PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 2, 1974

ON SOME GROUP ALGEBRA MODULES RELATED TO WIENER'S ALGEBRA M_1

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Along with his study of the general Tauberian theorem in L_1, N. Wiener introduced the algebra M_1 which consists of all those continuous functions f on the real line R for which

$$\sum_{n=-\infty}^{\infty} \max_{x\in[n,n+1]} |f(x)| < \infty .$$

He proved that many features of L_1, including the general Tauberian theorem, are shared by M_1. In this paper to generalize M_1 to an arbitrary locally compact group G. While doing this, a host of $L_1(G)$-modules mutually related by conjugation and the operation of forming multiplier modules. $\mathscr{M}_1(G)$ is among them. In case G is abelian, $\mathscr{M}_1(G)$ is a Segal algebra, so that it has the same ideal-theoretical structure as $L_1(G)$. If further $G = R$, $\mathscr{M}_1(G)$ reduces to the Wiener algebra M_1 with an equivalent norm.

1. Our notations are basically the same as those used in [3]. We use, however, C to denote the complex number field. Throughout the paper, G is a locally compact group with a left Haar measure λ. Instead of $C_{00}(G)$, $L_p(G)$ etc. we write C_{00}, L_p etc. We view L_1 as a subspace of M. We identify two functions that are equal almost everywhere.

For a function f on G define f' by

$$f'(x) = f(x^{-1})\Delta(x^{-1}) ,$$

where Δ denotes the modular function of G. Then $f'' = f$ and $(f*g)' = g'*f'$ for $f, g \in L_1$.

If B is a left Banach module over L_1 (see [3; 32.14]), then B^* becomes a left Banach module by

$$(j, f*\phi) = (f'*j, \phi) \qquad (j \in B; \phi \in B^*; f \in L_1) .$$

If $B = L_p(1 \leqq p < \infty)$ or $B = C_0$ the module operation on B^* coincides with the convolution operation on $L_q(q = p/(p-1))$ or M.

Let B be a left Banach module over L_1. By [3; 32.22], $\{f*j: f \in L_1; j \in B\}$ is a closed submodule of B. We denote this submodule by L_1*B or B_{abs}. We call B *absolutely continuous* if $B_{abs} = B$.

Suppose B is a Banach space, and there is a map $(j, x) \mapsto j_x$ of $B \times G$ into B such that

(1) $j_1 = j (j \in B)$,

(2) $(j_x)_y = j_{xy}(j \in B;\, x,\, y \in G)$,
(3) for every $x \in G$, $j \mapsto j_x$ is a linear isometry,
(4) for every $j \in B$, $x \mapsto j_x$ is a continuous map $G \to B$.

Then $(j, x) \mapsto j_x$ is called a *continuous shift* in B. Such a continuous shift makes B into an absolutely continuous left Banach module over L_1 by

$$(f * j, \phi) = \int f(x)(j_{x^{-1}}, \phi)dx,\ (f \in L_1;\, j \in B;\, \phi \in B^*)\ .$$

For details, see [2], [4].

We can define continuous shifts in $L_p(1 \leqq p < \infty)$ and C_0 by

$$j_x(y) = j(xy) \quad (j \in L_p \text{ or } C_0;\, x,\, y \in G)\ .$$

The resulting module operation is ordinary convolution.

Let B be a left Banach module over L_1. The continuous module-homomorphism $L_1 \to B$ (the multipliers of B) form a Banach space Mult B that can be turned into a left Banach module by

$$(f * T)(g) = T(g * f) \quad (f,\, g \in L_1;\, T \in \text{Mult}\, B)\ .$$

Every $j \in B$ induces the multiplier $f \mapsto f * j$. The following theorem is essentially due to Rieffel [6] and is proved in [2] as Theorem 5.2:

THEOREM 1.1. *Let B be an absolutely continuous module. For $\phi \in B^*$ let T_ϕ be the multiplier $f \mapsto f * \phi$ of B^*. Then T is a module isomorphism and a linear homeomorphism of B^* onto* Mult B^*.

A *Radon measure* on G is a linear functional $\mu: C_{00} \to \boldsymbol{C}$ such that for every compact set $C \subset G$ there exists a number c such that

$$|(j, \mu)| \leqq c\, \|j\|_\infty \qquad (j \in C_{00},\, \text{Supp}\, j \subset C)\ .$$

The Radon measures form a vector space which we denote by R.

For $\mu \in R$ and for an $f \in L_1$ with compact support we define $f * \mu \in R$ and $\mu * f \in R$ by

$$\left.\begin{aligned}(j, f * \mu) &= (f' * j, \mu)\\ (j, \mu * f) &= (j * \Delta f', \mu)\end{aligned}\right\}(j \in C_{00})\ .$$

These formulas reduce to the familiar convolution formulas in case $\mu \in L_p$.

Every Radon measure is a linear combination of positive Radon measures [1]. Thus, if $\mu \in R$ and if X is a relatively compact Borel subset of G, we can in a natural way define $\mu(X)$. Further, if $\mu \in R$ and if A is any Borel set there is a unique $\xi_A\mu \in R$ such that $\xi_A\mu(X) = \mu(X \cap A)$ for all relatively compact Borel sets X. There exists a unique $|\mu| \in R$ such that $\xi_A|\mu| = |\xi_A\mu|$ for every compact set A (see [1; Ch. 13]).

