

## THE BACK-AND-FORTH ISOMORPHISM CONSTRUCTION

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Frequently in model theory and occasionally elsewhere, a back-and-forth construction is used to show that any two countable structures satisfying a given relation are isomorphic. Such a construction is used to show that any two countable dense linear orders without end points are isomorphic (Cantor), that any two countable reduced  $p$ -primary abelian groups with the same Ulm invariants are isomorphic (see Kaplansky, *Infinite Abelian Groups*), and that any two countable elementarily equivalent saturated structures are isomorphic (Morley and Vaught). The back-and-forth arguments using these constructions can often be reduced to an application of the following result: If  $R$  is a symmetric relation between countable structures such that (1)  $\mathfrak{A}R\mathfrak{B}$  implies  $\mathfrak{A}$  and  $\mathfrak{B}$  satisfy the same atomic sentences and (2)  $\mathfrak{A}R\mathfrak{B}$  and  $a \in \mathfrak{A}$  implies there is a  $b \in \mathfrak{B}$  such that  $(\mathfrak{A}, a)R(\mathfrak{B}, b)$ , then  $\mathfrak{A}R\mathfrak{B}$  implies  $\mathfrak{A} \cong \mathfrak{B}$ .

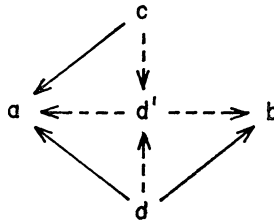
Loosely, the second condition requires that related structures have enough related expansions by constants. We prove a similar result in which the second condition requires, loosely, that related structures have enough similar decompositions into related components. The prototype of our result is a theorem of Vaught's on Boolean algebras mentioned in the last section. In order to suitably formalize "decomposition" we use category theory.

In a subsequent paper, our two theorems will be used to solve the problem of determining when two free locally-finite cylindric algebras are isomorphic. In particular, the answer to both questions in problem 2.8 of [4, p. 463] is no. The answer to the first is yes if  $|\mathcal{A}\xi| = \kappa$  and  $|\mathcal{A}'\xi| = \kappa$  are replaced by  $|\mathcal{A}\xi| \geq \kappa$  and  $|\mathcal{A}'\xi| \geq \kappa$  respectively.

*Constructors and decomposable relations.* We shall use [7] as our notational reference. A *directed* category is a small category generated by a preorder for which every pair of elements has an upper bound. Let  $\underline{C}$  be a category whose arrows are monics. Let  $\underline{C}^*$  be the category whose objects are functors from directed categories to  $\underline{C}$  and for which an arrow from an object  $A: \alpha \rightarrow \underline{C}$  to an object  $B: \beta \rightarrow \underline{C}$  is an ordered pair  $\langle f, \tau \rangle$  such that  $f: \alpha \rightarrow \beta$  is a functor and  $\tau: A \rightarrow f \circ B$  is a natural transformation. The *components* of  $\langle f, \tau \rangle: A \rightarrow B$  are those of  $\tau$ , i.e., the maps  $\tau_i$  for  $i$  in the domain of  $A$ . If  $\langle f,$

$\tau\rangle: A \rightarrow B$  and  $\langle g, \sigma\rangle: A \rightarrow D$  and  $i$  is in the domain of  $A$ , then  $\tau_i$  and  $\sigma_i$  are *corresponding* components. By identifying an object  $a$  of  $\underline{C}$  with the functor  $1 \rightarrow \underline{C}$  whose unique object value is  $a$ , we can consider  $\underline{C}$  to be a subcategory of  $\underline{C}^*$ . For an object  $A$  of  $\underline{C}^*$ , an object  $\bigcup A$  of  $\underline{C}$  is a *colimit* of  $A$  iff there is an arrow  $A \rightarrow \bigcup A$  which is universal from  $A$  to  $\underline{C}$ . Such an arrow is called a *universal cone*. An object  $c$  in  $\underline{C}$  is *compact* iff for every universal cone  $A \rightarrow \bigcup A$ , every  $c \rightarrow \bigcup A$  factors through  $A \rightarrow \bigcup A$ ; an arrow is *compact* iff its domain is. (See the last section for examples.) A category is *compactly generated* iff for any two arrows  $a \rightarrow_1 b$  and  $a \rightarrow_2 b$ ,  $a \rightarrow_1 b \neq a \rightarrow_2 b$  implies  $c \rightarrow a \rightarrow_1 b \neq c \rightarrow a \rightarrow_2 b$  for some compact arrow  $c \rightarrow a$ . An object in  $\underline{C}$  is *countable* iff it is a colimit  $\bigcup C$  of some  $C$  in  $\underline{C}^*$  with countable domain and compact values.

