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A "GOING DOWN" THEOREM FOR CERTAIN REFLECTED RADICALS

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In a category $\mathscr{K}$ suitable for radical theory, a functor $\Phi: \mathscr{K} \to \mathscr{K}$ is studied which is associated with a natural transformation $1_{\mathscr{K}} \to \Phi$ in a way which bears a formal resemblance to the behavior of certain "extension" functors of rings, such as that which assigns to each A the polynomial ring $A[x]$: every normal subobject $N \to \Phi(A)$ has a "contraction" $N^c \to A$. For a radical class $\mathscr{R}$ in $\mathscr{K}$ such that $\mathscr{R}^* = \{A \mid \Phi(A) \in \mathscr{R}\}$ is also radical, some conditions are obtained which imply that $\mathscr{R}^*(A) = \mathscr{R}(\Phi(A))^c$.

1. Preliminaries. We shall work in a category $\mathscr{K}$ for which the general theory of radicals can be developed (for a set of conditions on $\mathscr{K}$ which ensure this and for some other remarks on radicals in categories, see [9]) and shall consider a left-exact functor $\Phi: \mathscr{K} \to \mathscr{K}$ which has associated with it a natural transformation $1_{\mathscr{K}} \to \Phi$, which will be fixed throughout the discussion. We shall further assume that for each normal subobject $N \to \Phi(A)$ there is a normal subobject $N^{cA} \to A$ and a pullback

$$\begin{array}{ccc} N^{cA} & \longrightarrow & A \\ \downarrow & & \downarrow \\ N & \longrightarrow & \Phi(A) \end{array}$$

where the right-hand vertical map is defined by the natural transformation mentioned above. When no confusion can result, N^{cA} will be abbreviated to N^c. We shall frequently find it convenient to write A^e for $\Phi(A)$. A prototypical example of such a functor is that which assigns to each ring A its polynomial ring $A[x]$, in which case $A^e = A[x]$ ("extension") and $N^c = N \cap A$ ("contraction"). The symbol $A \to A^e$ will always denote a map defined by the given natural transformation.

Our category-theoretic terminology is essentially that of [2]. We shall not distinguish notationally between a subobject and a representative map. In particular if $A \in \mathscr{K}$ and $\mathscr{R}$ is a radical class, $\mathscr{R}(A) \to A$ will denote the $\mathscr{R}$-radical of A.

PROPOSITION 1.1.
(a) *If $N \to A$ is a normal subobject, then $N \to A \subseteqq N^{ec} \to A$.*
(b) *If $N_1 \to A^e \subseteqq N_2 \to A^e$ are normal subobjects then $N_1^c \to A \subseteqq N_2^c \to A$.*

(c) $A^{ec} = A$.

(d) *If* $I \to A$ *and* $J \to A^e$ *are normal subobjects, with* $J \to A^e \subseteqq I^e \to A^e$, *then there is a map* $J^{cI} \to J^{cA}$ *such that*

$$\begin{tikzcd} J^{cI} \arrow[r] \arrow[d] & I \arrow[d] \\ J^{cA} \arrow[r] & A \end{tikzcd}$$

commutes.

(e) *If*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & N_1 \arrow[d] \\ N_2 \arrow[r] & A^e \end{tikzcd}$$

is a pullback and $N_1 \to A^e$, $N_2 \to A^e$, $P \to A^e$ *are normal subobjects, then*

$$\begin{tikzcd} P^c \arrow[r] \arrow[d] & N_1^c \arrow[d] \\ N_2^c \arrow[r] & A \end{tikzcd}$$

is also a pullback.

Proof. (a) follows from consideration of the diagram

$$\begin{tikzcd} N \arrow[drr] \arrow[dr, dashed] \arrow[ddr] & & \\ & N^{ec} \arrow[r] \arrow[d] & A \arrow[d] \\ & N^e \arrow[r] & A^e \end{tikzcd} .$$

(b) follows from consideration of

$$\begin{tikzcd} N_1^c \arrow[drr] \arrow[dr, dashed] \arrow[ddd] & & \\ & N_2^c \arrow[r] \arrow[d] & A \arrow[d] \\ & N_2 \arrow[r] & A^e \\ N_1 \arrow[ur] \arrow[urr] & & \end{tikzcd} .$$

(c) is obtained from (a) by taking $N = A$.