By the Radon-Nikodým Theorem [3; 12.17] we may identify $L_{1,\mathrm{loc}}$ with $\{\mu \in R: \mu \ll \lambda\}$.

THEOREM 1.2. *If $f \in L_1$ has compact support and if $\mu \in R$, then $f * \mu$ and $\mu * f$ lie in $L_{1,\mathrm{loc}}$. If, in addition, f is bounded, and $\mu \in L_{1,\mathrm{loc}}$, then $f * \mu$ and $\mu * f$ are continuous.*

Proof. Let $C = \operatorname{Supp} f$. If $D \subset G$ is compact, then

$$(f * \xi_{C^{-1}D}\mu)(X) = (f * \mu)(X)$$

for all Borel set $X \subset D$. It follows that $f * \mu \ll \lambda$. Further by [3; 20.16], if f is bounded and $\mu \in L_{1,\mathrm{loc}}$, then $f * \mu$ is continuous on every compact $D \subset G$.

The proof for $\mu * f$ is similar.

From now on, K will be a nonempty compact subset of G which is the closure of its interior.

For $\mu \in R$ we define $\mu^\natural: G \to [0, \infty)$ by

$$\mu^\natural(x) = \|\xi_{xK}\mu\| = |\mu|(xK) \qquad (x \in G)\ .$$

It is easy to check that

$$\mu^\natural = |\mu| * \xi_{K^{-1}}\ .$$

Thus, $\mu^\natural \in L_{1,\mathrm{loc}}$, and $\mu^\natural$ is continuous if $\mu \in L_{1,\mathrm{loc}}$.

Further, if f is a measurable function on G, define $f^\#: G \to [0, \infty]$ by

$$f^\#(x) = \|f\xi_{xK}\|_\infty\ .$$

$f^\#$ is lower semicontinuous, hence measurable. (*Proof*: For $r \in \boldsymbol{R}$ put $f_r(x) = \min(r, |f(x)|)$; then $f^* = \sup_r f_r^\#$. Thus, we may assume that f is bounded and $\geqq 0$. For $j \in L_1$, $j \geqq 0$, $\int j = 1$, the function $fj * \xi_{K^{-1}}$ is continuous [3; 20.16], and for every $x \in G$,

$$f^\#(x) = \sup_j \int \xi_{xK} fj = \sup_j fj * \xi_{K^{-1}}\ .$$

Thus, $f^\#$ is a supremum of continuous functions.)

LEMMA 1.3. *Let $\mu \in R$, let f be a measurable function. Assume* *that either* $\mu \ll \lambda$ *or* f *is continuous. Then*

$$(f^{\sharp}, \mu^{\natural}) \geqq \lambda(K^{-1})(|f|, |\mu|) .$$

Proof. By the assumption, for every $x \in G$ we have $|f| \leqq f^{\sharp}(x)|\mu|$ a.e. on xK, so that

$$f^{\sharp}(x)\mu^{\natural}(x) \geqq \int_{xK} |f(y)|\, d\,|\mu|\,(y) .$$

We may assume $f^{\sharp}\mu^{\natural} \in L_1$. Then there exists a σ-compact set X such that $f^{\sharp}\mu^{\natural} = 0$ a.e. outside X. Since the X is σ-compact, it follows from [3; 5.7] that there exists a closed and open σ-compact subgroup H of G containing X and K. Every relatively compact Borel subset C of $G\backslash H$ can be covered by finitely many cosets $a_1K, \cdots, a_mK$ where $f^{\sharp}(a_i)\mu^{\natural}(a_i) = 0$ for each i. Then

$$\int_C |f|\, d\,|\mu| \leqq \Sigma f^{\sharp}(a_i)\mu^{\natural}(a_i) = 0 .$$

Put $f_1 = \xi_H|f|$, $\mu_1 = \xi_H|\mu|$. Then $\int|f|\,d\,|\mu| = \int f_1 d\mu_1$. Further, $f_1^{\sharp}\mu_1^{\natural} = f^{\sharp}\mu^{\natural}$ on H and $f_1^{\sharp} \leqq f^{\sharp}$, $\mu_1^{\natural} \leqq \mu^{\natural}$ everywhere; therefore $(f^{\sharp}, \mu^{\natural}) = (f_1^{\sharp}, \mu_1^{\natural})$. It follows that we may replace f by f_1 and μ by μ_1; i.e., we may assume that $f \geqq 0$, $\mu \geqq 0$ and that G is σ-compact. This enables us to apply the Fubini Theorem:

$$\begin{aligned}(f^{\sharp}, \mu^{\natural}) &\geqq \iint \xi_{xK}(y)\,|f(y)|\, d\,|\mu|\,(y)dx \\ &= \iint \xi_{yK^{-1}}(x)\,|f(y)|\, dxd\,|\mu|\,(y) \\ &= \int \lambda(yK^{-1})\,|f(y)|\, d\,|\mu|\,(y) = \lambda(K^{-1})\int |f(y)|\, d\,|\mu|\,(y) .\end{aligned}$$

Note. This lemma, and also its applications, Theorems 3.1 and 6.1, should be read with some caution. In the case where f is continuous the "f" in the right hand member of the formula denotes a single function, but the "f" in the left hand member stands for the class of all functions that are l.a.e. equal to f.