A *partial isomorphism* between objects  $a$  and  $b$  of  $\underline{C}$  is a pair of arrows  $a \leftarrow d \rightarrow b$  in  $\underline{C}$  (recall that arrows of  $\underline{C}$  are monics). Two objects  $a$  and  $b$  are *partially isomorphic*, written  $a \cong_p b$ , iff there is a nonempty set  $I$  of partial isomorphisms between  $a$  and  $b$  such that for every  $a \leftarrow d \rightarrow b$  in  $I$ , every compact arrow to  $a$  or  $b$  factors through an extension in  $I$  of  $a \leftarrow d \rightarrow b$ , i.e.,



is completable with  $a \leftarrow d' \rightarrow b$  in  $I$ . If  $I$  is as above, we write  $I: a \cong_p b$ . This definition is due to Karp [6]; see [1] for details. Two objects are partially isomorphic iff player II has a winning strategy in an appropriate Ehrenfeucht-Fraïssé game.

For any functor  $\varphi: \underline{C} \times \dots \times \underline{C} \rightarrow \underline{C}$  and arrows  $a_1 \rightarrow b_1, \dots, a_n \rightarrow b_n$ , let  $\varphi(a_1, \dots, a_n) \rightarrow \varphi(b_1, \dots, b_n)$  be the arrow  $\varphi(a_1 \rightarrow b_1, \dots, a_n \rightarrow b_n)$ . A functor  $\varphi: \underline{C} \times \dots \times \underline{C} \rightarrow \underline{C}$  is a *constructor* iff for any objects  $a_1, \dots, a_n \in \underline{C}$ , every compact arrow  $c \rightarrow \varphi(a_1, \dots, a_n)$  factors through  $\varphi(c_1, \dots, c_n) \rightarrow \varphi(a_1, \dots, a_n)$  for some compact arrows  $c_1 \rightarrow a_1, \dots, c_n \rightarrow a_n$ . Zero-ary functors  $1 \rightarrow \underline{C}$  are also regarded as constructors. Evidently the identity functor on  $\underline{C}$  is a constructor and a composition of constructors is a constructor. For any set  $\Phi$  of constructors let  $\bar{\Phi}$  be the smallest set including  $\Phi$  and the identity functor on  $\underline{C}$  and closed under composition.

It is not hard to verify that if every object of  $\underline{C}$  is a directed colimit of compacts, then  $\varphi$  is a constructor iff  $\varphi$  preserves directed colimits.

Suppose each component of  $\underline{C}$  has an initial object. Let  $\Phi$  be a set of constructors on  $\underline{C}$ . For any  $\varphi \in \Phi$ , any objects  $a_1, \dots, a_n \in \underline{C}$  whose components have initial objects  $p_1, \dots, p_n$  respectively, let  $\varphi(p_1, \dots, p_n) \rightarrow \varphi(a_1, \dots, a_n)$  be  $\varphi(p_1 \rightarrow a_1, \dots, p_n \rightarrow a_n)$  where  $p_i \rightarrow a_i$  is the unique arrow from  $p_i$  to  $a_i$ . A symmetric binary relation  $R$  on the objects of  $\underline{C}$  is  $\Phi$ -decomposable iff for any objects  $a$  and  $b$  in  $\underline{C}$  (1)  $aRb$  implies  $a$  and  $b$  are in the same component of  $\underline{C}$  and (2)  $aRb$  and  $c \rightarrow a$  compact implies there is a  $\varphi$  in  $\bar{\Phi}$ ,  $a_1, \dots, a_n, b_1, \dots, b_n$  in  $\underline{C}$ , and isomorphisms  $a \cong \varphi(a_1, \dots, a_n)$  and  $\varphi(b_1, \dots, b_n) \cong b$  such that  $a_1Rb_1, \dots, a_nRb_n$ , and  $c \rightarrow a$  factors through  $\varphi(p_1, \dots, p_n) \rightarrow \varphi(a_1, \dots, a_n) \cong a$  where  $p_i$  is an initial object of the common component of  $a_i$  and  $b_i$ . (See the last section for examples.)