(d) Since
$$\begin{tikzcd} I^e \arrow[r] & A^e \\ J \arrow[u] \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} J^{cI} \arrow[r] \arrow[d] & I \arrow[d] \\ J \arrow[r] & I^e \end{tikzcd}$$
and
$$\begin{tikzcd} J^{cA} \arrow[r] \arrow[d] & A \arrow[d] \\ J \arrow[r] & A^e \end{tikzcd}$$
commute and the last square is a pullback, consideration of the following diagram establishes the result.

$$\begin{tikzcd} & J^{cI} \arrow[rr] \arrow[dl] & & I \arrow[dl] \arrow[dd] \\ J^{cA} \arrow[rr] \arrow[dd] & & A \arrow[dd] & \\ & & & I^e \arrow[dl] \\ J \arrow[rr] & & A^e & \end{tikzcd}$$

(e) Consider the diagram

$$\begin{tikzcd} P' \arrow[dd] \arrow[dr, dashed] \arrow[drrr] \arrow[dddr, dashed] & & & \\ & P^c \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & N_1^c \arrow[dl] \arrow[dd] \\ N_2^c \arrow[rr] \arrow[dd] & & A \arrow[dd] & \\ & P \arrow[rr] \arrow[dl] \arrow[dr] & & N_1 \arrow[dl] \\ N_2 \arrow[rr] & & A^e & \end{tikzcd} .$$

By (b) the two triangles on the top of the cube commute and so the square on the top of the cube commutes. Suppose

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & N_1^c \arrow[d] \\ N_2^c \arrow[r] & A \end{tikzcd}$$

commutes. Since the base of the cube is a pullback we obtain an appropriate map $P' \dashrightarrow P$ and since

$$\begin{tikzcd} P^c \arrow[r] \arrow[d] & A \arrow[d] \\ P \arrow[r] & A^e \end{tikzcd}$$

is a pullback, we obtain the required map $P' \dashrightarrow P^c$ which is unique since $P^c \to A$ is monic.

2. **The results.** If $\mathscr{R}$ is a radical class in $\mathscr{K}$, we denote by $\mathscr{R}^*$ the class $\{A \in \mathscr{K} \mid \Phi(A) \in \mathscr{R}\}$. Henceforth we shall only consider radical classes $\mathscr{R}$ for which $\mathscr{R}^*$ is also a radical class. Some conditions on Φ which imply that $\mathscr{R}^*$ is radical for some or all radical classes $\mathscr{R}$ are given in [5].

PROPOSITION 2.1. *For every $A \in \mathscr{K}$, we have $\mathscr{R}^*(A) \to A \subseteqq \mathscr{R}(A^e)^c \to A$.*

Proof. Since $\mathscr{R}^*(A)^e \to A^e$ is a normal $\mathscr{R}$-subobject we have $\mathscr{R}^*(A)^e \to A^e \subseteqq \mathscr{R}(A^e) \to A^e$. The result now follows from (a) and (b) of Proposition 1.1.

A radical class $\mathscr{R}$ is *hereditary* if $N \in \mathscr{R}$ whenever $M \in \mathscr{R}$ and $N \to M$ is a normal subobject. $\mathscr{R}$ is *normally-hereditary* if for every normal subobject $N \to A$ we have

$$\mathscr{R}(N) \longrightarrow N \longrightarrow A = (N \to A) \cap (\mathscr{R}(A) \longrightarrow A) .$$

Clearly normally-hereditary radical classes are hereditary, but the converse need not be true.

In what follows we shall be concerned with the following conditions involving Φ and $\mathscr{R}$.

(α) If $\mathscr{R}(A^e)^c = A$, then $A^e \in \mathscr{R}$.

(α') $\mathscr{R}(A^e)^{ce} \to A^e \subseteqq \mathscr{R}(A^e) \to A^e$ for each $A \in \mathscr{K}$.

(α'') $\mathscr{R}(A^e)^{ce} \in \mathscr{R}$ for each $A \in \mathscr{K}$.

It is easy to see that $(\alpha'') \Rightarrow (\alpha') \Rightarrow (\alpha)$.