COROLLARY 1.4. *For all* $\mu \in R$, $\int \mu^{\natural} = \lambda(K^{-1})\,|\mu|\,(G)$. *Thus, if* $\mu^{\natural} = 0$ a.e., *then* $\mu = 0$.

Proof. It is clear that $\int \mu^{\natural} \leqq \|\xi_{K^{-1}}\|_1 |\mu|(G) = \lambda(K^{-1})|\mu|(G)$. On taking $f = 1$, we get $\int \mu^{\natural} \geqq \lambda(K^{-1})|\mu|(G)$. Hence the equality. If C

is compact, then

$$\int \mu^\natural \geqq \int (\xi_C \mu)^\natural = \int |\xi_C \mu| * \xi_{K^{-1}} = \lambda(K^{-1}) |\xi_C \mu|(G) = \lambda(K^{-1}) |\mu|(C) .$$

Hence we have the second statement.

2. For $1 \leqq p < \infty$ let

$$\mathscr{V}_p = \mathscr{V}_{p0} = \{j \in L_{1,\mathrm{loc}}: j^\natural \in L_p\} .$$

Further, put

$$\mathscr{V}_\infty = \{j \in L_{1,\mathrm{loc}}: j^\natural \in L_\infty\} ,$$
$$\mathscr{V}_{\infty 0} = \{j \in L_{1,\mathrm{loc}}; j^\natural \in C_0\} .$$

Clearly $\mathscr{V}_{\infty 0} \subset \mathscr{V}_\infty$. As $\int j^\natural = \lambda(K^{-1}) \int |j|$, we have $\mathscr{V}_1 = L_1$. For $j \in \mathscr{V}_p$, we set

$$||j||_p^\natural = ||j^\natural||_p .$$

THEOREM 2.1. *Let $1 \leqq p \leqq \infty$. $\mathscr{V}_p$ and $\mathscr{V}_{p0}$ are Banach spaces. L_1 is a dense subset of $\mathscr{V}_{p0}$: the natural injection $L_1 \to \mathscr{V}_{p0}$ is continuous. The formula*

$$j_x(y) = j(xy) \qquad (j \in \mathscr{V}_{p0};\ x,\ y \in G)$$

defines a continuous shift in $\mathscr{V}_{p0}$.

Proof. Clearly $\mathscr{V}_p$, $\mathscr{V}_{p0}$ are vector spaces and $||\ ||_p^\natural$ is a norm. To prove completeness of $\mathscr{V}_p$, let $\{j_n\}$ be a sequence in $\mathscr{V}_p$ such that $\sum ||j_n||_p^\natural < \infty$; it suffices to prove that $\sum j_n$ converges in $\mathscr{V}_p$. We know that $\sum j_n^\natural(x)$ converges for all $x \in G$ outside a certain locally null set X. Take a compact set $C \subset G$. C is covered by finitely many translates $a_1 K, \cdots, a_m K$ of K and we can choose all a_i in $G \backslash X$. Then

$$\sum_n \int_C |j_n| \leqq \sum_n \sum_i j_n^\natural(a_i) < \infty ,$$

so that $\Sigma |j_n| < \infty$ a.e. on C. By [3; 11.39 and 11.42] there exists a measurable function j such that $\Sigma\ j_n = j$ l.a.e. Then $\int_C |j| \leqq$ $\leqq \Sigma \int_C |j_n|$ for every compact C, so $j \in L_{1,\mathrm{loc}}$ and $j^\natural \leqq \Sigma j_n^\natural$. Hence, $j \in \mathscr{N}_p$. We also obtain $(j - \sum_1^N j_n)^\natural \leqq \sum_{N+1}^\infty j_n^\natural$, so

$$\left|\left| j - \sum_1^N j_n \right|\right|_p^\natural \leqq \left|\left| \sum_{N+1}^\infty j_n^\natural \right|\right|_p \leqq \sum_{N+1}^\infty ||j_n^\natural||_p = \sum_{N+1}^\infty ||j_n||_p^\natural .$$

Therefore, $j = \Sigma j_n$ in the sense of $\mathscr{N}_p$.

For $\mathscr{T}_{\infty 0}$ we simply observe that $j \mapsto j^\natural$ is a continuous map $\mathscr{T}_\infty \to L_\infty$, so that $\mathscr{T}_{\infty 0}$ is closed in $\mathscr{T}_\infty$.

Next, take $j \in L_1$. Since $\xi_{K^{-1}} \in L_p$, $j^\natural = |j| * \xi_{K^{-1}} \in L_p$ and $\|j\|_p^\natural \leqq \|j\|_1 \|\xi_{K^{-1}}\|_p$. It is easy to see that $j^\natural \in C_0$. Hence, $L_1 \subset \mathscr{T}_{p0}$ and the injection is continuous. To prove that L_1 is dense, take $j \in \mathscr{T}_{p0}$, $\varepsilon > 0$. There is a compact set C such that $\|j^\#(1 - \xi_C)\|_p \leqq \varepsilon$. Then $(1 - j\xi_{CK})^\natural = 0$ on C and $(j - j\xi_{CK})^\natural \leqq j^\natural$ everywhere. For $p < \infty$ it follows that $\|j - j\xi_{CK}\|_p^\natural \leqq \|j^\natural(1 - \xi_C)\|_p \leqq \varepsilon$. But $j\xi_{CK} \in L_1$. Thus, for $p < \infty$, L_1 is dense in $\mathscr{T}_{p0}$. A similar proof works for $p = \infty$.