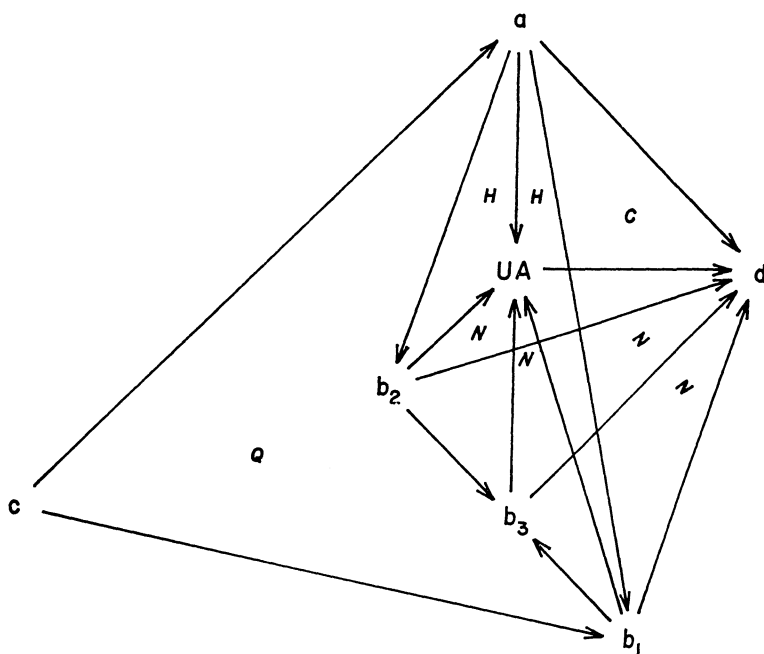
*The back-and-forth theorems.* The following lemma is related to the final functor theorem [7, p. 213].

LEMMA 1. *For any compactly generated category  $\underline{C}$  of monics, any  $A$  in  $\underline{C}^*$  with universal cone  $A \rightarrow \bigcup A$ , and any  $B \rightarrow \bigcup A$  in  $\underline{C}^*$ , if every component of  $A \rightarrow \bigcup A$  factors through a component of  $B \rightarrow \bigcup A$ , then  $\bigcup A$  is a colimit of  $B$ .*

*Proof.* Suppose  $\underline{C}$ ,  $A \rightarrow \bigcup A$ , and  $B \rightarrow \bigcup A$  are as hypothesized. We claim that  $B \rightarrow \bigcup A$  is a universal cone. Given  $B \rightarrow d$ , let  $A \rightarrow d$  be the unique arrow such that for any  $A \rightarrow \bigcup A$  component  $a \rightarrow \bigcup A$ , the corresponding  $A \rightarrow d$  component is  $a \rightarrow d = a \rightarrow b \rightarrow d$  where (\*)  $b \rightarrow d$  is a  $B \rightarrow d$  component and  $a \rightarrow b$  is an arrow such that  $a \rightarrow b \rightarrow \bigcup A = a \rightarrow \bigcup A$  where  $b \rightarrow \bigcup A$  is the  $B \rightarrow \bigcup A$  component corresponding to  $b \rightarrow d$ . To show that  $a \rightarrow d$  is independent of the choice of  $b$ , let  $a \rightarrow b_1 \rightarrow d$  and  $a \rightarrow b_2 \rightarrow d$  satisfy (\*). Thus, in the diagram below with  $c \rightarrow a, c \rightarrow b_1$ , and  $\bigcup A \rightarrow d$  omitted, the triangles  $H$  commute by hypothesis. Since  $B$  has a directed domain, there are arrows  $b_1 \rightarrow b_3 \leftarrow b_2$  and components  $b_3 \rightarrow \bigcup A$  and  $b_3 \rightarrow d$  such that the triangles  $N$  commute by naturality.

Hence  $a \rightarrow b_1 \rightarrow b_3 \rightarrow \bigcup A = a \rightarrow b_2 \rightarrow b_3 \rightarrow \bigcup A$  and, since  $b_3 \rightarrow \bigcup A$  is monic,  $a \rightarrow b_1 \rightarrow b_3 = a \rightarrow b_2 \rightarrow b_3$ . Hence  $a \rightarrow b_1 \rightarrow d = a \rightarrow b_2 \rightarrow d$ . Finally, these components “add up” to an arrow (cone)  $A \rightarrow d$  since if  $A$  assigns  $a \rightarrow a'$  to an arrow of its domain, if  $a \rightarrow \bigcup A$  and  $a' \rightarrow \bigcup A$  are the associated  $A \rightarrow \bigcup A$  components and if  $a \rightarrow d = a \rightarrow b \rightarrow d$  and  $a' \rightarrow d = a' \rightarrow b' \rightarrow d$  are the corresponding arrows satisfying (\*); then, letting  $a \rightarrow b' = a \rightarrow a' \rightarrow b'$  and noting that  $a \rightarrow b' \rightarrow d$  satisfies (\*), we have  $a \rightarrow a' \rightarrow d = a \rightarrow a' \rightarrow b' \rightarrow d = a \rightarrow b' \rightarrow d = a \rightarrow d$ , the last equality due to the independence of  $a \rightarrow d$  from the choice of  $b$ .