PROPOSITION 2.2. *If every pair of normal subobjects of each object in $\mathscr{K}$ has a normal intersection, then for normally-hereditary radical classes $\mathscr{R}$, (α) and (α') are equivalent.*

Proof. We need only show that (α) implies (α'). Assume (α) is satisfied. The pullback

$$(*) \qquad \begin{array}{ccc} P & \longrightarrow & \mathscr{R}(A^e)^{ce} \\ \downarrow & & \downarrow \\ \mathscr{R}(A^e) & \longrightarrow & A^e \end{array}$$

exists. By Proposition 1.1(e)

$$\begin{tikzcd}
P^c \arrow[r] \arrow[d] & \mathscr{R}(A^e)^{cec} \arrow[d] \\
\mathscr{R}(A^e)^c \arrow[r] & A
\end{tikzcd}$$

is a pullback and by Proposition 1.1(a), $\mathscr{R}(A^e)^c \to A \subseteqq \mathscr{R}(A^e)^{cec} \to A$ and so $P^c = \mathscr{R}(A^e)^c$. Since $\mathscr{R}$ is normally-hereditary, it follows from (*) that $P = \mathscr{R}(\mathscr{R}(A^e)^{ce})$ and so $P^c = \mathscr{R}(\mathscr{R}(A^e)^{ce})^c$. We conclude that $\mathscr{R}(A^e)^c = \mathscr{R}(\mathscr{R}(A^e)^{ce})^c$, i.e., $\mathscr{R}(I^e)^{cA} = I$, where $I = \mathscr{R}(A^e)^c$.

Consider the diagram

$$\begin{tikzcd}
\mathscr{R}(I^e)^{cA} \arrow[rrr] \arrow[dr, dashed] \arrow[drr, equal] \arrow[ddr] & & & A \arrow[dd] \\
& \mathscr{R}(I^e)^{cI} \arrow[r] \arrow[d] & I \arrow[d] & \\
& \mathscr{R}(I^e) \arrow[r] & I^e \arrow[r] & A^e .
\end{tikzcd}$$

Since $I^e \to A^e$ is monic and

$$\begin{tikzcd}
\mathscr{R}(I^e)^{cI} \arrow[r] \arrow[d] & I \arrow[d] \\
\mathscr{R}(I^e) \arrow[r] & I^e
\end{tikzcd}$$

is a pullback we obtain a map $\mathscr{R}(I^e)^{cA} \to \mathscr{R}(I^e)^{cI}$ such that

$$\begin{tikzcd}
\mathscr{R}(I^e)^{cA} \arrow[rr, equal] \arrow[dr] & & I \\
& \mathscr{R}(I^e)^{cI} \arrow[ur] &
\end{tikzcd}$$

commutes. Thus $\mathscr{R}(I^e)^{cI} = I$ and so by (α) we have $\mathscr{R}(A^e)^{ce} = I^e \in \mathscr{R}$, which establishes (α').

PROPOSITION 2.3. *If $\mathscr{R}$ is hereditary, then (α') and (α'') are equivalent.*

Proof. Obvious.

PROPOSITION 2.4. *If $\mathscr{R}$ satisfies (α) then*

$$\mathscr{R}^* = \{A \mid \mathscr{R}(A^e)^c = A\} .$$

Proof. If $\mathscr{R}(A^e)^c = A$, then $A^e \in \mathscr{R}$, i.e., $A \in \mathscr{R}^*$, by (α). Conversely, if $A \in \mathscr{R}^*$, then $A^e \in \mathscr{R}$ and so $\mathscr{R}(A^e) = A^e$. Hence $\mathscr{R}(A^e)^c = A^{ec} = A$ by Proposition 1.1(c).