Trivially, if $j \in \mathscr{T}_{p0}$, then $j_x \in \mathscr{T}_{p0}$ for every x. We only have to prove that $x \mapsto j_x$ is continuous. As L_1 is dense, we may assume $j \in L_1$. Now for such j we know that the formula $x \mapsto j_x$ defines a continuous map $G \to L_1$. Now observe that the injection $L_1 \to \mathscr{T}_{p0}$ is continuous.

THEOREM 2.2. *For $1 \leqq p \leqq \infty$, L_p is a dense subset of $\mathscr{T}_p$, and the injection $L_p \to \mathscr{T}_p$ is continuous. Further, C_0 is a dense subset of $\mathscr{T}_{\infty 0}$ and $C_0 \to \mathscr{T}_{\infty 0}$ is continuous.*

Proof. The first statement follows from the formula

$$j^\natural = |j| * \xi_{K^{-1}}$$

and [3; 20.13]. Further, for $j \in C_0$ we have $|j| * \xi_{K^{-1}} \in C_0$ as is easy to prove.

According to the remarks made in §1, $\mathscr{T}_{p0}$ can be made into an absolutely continuous module. The module operation $*$ is given by the familiar convolution formula:

THEOREM 2.3. *If $f \in L_1$ and $j \in \mathscr{N}_{p0}(1 \leqq p \leqq \infty)$, one has for locally almost every $x \in G$,*

$$f * j(x) = \int f(y) j(y^{-1}x) dy .$$

Proof. We may assume $f, j \geqq 0$. If $h \in C_{00}$ and $h \geqq 0$, then

$$\begin{aligned}(h, f * j) &= \int f(y)(h, j_{y^{-1}})dy = \iint f(y)h(x)j(y^{-1}x)dxdy \\ &= \int h(x) \int f(y)j(y^{-1}x)dydx .\end{aligned}$$

3. For $1 \leqq p \leqq \infty$ let $\mathscr{N}_p$ be the set of all measurable functions f on G for which $f^\# \in L_p$. Then $\mathscr{N}_p$ is a vector space, and the formula

$$\|f\|_p^\# = \|f^\#\|_p \qquad (f \in \mathscr{N}_p)$$

defines a norm on $\mathscr{N}_p$.

THEOREM 3.1. *Let* $1 \leqq p \leqq \infty$, $q = p/(p-1)$. *Every* $f \in \mathscr{N}_q$ *determines a* $\Phi f \in \mathscr{T}_{p0}^*$ *by*

$$(j, \Phi f) = \int j(x)f(x)dx \qquad (j \in \mathscr{T}_{p0}) .$$

Φ *is a linear homeomorphism on* $\mathscr{N}_q$ *onto* $\mathscr{T}_{p0}^*$. *We have*

$$\|\Phi\| \leqq 1/\lambda(K^{-1}) \quad and \quad \|\Phi^{-1}\| \leqq \int_{KK^{-1}} \Delta^{-1/q} .$$

Proof. It follows directly from Lemma 1.3 that Φ maps $\mathscr{N}_q$ into $\mathscr{T}_{p0}^*$ and that $\|\Phi\| \leqq \lambda(K^{-1})^{-1}$. For the converse, take $\phi \in \mathscr{T}_{p0}^*$. As $L_1 \subset \mathscr{T}_{p0}$ and as the injection $L_1 \to \mathscr{T}_{p0}$ is continuous, there is an $f \in L_\infty$ such that

$$(j, \phi) = \int j(x)f(x)dx \qquad (j \in L_1) .$$

If $f \in \mathscr{N}_q$, then $\phi = \Phi f$ on a dense subspace of $\mathscr{T}_{p0}$, hence on all of $\mathscr{T}_{p0}$. We proceed to prove that $f \in \mathscr{N}_q$ and $\|f\|_q^\# \leqq \|\phi\| \int_{KK^{-1}} \Delta^{-1/q}$. Assume for the time being that $f \geqq 0$. Take $\varepsilon \in (0, 1)$. The set

$$S = \{(x, y) \in G^2 \colon y \in xK;\ f(y) \geqq (1-\varepsilon)f^\#(x)\}$$

is measurable. Then for l.a.e. $x \in G$ the set

$$S_x = \{y \in G \colon (x, y) \in S\} = \{y \in xK \colon f(y) \geqq (1-\varepsilon)f^\#(x)\}$$

is measurable. Moreover, the function $x \to \lambda(S_x)$ is measurable, and $\lambda(S_x) > 0$ l.a.e.