By universality of  $A \rightarrow \bigcup A$ , there is a  $\bigcup A \rightarrow d$  such that  $A \rightarrow$

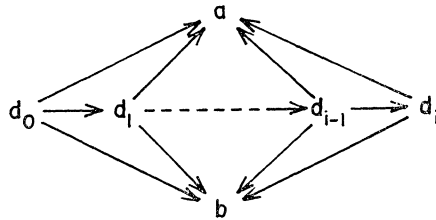


$\bigcup A \rightarrow d = A \rightarrow d$ . We show that  $B \rightarrow \bigcup A \rightarrow d = B \rightarrow d$ . Since  $\underline{C}$  is compactly generated, it suffices to show that  $c \rightarrow b \rightarrow \bigcup A \rightarrow d = c \rightarrow b \rightarrow d$  for every pair  $b \rightarrow \bigcup A$  and  $b \rightarrow d$  of corresponding components of  $B \rightarrow \bigcup A$  and  $B \rightarrow d$  and every compact arrow  $c \rightarrow b$ . Suppose  $b_1 \rightarrow \bigcup A$  and  $b_1 \rightarrow d$  are corresponding components of  $B \rightarrow \bigcup A$  and  $B \rightarrow d$  and  $c \rightarrow b_1$  is compact. By compactness  $c \rightarrow b_1 \rightarrow \bigcup A = c \rightarrow a \rightarrow \bigcup A$  for some  $A \rightarrow \bigcup A$  component  $a \rightarrow \bigcup A$ . Let  $a \rightarrow d$  be the  $A \rightarrow d$  component corresponding to  $a \rightarrow \bigcup A$ . Then by definition of  $A \rightarrow d$ ,  $a \rightarrow d = a \rightarrow b_2 \rightarrow d$  for some  $B \rightarrow d$  component  $b_2 \rightarrow d$  and some  $a \rightarrow b_2$  such that  $a \rightarrow b_2 \rightarrow \bigcup A = a \rightarrow \bigcup A$ . Thus in the diagram above, this time with  $a \rightarrow b_1$  omitted, the triangle  $H$  commutes by hypothesis and the triangle  $C$  by choice of  $\bigcup A \rightarrow d$ . Since  $B$  has a directed domain, there are arrows  $b_1 \rightarrow b_3 \leftarrow b_2$  and components  $b_i \rightarrow \bigcup A$  and  $b_i \rightarrow d$  such that the triangles  $N$  commute by naturality. Now  $c \rightarrow b_1 \rightarrow b_3 \rightarrow \bigcup A = c \rightarrow b_1 \rightarrow \bigcup A = c \rightarrow a \rightarrow \bigcup A = c \rightarrow a \rightarrow b_2 \rightarrow b_3 \rightarrow \bigcup A$  and  $b_3 \rightarrow \bigcup A$  is monic so  $c \rightarrow b_1 \rightarrow b_3 = c \rightarrow a \rightarrow b_2 \rightarrow b_3$  and the quintilateral  $Q$  commutes. Using the commutativity of the triangles  $H$ ,  $N$ ,  $C$ , and  $ab_2d$  and the quadrilateral  $cb_1\bigcup Aa$ ,  $c \rightarrow b_1 \rightarrow \bigcup A \rightarrow d = c \rightarrow a \rightarrow \bigcup A \rightarrow d = c \rightarrow a \rightarrow d = c \rightarrow a \rightarrow b_2 \rightarrow d = c \rightarrow a \rightarrow b_1 \rightarrow b_3 \rightarrow d = c \rightarrow b_1 \rightarrow b_3 \rightarrow d = c \rightarrow b_1 \rightarrow d$ . Hence  $B \rightarrow \bigcup A \rightarrow d = B \rightarrow d$ .

To see that  $\bigcup A \rightarrow d$  is the unique such arrow, use the same diagram, this time with  $c$ ,  $b_1$ , and  $b_3$  omitted and  $\bigcup A \rightarrow d$  replaced by  $\bigcup A \rightarrow' d$ , to show that if  $B \rightarrow \bigcup A \rightarrow' d = B \rightarrow d$ , then  $A \rightarrow \bigcup A \rightarrow' d = A \rightarrow d$  and hence, by the universality of  $A \rightarrow \bigcup A$ ,  $\bigcup A \rightarrow' d = \bigcup A \rightarrow d$ .

