume h', where $h = h_1, h_2, h_3, \cdots$, is a point of limit point of d', i.e., of j'_t. Thus $g_1(h')', g_2(h')', g_3(h')', \cdots$ is adjacent to j'_t, and hence $h_1, h_2, h_3, \cdots$ is adjacent j_t. Thus D must cover $h_1, h_2, h_3, \cdots$ and $h' \in D'$. This shows that K' is regular.

From techniques used in the proof of Theorem 5, we see that K' is T_2, contains X as a dense subspace, and the development Q is a semicomplete nested development for K'. This completes the proof.

THEOREM 13. *A dense subspace of a semicomplete Moore space S satisfies Axiom K.*

Proof. In Theorem 2 of [5], M. E. Rudin proves the following result. If S is a Moore space with nested development G, S' is a subspace of S and G' is a nested development for S', there are sequences $T_1, T_2, T_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ such that (1) T_i and Q_i are subcollections of G_i and G'_i containing T_{i+1} and Q_{i+1} respectively covering S', (2) if $q_1, q_2, q_3, \cdots$ is a sequence such that for each i, $q_i \in Q_i$ and contains q_{i+1}, then there are sequences $w_1, w_2, \cdots$ of increasing integers and $g_1, g_2, g_3, \cdots$ of regions such that for each i, $g_i \in T_i$ and contains $\overline{g_{i+1}}$ and $\overline{q_{w_i}}$ (in S), and (3) if $x_1, x_2, x_3, \cdots$ is a sequence such that for each i, $x_i \in T_i$ and contains x_{i+1}, there are sequences $z_1, z_2, z_3, \cdots$ of increasing integers and $h_1, h_2, h_3, \cdots$ of regions such that for each i, $h_i \in Q_i$ and contains $\overline{h_{i+1}}$ and $\overline{x_{z_i}} \cap S'$.

Let S' be a dense subspace of a semicomplete Moore space S. Let M be a semicomplete nested development for S. Let $G_1 = M_1$. For each positive integer $n \geqq 2$, let $G_n = \{g \mid g \in M_n$ and g is a subset of some element of $G_{n-1}\}$. Thus $G = G_1, G_2, G_3, \cdots$ is also a semicomplete nested development for S. Let G' be the corresponding development for S'. Applying the preceding theorem to S, G, S', and G', we get the sequences $T = T_1, T_2, T_3, \cdots$ and $Q = Q_1, Q_2, Q_3, \cdots$. Notice that Q is a nested development for S'.

For each element x of S such that $x \in \bigcap_{i=1}^{\infty} Cl\, S\, q_i$ for some sequence $q_1, q_2, q_3, \cdots$ where $q_i \in Q_i$ and $q_{i+1} \subseteqq q_i$, let d_x denote one such sequence, $q_1, q_2, q_3, \cdots$. Let $w_1, w_2, w_3, \cdots$ and $g_1(x), g_2(x), \cdots$ be sequence such that $g_i(x) \in T_i$ and $g_i(x)$ contains $\overline{g_{i+1}(x)}$ and $\overline{q_{w_i}}$ in S. Notice $x \in \bigcap_{i=1}^{\infty} g_i(x)$. Now, let $z_1, z_2, z_3, \cdots$ and $h(x) = h_1(x), h_2(x), \cdots$ be sequences such that $h_i(x) \in Q_i$, contains $\overline{h_{i+1}(x)}$ and $\overline{g_{z_i}} \cap S'$. For each n, x is a point or limit point of $h_n(x)$ in S, since for each $k \geqq n$, $g_{z_k}(x)$ contains x and intersects $h_n(x)$.

If $x \neq y$ and h_x and h_y exist, then $h(x)$ and $h(y)$ are mutually separated. Let D_x and D_y be mutually separated open sets in S containing x and y respectively. Let n be a positive integer such that any element of G_n containing x is a subset of D_x and any element of G_n containing y is a subset of D_y. Let $m > n + 1$. Now $x \in \overline{h_m(x)}$ and $h_m(x) \in G'_m$; $y \in \overline{h_m(y)}$ and $h_m(y) \in G'_m$. In S, $\overline{h_m(x)}$ is a subset of some element of G_{m-1} containing x and $\overline{h_m(y)}$ is a subset of some element of G_{m-1} containing y. Thus $h_m(x) \subseteqq D_x$ and $h_m(y) \subseteqq D_y$. So $h_m(x) \cap h_m(y) = \varnothing$.