Let $h \in L_1 \cap L_p$, $h \geqq 0$, Supp h compact. Then

$$(1-\varepsilon)\int f^\#(x)h(x)dx \leqq \int\Big[\int \frac{f(y)}{\lambda(S_x)}\xi_{S_x}(y)dy\Big]h(x) = dx = \int f(y)j(y)dy ,$$

where

$$j(y) = \int \frac{h(x)}{\lambda(S_x)}\xi_{S_x}(y)dx .$$

One easily sees that $j \in L_1$ so that

$$(1-\varepsilon)\int f^\#(x)h(x)dx \leqq \int f(y)j(y)dy = (j, \phi) \leqq \|\phi\| \, \|j\|_p^\natural .$$

To find an estimate for $\|j\|_p^\natural$ we observe that for every $a \in G$,

$$j^{\natural}(a) = \iint \xi_{aK}(y) \frac{h(x)}{\lambda(S_x)} \xi_{S_x}(y) dy dx = \int h(x) \frac{\lambda(aK \cap S_x)}{\lambda(S_x)} dx \ .$$

As $S_x \subset xK$ we have $\lambda(aK \cap S_x) = 0$ unless $x \in aKK^{-1}$, so that

$$j^{\natural}(a) \leqq \int_{aKK^{-1}} h(x) dx = h^{!} * \xi_{KK^{-1}}(a) \ .$$

By again applying [3; 20.13] we find

$$\|j\|_p^{\natural} \leqq \|h * \xi_{KK^{-1}}\|_p \leqq \|h\|_p \int_{KK^{-1}} \Delta(y)^{-1/q} dy \ .$$

Thus, for all $\varepsilon > 0$ and all $h \in L_1 \cap L_p$, $h \geqq 0$ we get

$$(1 - \varepsilon) \int f^{\#} h \leqq \|\phi\| \, \|h\|_p \int_{KK^{-1}} \Delta^{-1/q} \ .$$

Then $f^{\#} \in L_q$, i.e., $f \in \mathscr{N}_q$, and $\|f\|_q^{\#} \leqq \|\phi\| \int_{KK^{-1}} \Delta^{-1/q}$.

We have proved our point for the case $f \geqq 0$. For the general case we notice that there exists a measurable function $\tau: G \to \boldsymbol{C}$ such that $|\tau(x)| = 1$ for all x and $\tau f \geqq 0$. Define $\psi: \mathscr{V}_{p0} \to \boldsymbol{C}$ by

$$(j, \psi) = (j\tau^{-1}, \phi) \ .$$

Then $\psi \in \mathscr{V}_{p0}^{*}$, $\|\psi\| = \|\phi\|$, and $(j, \psi) = \int j(x)(\tau f)(x) dx$ for $j \in L_1$. It follows that $\tau f \in \mathscr{N}_q$ and $\|\tau f\|_q^{\#} \leqq \|\psi\| \int_{KK^{-1}} \Delta^{-1/q}$; so $f \in \mathscr{N}_q$ and $\|f\|_q^{\#} \leqq \|\phi\| \int_{KK^{-1}} \Delta^{-1/q}$.

COROLLARY 3.2. *$\mathscr{N}_q \subset L_\infty \cap L_q$. The injections $\mathscr{N}_q \to L_\infty$ and $\mathscr{N}_q \to L_q$ are continuous.*

Proof. By Theorem 2.1, there is a constant c such that for every $j \in L_1$, $\|j\|_p^{\natural} \leqq c\|j\|_1$. Then if $f \in \mathscr{N}_q$, we have

$$\int jf = (j, \Phi f) \leqq c\|\Phi\| \, \|f\|_q^{\#} \|j\|_1 \ .$$

Hence $f \in L_\infty$ and $\|f\|_\infty \leqq c\|\Phi\| \, \|f\|_q^{\#}$. The proof for the statements concerning L_q is simimilar, using Theorem 2.2 instead of Theorem 2.1.

COROLLARY 3.3. *$\mathscr{N}_q$ is a Banach space which is also an L_1-module under convolution.*

Proof. We know that $\mathscr{V}_{p0}^{*}$ can be made into a left Banach module over L_1. After Theorem 3.1 we only have to show that the induced module operation in $\mathscr{N}_q$ is convolution. Now for $f \in L_1$, $g \in \mathscr{N}_q$ and

$j \in \mathscr{T}_{p0}$ we have

$$\begin{aligned}(j, f_* g) = (f' * j, g) &= \iint f'(x) j(x^{-1}y) g(y) dx dy \\ &= \iint f(x^{-1}) j(x^{-1}y) g(y) \Delta(x^{-1}) dx dy \\ &= \iint f(x) j(xy) g(y) dy dx = \iint f(x) j(y) g(x^{-1}y) dy dx \\ &= \int j(y) \int f(x) g(x^{-1}y) dx dy .\end{aligned}$$

4. For $1 \leqq p < \infty$ let

$$\mathscr{M}_p = \{f \in \mathscr{N}_p : f \text{ is continuous}\} .$$

THEOREM 4.1. *C_{00} is a dense subspace of $\mathscr{M}_p$.*

Proof. Take $f \in \mathscr{M}_p$, $\varepsilon > 0$. There exists a compact $C \subset G$ such that $\|f^\#(1 - \xi_C)\|_p \leqq \varepsilon$, and there exists a $g \in C_{00}$, $g = f$ on CK, such that $|g| \leqq |f|$. Then $(f - g)^\# = 0$ on C, and

$$\|f - g\|_p^\# \leqq \|(f - g)^\# \xi_C\|_p + \|f^\#(1 - \xi_C)\|_p + \|g^\#(1 - \xi_C)\|_p \leqq 2\varepsilon .$$

THEOREM 4.2. *For $1 \leqq p < \infty$, $\mathscr{M}_p = (\mathscr{N}_p)_{abs}$. The formula*

$$T_f g = g * f \qquad (f \in \mathscr{N}_p;\ g \in L_1)$$

establishes a one-to-one correspondence between $\mathscr{N}_p$ and Mult $\mathscr{M}_p$.