**THEOREM 2.** (Countable and  $\cong_p$  implies  $\cong$ .) *If  $a$  and  $b$  are countable objects in a compactly generated category of monics, then  $a \cong_p b$  implies  $a \cong b$ .*

*Proof.* Let  $A \rightarrow a$  and  $B \rightarrow b$  be universal cones from functors with countable domains and compact values; let  $a_1 \rightarrow a, a_3 \rightarrow a, a_5 \rightarrow a, \dots$  and  $b_2 \rightarrow b, b_4 \rightarrow b, \dots$  be enumerations of their (compact) components; and let  $I: a \cong_p b$ . Let  $a \leftarrow d_0 \rightarrow b$  be any element of  $I$ . Given arrows  $a \leftarrow d_k \rightarrow b$  in  $I$  and  $d_{k-1} \rightarrow d_k, k = 1, \dots, i - 1$ , such that the solid part of the diagram below commutes, let  $a \leftarrow d_i \rightarrow b$  in  $I$  be an extension of  $a \leftarrow d_{i-1} \rightarrow b$  via some  $d_{i-1} \rightarrow d_i$  such that  $a_i \rightarrow a$ , if  $i$  is odd, or  $b_i \rightarrow b$ , if  $i$  is even, factors through  $a \leftarrow d_i \rightarrow b$ .



Let  $D$  in  $\underline{C}^*$  be the functor from the directed category  $\omega$  for which  $D(i) = d_i$  and  $D(i \rightarrow i + 1) = d_i \rightarrow d_{i+1}$ , i.e.,  $D = "d_0 \rightarrow d_1 \rightarrow d_2 \dots"$ , and let  $D \rightarrow a$  be the arrow of  $\underline{C}^*$  whose  $i$ th component is  $d_i \rightarrow a$ . By construction, every component of  $A \rightarrow a$  factors through a component of  $D \rightarrow a$  and so, by Lemma 1,  $a$  is a colimit of  $D$ . Likewise  $b$  is a colimit of  $D$  and hence  $a \cong b$ .

**THEOREM 3.** (Decomposably related implies  $\cong_p$ .) *If every component of the category  $\underline{C}$  has an initial object, if  $\Phi$  is a set of constructors on  $\underline{C}$ , and if  $R$  is a  $\Phi$ -decomposable relation, then  $aRb$  implies  $a \cong_p b$ .*

*Proof.* Suppose  $\Phi$  and  $R$  are as hypothesized and suppose  $aRb$ . Let  $I$  be the set of partial isomorphisms from  $a$  to  $b$  of the form  $a \cong \varphi(a_1, \dots, a_n) \leftarrow \varphi(p_1, \dots, p_n) \rightarrow \varphi(b_1, \dots, b_n) \cong b$ , where  $\varphi$  is in  $\bar{\Phi}$ ,  $a_i R b_i, p_i$  is an initial object of the common component of  $a_i$  and  $b_i, a \cong \varphi(a_1, \dots, a_n)$  and  $\varphi(b_1, \dots, b_n) \cong b$  are isomorphisms, and for  $d_i$  in the component of  $p_i$  and  $p_i \rightarrow d_i$  the unique arrows,  $\varphi(p_1, \dots, p_n) \rightarrow \varphi(d_1, \dots, d_n)$  is  $\varphi(p_1 \rightarrow d_1, \dots, p_n \rightarrow d_n)$ .

If  $p$  is an initial object of the common component of  $a$  and  $b$ , then, since the identity is in  $\bar{\Phi}, a \leftarrow p \rightarrow b$  is in  $I$  and hence  $I$  is nonempty. If  $a \cong \varphi(a_1, \dots, a_n) \leftarrow \varphi(p_1, \dots, p_n) \rightarrow \varphi(b_1, \dots, b_n) \cong b$  is in  $I$  and  $c \rightarrow a$  compact, then, since  $\varphi$  is a constructor, the compact arrow  $c \rightarrow \varphi(a_1, \dots, a_n) = c \rightarrow a \cong \varphi(a_1, \dots, a_n)$  factors through  $\varphi(c_1,$