Let $K = \{h(x) \mid x \in S, h(x) \text{ exists}\}$. Thus K is a collection of mutually separated g-sequences wrt Q, where Q is a nested development for S'.

To examine #3 of Axiom K, let D be an open set of S' covering $h(x) = h_1(x), h_2(x), h_3(x), \cdots$ of K. Let m be the maximal open set in

S such that $m \cap S' = D$. D contains $h_n(x)$ for some n. Thus D contains $\overline{g_{z_n}(x)} \cap S'$, and m contains $g_{z_n}(x)$. Thus m contains x. Let k be a positive integer such that $x \in g \in G_k$ implies $g \subseteqq m$. Let $j > k + 1$. Let $h_1(y), h_2(y), h_3(y), \cdots$ be an element of K such that $h_j(y)$ covers $h(x)$. Since $x \in \overline{h_i(x)}$ in S, for each i, $x \in \overline{h_i(y)}$ in S. Since $\overline{h_j(y)}$ in S is a subset of some element g of G_{j-1}, then $x \in g$ and $g \subseteqq m$. Thus $\overline{h_j(y)} \subseteqq m$ in S, and $y \in m$. Now let $h_1(p), h_2(p), h_3(p), \cdots$ be an element of K adjacent to $h_j(y)$. The point p is an element of $\overline{h_i(p)}$ in S for each i. Let $m_1, m_2, m_3, \cdots$ be a sequence such that for each i, $m_i \in G_i$ and contains $\overline{h_{i+1}(p)}$ in S. Thus for each i, $m_i \cap h_j(y) \neq \emptyset$, since $h_{i+1}(p) \cap h_j(y) \neq \emptyset$. So $p \in \overline{h_j(y)}$ in S. Since $\overline{h_j(y)} \subseteqq m$, $p \in m$. Thus for some i, m contains $h_i(p)$. Hence D contains $h_i(p)$ and covers $h(p)$.

To examine $\sharp 2$ of Axiom K, let $q_1, q_2, q_3, \cdots$ be a nested sequence wrt Q in S'. Thus for each n, $q_n \in Q_n$ and contains $\overline{q_{n+1}}$. Let $w_1, w_2, w_3, \cdots$ and $g_1, g_2, g_3, \cdots$ be such that $g_n \in T_n$ and $g_n \supseteqq \overline{g_{n+1}}$ and $\overline{q_{w_n}}$ in S. Since G is a semicomplete development for S, there is an x such that $x \in \bigcap_{i=1}^{\infty} g_i$. Thus for each i, $x \in \overline{q_i}$ in S since for $j \geqq i$ g_j contains x and intersects q_i. Thus $x \in \bigcap_{i=1}^{\infty} \overline{q_i}$ in S, where $q_i \in Q_i$ and q_i contains q_{i+1}. Examine $h(x)$. Let $q_1(x)', q_2(x)', q_3(x)', \cdots$ be the defining sequence for $h(x)$, i.e., $x \in \bigcap_{i=1}^{\infty} q_i(x)'$ in S and $q_i(x)' \in Q_i$ and $q_{i+1}(x)' \subseteqq q_i(x)'$. The sequence $w_1, w_2, w_3, \cdots$ and $g_1(x), g_2(x), g_3(x), \cdots$ were chosen such that $g_n(x) \supseteqq \overline{g_{n+1}(x)}$ and $\overline{q_{w_n}(x)'}$ in S. Since for each n, $x \in g_n(x)$ and $\overline{x \in q_n(x)}$ in S, then $g_n(x) \cap q_n(x) \neq \emptyset$. Since $h_n(x)$ contains $\overline{g_{z_n}(x)} \cap S'$, we have $h_n(x) \cap q_{z_n}(x) \neq \emptyset$, and hence, $h_n(x) \cap q_n(x) \neq \emptyset$ for each n. Thus there is an element $h(x)$ of K such that $h(x)$ and $q_1, q_2, q_3, \cdots$ are not mutually separated.

This completes the proof that a Moore space is semicompletable if and only if it satisfies Axiom K.

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Received March 30, 1973. This paper was written while the author was under the direction of Professor Phillip L. Zenor.

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