Proof. First, take $f \in C_{00}$. Let $S = \operatorname{Supp} f$. Let U be a compact neighborhood of S. For every $\varepsilon > 0$ there exists a neighborhood V_ε of 1 such that $V_\varepsilon^{-1} S \subset U$ and $\|f - f_x\|_\infty \leqq \varepsilon$ for all $x \in V_\varepsilon$. Then for $x \in V_\varepsilon$, we have $\|(f - f_x)^\#\|_\infty \leqq \varepsilon$ and $(f - f_x)^\# = 0$ off UK^{-1}. Thus, for $x \in V_\varepsilon$, $\|f - f_x\|_p^\# \leqq \varepsilon[\lambda(UK^{-1})]^{1/p}$.

It follows that $x \mapsto f_x$ is a continuous map $G \to \mathscr{M}_p$ for every $f \in C_{00}$, hence for every $f \in \mathscr{M}_p$ (see Theorem 4.1). Thus, $\mathscr{M}_p$ can be made into an absolutely continuous left Banach module over L_1. It is easy to see that the induced module operation is convolution, which is the same as the module operation in $\mathscr{N}_p$. Thus, $\mathscr{M}_p = L_1 * \mathscr{M}_p = (\mathscr{N}_p)_{abs}$.

Now $(\mathscr{N}_p)_{abs} = L_1 * \mathscr{N}_p \subset L_1 * L_\infty$ and every element of $L_1 * L_\infty$ is continuous [3; 20.16]. Hence, $(\mathscr{N}_p)_{abs} \subset \mathscr{M}_p$, so that $(\mathscr{N}_p)_{abs} = \mathscr{M}_p$.

We also see that every T_f maps L_1 into $\mathscr{M}_p$, hence is an element of Mult $\mathscr{M}_p$. Let $q = p/(p-1)$, $(q = \infty$ if $p = 1)$. As $\mathscr{T}_{p0}$ is an absolutely continuous module, it follows from Theorem 1.1 that for every $T \in$ Mult $\mathscr{T}_{p0}^*$ there exists a $\phi \in \mathscr{T}_{p0}^*$ such that $Tf = f * \phi (f \in L_1)$.

The rest of the theorem follows from Theorem 3.1 and the trivial observation that Mult $\mathscr{M}_p \subset$ Mult $\mathscr{N}_p$.

COROLLARY 4.3. *$\mathscr{M}_p$ is a Banach space.*

Let us consider the case $p = \infty$. Obviously, $\mathscr{N}_\infty = L_\infty$. Thus, $\{f \in N_\infty : f$ is continuous$\}$ is the space of all bounded continuous functions. It is known, however, that $(L_\infty)_{abs} = C_{ru}$(see [3; 32.45]). We could save the situation by defining, for $1 \leqq p \leqq \infty$,

$$\mathscr{M}_p = C_{ru} \cap \mathscr{N}_p .$$

Then Theorem 4.2 remains valid if we change "$1 \leqq p < \infty$" into "$1 \leqq p \leqq \infty$".

5. $\mathscr{M}_1$ deserves some special attention.

THEOREM 5.1. *$\mathscr{N}_1$ and $\mathscr{M}_1$ are Banach algebras under convolution. If G is abelian, $\mathscr{M}_1$ is a Segal algebra (as defined in [5; Ch. 6, §2]).*

Proof. The injection $I: \mathscr{N}_1 \to L_1$ is continuous (Corollary 3.2). For $f, g \in \mathscr{N}_1$ we have $\|f * g\|_1^\# \leqq \|f\|_1 \|g\|_1^\# \leqq \|I\| \, \|f\|_1^\# \|g\|_1^\#$. Further, $\mathscr{M}_1$ is a left ideal in L_1 by Theorem 4.2. The second statement follows from Theorem 4.1 and the continuity of the shift (see proof of Theorem 4.2).

Consider in particular the case $G = \boldsymbol{R}$, the additive reals. Wiener defines his Banach algebra M_1 as the set of all continuous functions f on $\boldsymbol{R}$ for which $\|f\|_{M_1} < \infty$, where

$$\|f\|_{M_1} = \sum_{n=-\infty}^{\infty} \max_{[n,n+1]} |f(x)| .$$

M_1 was discussed in [7; Ch. 2] and [5; pp. 12, 127], [3; II pp. 506, 600]. To show that $M_1 = \mathscr{M}_1$, for a continuous function f on $\boldsymbol{R}$ define $f^\flat$ on $\boldsymbol{R}$ by

$$f^\flat(x) = \sup_{[n,n+1]} |f| \quad \text{if} \quad n \in \boldsymbol{Z},\, n \leqq x < n+1 .$$

Then $\|f\|_{M_1} = \|f^\flat\|_1$. By taking $K = [0, 1]$ we find $f^\#(x) \leqq f^\flat(x) + f^\flat(x+1)$ and $f^\flat(x) \leqq f^\#(x-1) + f^\#(x)$ for all x. Hence $M_1 = \mathscr{M}_1$ and

$$\frac{1}{2}\| \;\|_1^\# \leqq \| \;\|_{M_1} \leqq 2 \| \;\|_1^\# .$$

6. Finally, for $1 \leqq p \leqq \infty$ we set

$$\mathscr{W}_p = \{\mu \in R : \mu^\natural \in L_p\} , \quad \|\mu\|_p^\natural = \|\mu^\natural\|_p .$$

For all $\mu \in R$ we have $\int \mu^{\natural}(x)dx = \lambda(K^{-1})\,|\mu|\,(G)$. Hence $\mathscr{W}_1 = M$.