$\dots, c_n) \rightarrow \varphi(a_1, \dots, a_n)$  for some compact  $c_i \rightarrow a_i$ . Since  $R$  is  $\Phi$ -decomposable, each  $c_i \rightarrow a_i$  factors through  $\varphi(p_{i1}, \dots, p_{ip_i}) \rightarrow \varphi_i(a_{i1}, \dots, a_{in_i}) \cong a_i$  for some  $\varphi_i$  in  $\bar{\Phi}$ , some  $a_{ij}$  and  $b_{ij}$  with  $a_{ij} R b_{ij}$  and some  $a_i \cong \varphi_i(a_{i1}, \dots, a_{in_i})$  and  $\varphi_i(b_{i1}, \dots, b_{in_i}) \cong b_i$ . Hence  $c \rightarrow a$  factors through  $a \cong \varphi(a_1, \dots, a_n) \cong \varphi(\varphi_1(a_{11}, \dots, a_{1n_1}), \dots, \varphi_n(a_{n1}, \dots, a_{nn_n})) \leftarrow \varphi(\varphi_1(p_{11}, \dots, p_{1n_1}), \dots, \varphi_n(p_{n1}, \dots, p_{nn_n})) \rightarrow \varphi(\varphi_1(b_{11}, \dots, b_{1n_1}), \dots, \varphi_n(b_{n1}, \dots, b_{nn_n})) \cong \varphi(b_1, \dots, b_n) \cong b$  which is in  $I$  and which extends  $a \cong \varphi(a_1, \dots, a_n) \leftarrow \varphi(p_1, \dots, p_n) \rightarrow \varphi(b_1, \dots, b_n) \cong b$  via  $\varphi(p_1, \dots, p_n) \rightarrow \varphi(\varphi_1(p_{11}, \dots, p_{1n_1}), \dots, \varphi_n(p_{n1}, \dots, p_{nn_n}))$ . Hence  $I: a \cong_p b$ .

Suppose for the remainder of the section that  $\underline{C}$  has an initial object  $p$ . Then condition (1) of the  $\Phi$ -decomposability definition is trivial.

**DEFINITION.** With respect to a given set  $\Phi$  of constructors on  $\underline{C}$ , an object  $b$  is a *factor* of an object  $a$  iff  $a \cong \varphi(a_1, \dots, a_n, b)$  for some  $a_1, \dots, a_n$  and some  $\varphi \in \bar{\Phi}$ .  $\underline{C}$  is  $\Phi$ -decomposable iff every compact arrow  $c \rightarrow a$  factors through  $\varphi(p, \dots, p) \rightarrow \varphi(a_1, \dots, a_n) \cong a$  for some  $a_1, \dots, a_n$ , some  $\varphi \in \bar{\Phi}$ , and some  $\varphi(a_1, \dots, a_n) \cong a$ .

**NOTE.**  $\underline{C}$  is  $\Phi$ -decomposable iff the equality relation on  $\underline{C}$  is  $\Phi$ -decomposable.

**DEFINITION.** For any set  $\Phi$  of constructors and any object  $a$ , the  $\Phi$ -factor algebra of  $a$  is the partial algebra consisting of isomorphism types of factors of  $a$  under the operations of  $\Phi$  plus a constant for the isomorphism type of  $a$ . (See the next section for examples.)

**COROLLARY 4.** If  $\underline{C}$  is  $\Phi$ -decomposable and objects  $a$  and  $b$  have isomorphic factor algebras, then  $a \cong_p b$ .

*Proof.* Let  $R$  be the relation on  $\underline{C}$  such that  $aRb$  iff  $a$  and  $b$  have isomorphic  $\Phi$ -factor algebras. Suppose  $aRb$  and  $c \rightarrow a$  is a compact arrow. Then by decomposability of  $\underline{C}$ ,  $c \rightarrow a$  factors through  $\varphi(p, \dots, p) \rightarrow \varphi(a_1, \dots, a_n) \cong a$  for some  $\varphi \in \bar{\Phi}$  and some  $\varphi(a_1, \dots, a_n) \cong a$ . Let  $f$  be an isomorphism from the  $\Phi$ -factor algebra of  $a$  to that of  $b$ , let  $\bar{a}_i$  be the isomorphism type of  $a_i$ , and let  $b_i$  be of the type  $f(\bar{a}_i)$ . Then  $b \cong \varphi(b_1, \dots, b_n)$  and  $a_i R b_i$ . Hence  $R$  is  $\Phi$ -decomposable and, by Theorem 3,  $aRb$  implies  $a \cong_p b$ .

**EXAMPLES.** In the following we give examples of constructors, decomposable relations, and factor algebras in the categories of sets, linear orders, Boolean algebras, bordered surfaces, and structures of a first order language.

*Sets.* Let **Set** be the category of sets and 1 – 1 functions. The empty set is the initial object and the finite and countable sets are the compact and countable objects respectively. Let  $\varphi: \mathbf{Set} \rightarrow \mathbf{Set}$  be the functor such that  $\varphi(X) = X \cup \{X\}$  for any set  $X$  and  $\varphi(f) = f \cup \{\langle \text{domain } f, \text{range } f \rangle\}$  for any function  $f$ . Then  $\varphi$  is a constructor. If  $R$  is a relation on sets such that  $XY$  iff  $X$  and  $Y$  have the same finite cardinality or are both infinite, then  $R$  is  $\{\varphi\}$ -decomposable. The  $\{\varphi\}$ -factor algebra of the set  $\{0, 1, \dots, n - 1\}$  is  $\langle 0, 1, \dots, n, S, n \rangle$  where  $S(i)$  is  $i + 1$  if  $i < n$  and undefined if  $i = n$ . The  $\{\varphi\}$ -factor algebra of  $\omega = \{0, 1, \dots\}$  is (isomorphic to)  $\langle \omega \cup \{\infty\}, S, \infty \rangle$  where  $S(\alpha) = \alpha + 1$  if  $\alpha \in \omega$  and  $\infty$  if  $\alpha = \infty$ .

*Linear orders.* Let **Lin** be the category of linear orders and 1 – 1 order preserving functions. The empty order is the initial object and the finite and countable orders are the compact and countable objects respectively. Let  $\varphi: \mathbf{Lin} \times \mathbf{Lin} \rightarrow \mathbf{Lin}$  be a functor such that  $\varphi(K, L)$  is an order consisting of an initial segment of type  $K$  followed by a new point followed by a final segment of type  $L$  and for functions  $f$  and  $g$ ,  $\varphi(f, g)$  is, essentially,  $f$  on the first segment, the new point goes to the new point, and  $g$  on the final segment. Then  $\varphi$  is a constructor. If  $R$  is the relation such that  $KRL$  iff  $K$  and  $L$  are both dense linear orders without end points, then  $R$  is  $\{\varphi\}$ -decomposable. If  $R$  is any symmetric relation on linear orders such that (1)  $KRL$  implies  $K$  is empty iff  $L$  is and (2)  $KRL$  and  $a \in K$  implies there is a  $b \in L$  such that  $\langle \{x \in K: x <_K a\}, <_K \rangle R \langle \{x \in L: x <_L b\}, <_L \rangle$  and  $\langle \{x \in K: x >_K a\}, <_K \rangle R \langle \{x \in L: x >_L b\}, <_L \rangle$ , then  $R$  is  $\{\varphi\}$ -decomposable. The  $\{\varphi\}$ -factor algebra of the empty order is  $\langle \{1\}, +, 1 \rangle$  where  $1 + 1$  is undefined. The  $\{\varphi\}$ -factor algebra of the rationals with the usual order is  $\langle \{\infty\}, +, \infty \rangle$  where  $\infty + \infty = \infty$ .

*Boolean algebras.* Let **Bool** be the category of Boolean algebras with two or more elements and monomorphisms. The two-element Boolean algebra is the initial object and the finite and countable Boolean algebras are the compact and countable objects respectively. Let  $\varphi: \mathbf{Bool} \times \mathbf{Bool} \rightarrow \mathbf{Bool}$  be the cartesian product functor. Then  $\varphi$  is a constructor. If  $R$  is the relation such that  $ARB$  iff  $A$  and  $B$  are atomless Boolean algebras, then  $R$  is  $\{\varphi\}$ -decomposable. If  $R$  is any symmetric relation on Boolean algebras such that (1)  $ARB$  implies  $A$  is the two-element algebra iff  $B$  is and (2)  $ARB$  and  $a \in A$  implies there is a  $b \in B$  such that  $A[a]RB[b]$  and  $A[-a]RB[-b]$  where  $A[a]$  is the ideal (considered as a Boolean algebra) generated by  $a$ , then  $R$  is  $\{\varphi\}$ -decomposable. Relations of this type were first studied by Vaught [13] who showed that any two countable Boolean algebras related by such a relation were isomorphic. Decomposability is

simply a categorical generalization of conditions (1) and (2). The  $\{\varphi\}$ -factor algebras of the two-element algebra and the atomless algebra are the partial algebras  $\langle\{1\}, +, 1\rangle$  and  $\langle\{\infty\}, +, \infty\rangle$  respectively defined in the previous example. For primitive Boolean algebras,  $\{\varphi\}$ -factor algebras are nothing more than algebraic versions of the structure diagrams of Hanf [3] and Pierce [10] which have played an important role in classifying such Boolean algebras and determining isomorphism types of Lindenbaum-Tarski algebras [3, 9, 12].