THEOREM 6.1. *Let* $1 \leqq p < \infty$, $q = p/(p-1)$. *The formula*

$$(f, \Psi\mu) = \int f d\mu \quad (f \in \mathscr{M}_p;\ \mu \in \mathscr{W}_q)$$

establishes a linear homeomorphism of $\mathscr{W}_q$ *onto* $\mathscr{M}_p^*$. *Further,*

$$||\Psi|| \leqq 1/\lambda(K^{-1})\ , \quad ||\Psi^{-1}|| \leqq \int_{KK^{-1}} \Delta^{-1/q}\ .$$

Proof. It follows from Lemma 1.3 that $\Psi\colon \mathscr{W}_q \rightarrow \mathscr{M}_p^*$ and that $||\Psi|| \leqq \lambda(K^{-1})^{-1}$. Conversely, take $\phi \in \mathscr{M}_q^*$. If $C \subset G$ is compact, then for every $f \in C_{00}$ that has its support in C we have

$$|(f, \phi)| \leqq ||\phi||\ ||f||_p^{\sharp} \leqq ||\phi||\,||f||_\infty[\lambda(CK^{-1})]^{-1/p}\ .$$

Thus, there is a $\mu \in R$ such that

$$(f, \phi) = \int f\, d\mu \quad (f \in C_{00})\ .$$

If $\mu^{\natural} \in L_q$ and $||\mu^{\natural}||_q \leqq ||\phi|| \int_{KK^{-1}} \Delta^{-1/q}$, then $\Psi\mu = \phi$ on a dense subset of $\mathscr{M}_p$, and we are done.

Take $h \in C_{00}$, $h \geqq 0$. Then

$$\begin{aligned}\int h(x)\mu^{\natural}(x)dx &= \iint h(x)\xi_{xK}(y)d\,|\mu|\,(y)dx\\ &= \iint h(x)\xi_K(x^{-1}y)dxd|\mu|(y) = (h * \xi_K,\ |\mu|)\ .\end{aligned}$$

By [3; 14.5] and the continuity of $h * \xi_K$ we obtain $(h * \xi_K, |\mu|) = \sup_{j \in \mathscr{F}} |(j, \mu)|$ where $\mathscr{F} = (f \in C_{00}\colon |f| \leqq h * \xi_K\}$. Observe that for every $f \in \mathscr{F}$ and $a \in G$,

$$f^{\sharp}(a) \leqq \sup_{x \in aK} h * \xi_K(x) = \sup_{x \in aK} \int_{xK^{-1}} h \leqq \int_{aKK^{-1}} h = h * \xi_{KK^{-1}}(a)\ .$$

By another application of [3; 20.13] we find

$$\begin{aligned}\int h(x)\mu^{\natural}(x)dx &= (h * \xi_K,\ |\mu|) = \sup_{f \in \mathscr{F}} |(f, \mu)| \leqq \sup_{f \in \mathscr{F}} |(f, \phi)|\\ &\leqq \sup_{f \in \mathscr{F}} ||\phi||\,||f||_p^{\sharp} \leqq ||\phi||\ \,|h| * \xi_{KK^{-1}}||_p \leqq ||\phi||\,||h||_p \int_{KK^{-1}} \Delta^{-1/q}\ .\end{aligned}$$

Thus, $\mu^{\natural} \in L_q$ and $||\mu^{\natural}||_q \leqq ||\phi|| \int_{KK^{-1}} \Delta^{-1/q}$.

COROLLARY 6.2. *$\mathscr{W}_p$ is a Banach space.*

By the general theory of Banach modules, $\mathscr{W}_p$ can be made into a left Banach module over L_1 (see §1, and §3 where we introduced a module operation on $\mathscr{N}_p$). An application of Theorem 4.1 yields the fact that for $f \in L_{1,\mathrm{loc}}$ and $\mu \in \mathscr{W}_p$ the module product and the convolution product $f * \mu$ coincide. It follows that $\mathscr{V}_p$ is a submodule of $\mathscr{W}_p$.

We close the circle by:

COROLLARY 6.3. *If $p < \infty$, then $\mathscr{V}_p = (\mathscr{W}_p)_{abs}$. For all p the formula*

$$S_\mu f = f * \mu \quad (\mu \in \mathscr{W}_p; f \in L_1)$$

yields a linear homeomorphism S of $\mathscr{W}_p$ onto Mult $\mathscr{V}_p$.