*Bordered surfaces.* Let **Sur** be the category whose objects are bordered orientable surfaces (orientable two-dimensional manifolds whose boundary is a disjoint union of simple closed curves) each with a designated homeomorphism from the closed unit disc onto a subspace of the surface's interior and whose arrows are homeomorphisms from one surface onto a subsurface of a second such that the designated map of the first composed with the arrow is the designated map of the second and such that a boundary curve of the first is either carried onto a boundary curve of the second or into the interior of the second. The closed disc with the identity map as the designated map is the initial object, the compact bordered surfaces are the compact objects (this is false if the boundary curve condition on arrows is dropped), and the separable bordered surfaces are the countable objects. Let  $C$  be a cylinder — finite length, closed, and open at both ends — with a designated (map from the) disc in its interior. Let  $\varphi: \mathbf{Sur} \times \mathbf{Sur} \rightarrow \mathbf{Sur}$  be a functor such that for any objects  $S$  and  $T$ ,  $\varphi(S, T)$  is a surface obtained by cutting out the interiors of the designated disc of  $S$  and  $T$  and gluing their perimeters to the open ends of  $C$  ( $C$  becomes a tube joining the two surfaces) and letting the designated disc of  $C$  be the designated disc of the result; and such that for any arrows  $f$  and  $g$ ,  $\varphi(f, g)$  is, essentially,  $f$  on the first surface,  $g$  on the second, and the identity on  $C$ . Let  $\sigma$  and  $\tau$  be the 0-ary functors whose unique values are the sphere and torus respectively with designated discs. Then  $\varphi$ ,  $\sigma$ , and  $\tau$  are constructors. If  $R$  is the relation between surfaces such that  $SRT$  iff  $S$  and  $T$  have the same number of boundary components, the same genus, and homeomorphic ideal boundaries (see [11]), then  $R$  is  $\{\varphi, \sigma, \tau\}$ -decomposable. The  $\{\varphi, \sigma, \tau\}$ -factor algebra of the surface of a solid semi-infinite rod is  $\langle\{s, t, 1\}, +, s, t, 1\rangle$  where  $s + s = s$ ,  $s + t = t + s = t$ ,  $s + 1 = 1 + s = 1$ , and all other sums are undefined. The  $\{\varphi, \sigma, \tau\}$ -factor algebra of the surface of an infinite binary tree with solid branches of nonzero width is  $\langle\{s, t, \infty\}, +, s, t, \infty\rangle$  where  $s + s = s$ ,  $s + t = t + s = t$ ,  $s + \infty = \infty + s = \infty$ ,  $\infty + \infty = \infty$ , and all other sums are undefined.

*Structures.* Given a first-order language with at least one constant



symbol, let  $\mathbf{Str}$  be the category whose objects are finite expansions of structures of the language via constants, i.e., structures of the form  $(\mathfrak{A}, a_1, \dots, a_n)$  where  $\mathfrak{A}$  is a structure of the language and  $a_1, \dots, a_n \in \mathfrak{A}$ , and whose arrows are isomorphisms from one structure to a substructure of a second, i.e., monomorphisms which preserve quantifier-free formulas. For any object  $\mathfrak{B}$  of  $\mathbf{Str}$  the initial object of its component is Prime ( $\mathfrak{B}$ ), the substructure of  $\mathfrak{B}$  generated by its constants. Finitely generated structures and countably generated structures are the compact and countable objects respectively. Let  $\varphi: \mathbf{Str} \rightarrow \mathbf{Str}$  be the functor such that for any structure  $\mathfrak{A}$  of the language  $\varphi(\mathfrak{A}) = \mathfrak{A}$  and for any  $a_1, \dots, a_n \in \mathfrak{A}$ ,  $\varphi((\mathfrak{A}, a_1, \dots, a_n)) = (\mathfrak{A}, a_1, \dots, a_{n-1})$ . Then  $\varphi$  is a constructor. If  $R$  is the relation such that  $\mathfrak{A}R\mathfrak{B}$  iff  $\mathfrak{A}$  and  $\mathfrak{B}$  are elementarily equivalent  $\omega$ -saturated structures, then  $R$  is  $\{\varphi\}$ -decomposable. If  $R$  is a symmetric relation such that (1)  $\mathfrak{A}R\mathfrak{B}$  implies Prime  $\mathfrak{A} \cong$  Prime  $\mathfrak{B}$  and (2)  $\mathfrak{A}R\mathfrak{B}$  and  $a \in \mathfrak{A}$  implies there is a  $b \in \mathfrak{B}$  such that  $(\mathfrak{A}, a)R(\mathfrak{B}, b)$ , then  $R$  is  $\{\varphi\}$ -decomposable.

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