Proof. For $p = 1$ we can apply Wendel's Theorem [3; 35.5], since $\mathscr{W}_1 = M$ and $\mathscr{V}_1 = L_1$. By the last part of Theorem 2.2, $\mathscr{V}_p = (\mathscr{V}_p)_{abs} \subset (\mathscr{W}_p)_{abs}$ if $p < \infty$. For any p and for an $f \in L_1$ that has compact support, we have

$$f * \mathscr{W}_p \subset (f * R) \cap \mathscr{W}_p \subset L_{1,\mathrm{loc}} \cap \mathscr{W}_p = \mathscr{V}_p .$$

As $\mathscr{V}_p$ is closed, $(\mathscr{W}_p)_{abs} = L_1 * \mathscr{W}_p \subset \mathscr{V}_p$. Thus $\mathscr{V}_p = (\mathscr{W}_p)_{abs}$ if $p < \infty$, and $S_\mu \in \mathrm{Mult}\ \mathscr{V}_p$ for all $\mu \in \mathscr{W}_p$. The proof of the facts that for $p \neq 1$ every element of Mult $\mathscr{V}_p$ is of the form S_μ, and that S is a homeomorphism is entirely analogous to the final part of the proof of Theorem 4.2 (using 6.1 instead of 3.1).

7. In order to see how the operations $\natural$ and $\sharp$ depend on K, take another nonempty compact set K_1 that is the closure of its interior and define

$$\mu_1^\natural(x) = |\mu|(xK_1) \quad (x \in G; \mu \in R) ,$$
$$f_1^\sharp(x) = \|f \xi_{xK_1}\|_\infty \quad (x \in G; f \in L_{1,\mathrm{loc}}) .$$

As K is compact and K_1 has nonempty interior, there exists $a_1, \cdots, a_n$ such that $K \subset a_1K_1 \cup \cdots \cup a_nK_1$. Then we see that

$$\mu^\natural \leqq (\mu_1^\natural)_{a_1} + \cdots + (\mu_1^\natural)_{a_n} \quad (\mu \in R) ,$$
$$f^\natural \leqq (f_1^\natural)_{a_1} + \cdots + (f_1^\natural)a_n \quad (f \in L_{1,\mathrm{loc}}) .$$

From these formulas it will follow directly that the Banach spaces $\mathscr{V}_p$, $\mathscr{V}_{\infty 0}$, $\mathscr{N}_p$, $\mathscr{M}_p$, $\mathscr{W}_p$ are essentially independent of K: A different choice of K will only lead to a different, but equivalent norm in the same space.

In fact the proof shows that we can relax the conditions on K. We only need to require that K is a relatively compact set with nonempty interior. Any such K will lead to the same spaces $\mathscr{V}_p$, $\mathscr{V}_{\infty 0}$, $\mathscr{N}_p$, $\mathscr{M}_p$, and $\mathscr{W}_p$ with equivalent norms. The inequalities in Theorem 6.1 will no longer be true for such a general K. (The analogous inequalities in Theorem 3.1 remain valid.)

The results obtained so far can be summarized in the following table where we use the equality sign to indicate linear homeomorphism. In each formula, $1/p + 1/q = 1$.

$$\begin{array}{lll} \mathscr{V}_p^* = \mathscr{N}_q, & \mathscr{V}_{\infty 0}^* = \mathscr{N}_1 & (1 \leqq p < \infty) \\ (\mathscr{N}_p)_{abs} = \mathscr{M}_p, & \text{Mult}\, \mathscr{M}_p = \mathscr{N}_p & (1 \leqq p \leqq \infty) \\ \mathscr{M}_p^* = \mathscr{W}_q & & (1 \leqq p < \infty) \\ (\mathscr{W}_p)_{abs} = \mathscr{V}_p & & (1 \leqq p < \infty) \\ \text{Mult}\, \mathscr{V}_p = \mathscr{W}_p & & (1 \leqq p \leqq \infty) \end{array}$$

The equalities in the first and third line clearly do not hold in general if we put $p = \infty$. For the fourth line this is less easy to see. Take $G = \boldsymbol{R}$, $K = [0, 1]$. Let j be the function that vanishes on $(-\infty, 0]$ and coincides with nth Rademacher function ϕ_n on $(n-1, n]$ for every positive integer n (see [8; Ch. I, §3]). Then $j \in \mathscr{V}_\infty$, but $j \notin (\mathscr{V}_\infty)_{abs}$. (It is not hard to prove that, if $j \in (\mathscr{V}_\infty)_{abs}$, then $\lim_{x \to 0} \|j - j_x\|_\infty^K = 0$, and that the latter formula is false.)

We do not have an adequate description of Mult $\mathscr{V}_{\infty 0}$.

Let us conclude with a table listing $\mathscr{V}_p$, $\mathscr{W}_p$, $\mathscr{M}_p$, $\mathscr{N}_p$ for compact G and for discrete G.

G	compact	discrete
$\mathscr{V}_p(1 \leqq p \leqq \infty)$	L_1	L_p
$\mathscr{V}_{\infty 0}$	L_1	C_0
$\mathscr{W}_p(1 \leqq p \leqq \infty)$	M	L_p
$\mathscr{W}_\infty$	M	L_∞
$\mathscr{M}_p(1 \leqq p \leqq \infty)$	C	L_p
$\mathscr{N}_p(1 \leqq p \leqq \infty)$	L_∞	L_p

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Received November 29, 1971 and in revised form January 31, 1974. The first authors' research was supported in part by the U. S. National Science Foundation.

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Caution: This was not proof read by the authors.