Proposition 2.4 gives a "global" description of $\mathscr{R}^*$. We can also give a "local" description of $\mathscr{R}^*$ under more restrictive conditions. We shall need

LEMMA 2.5. *If* $\mathscr{R}$ *satisfies* (α'') *then for each* $A \in \mathscr{K}$ *we have*

$$\mathscr{R}(A^e)^{cec} = \mathscr{R}(A^e)^c .$$

Proof. By (α''), $\mathscr{R}(A^e)^{ce} \in \mathscr{R}$, so

$$\mathscr{R}(\mathscr{R}(A^e)^{ce})^c = \mathscr{R}(A^e)^{cec} \quad \text{and} \quad \mathscr{R}(A^e)^{ce} \longrightarrow A^e \subseteqq \mathscr{R}(A^e) \longrightarrow A^e .$$

The latter implies, by Proposition 1.1(b), that $\mathscr{R}(A^e)^{cec} \to A \subseteqq \mathscr{R}(A^e)^c \to A$ and from Proposition 1.1(a) we see that $\mathscr{R}(A^e)^c \to A \subseteqq \mathscr{R}(A^e)^{cec} \to A$.

THEOREM 2.6. *If* $\mathscr{R}$ *satisfies* (α) *and* $\mathscr{R}(\mathscr{R}(A^e)^{ce})^c = \mathscr{R}(A^e)^c$, *then* $\mathscr{R}^*(A) = \mathscr{R}(A^e)^c$.

Proof. Applying Proposition 2.4 to $\mathscr{R}(A^e)^c$, we see that $\mathscr{R}(A^e)^c \in \mathscr{R}^*$. The equality now follows from Proposition 2.1.

COROLLARY 2.7. *If every pair of normal subobjects of each object of* $\mathscr{K}$ *has a normal intersection and if* $\mathscr{R}$ *is normally-hereditary and satisfies* (α), *then* $\mathscr{R}^*(A) = \mathscr{R}(A^e)^c$ *for each* $A \in \mathscr{K}$.

Proof. By Proposition 2.2, $\mathscr{R}$ satisfies (α') and hence, by Proposition 2.3, (α''). By Lemma 2.5, $\mathscr{R}$ satisfies the requirements of Theorem 2.6 for all $A \in \mathscr{K}$.

3. EXAMPLES. In the category of associative rings, the functor Φ which associates the semigroup ring $A[S]$ with a ring A(S is a fixed semigroup with identity) and acts on maps in the obvious way satisfies the requirements listed in §1, the natural transformation being defined by the standard embedding $A \to A[S]$. Moreover, $\mathscr{R}^* = \{A \mid A[S] \in \mathscr{R}\}$ is radical for every radical class $\mathscr{R}$ (cf. [5]) and $\mathscr{R}^* \subseteqq \mathscr{R}$.

The following result is essentially due to Krempa [7] who proved it in the special case where S is the free semigroup with identity on one generator, i.e., $A[S]$ is the polynomial ring $A[x]$.

PROPOSITION 3.1. *Every radical class of associative rings satisfies* (α') *for the functor defined by the correspondence* $A \mapsto A[S]$, *for any semigroup* S.

From Proposition 2.4 we see that $\mathscr{R}^* = \{A \mid \mathscr{R}(A[S]) \cap A = A\}$ for every radical class $\mathscr{R}$. Thus in the case $A^e = A[x]$, $\mathscr{R}^*$ coincides with the radical class discussed by Ortiz [8].

Proposition 2.1 and Theorem 2.6 therefore generalize Theorem 1 of [8]. By Corollary 2.7, $\mathscr{R}^*(A) = \mathscr{R}(A[S]) \cap A$ whenever $\mathscr{R}$ is hereditary. For $A[S] = A[x]$, this was proved by the first author in [4].

Another example of a functor defined on the category of associative rings which meets our requirements is that which assigns to each ring A the ring A_n of $n \times n$ matrices for some (fixed) n. Again action on maps is defined in the obvious way. The natural transformation is defined by the embedding of A in A_n as the subring of scalar matrices. In this case too, $\mathscr{R}^* = \{A \mid A_n \in \mathscr{R}\}$ is radical for all radical classes $\mathscr{R}$ [5].

The proof of the following result closely resembles that of Proposition 3.1.

PROPOSITION 3.2. *Every radical class $\mathscr{R}$ of associative rings satisfies (α') for the functor defined by the correspondence $A \mapsto A_n$.*

By Proposition 2.4, $\mathscr{R}^* = \{A \mid A \subseteqq \mathscr{R}(A_n)\}$ in this case and by Corollary 2.7, $\mathscr{R}^*(A) = \mathscr{R}(A_n) \cap A$ when $\mathscr{R}$ is hereditary. (Here we have identified A with the ring of scalar matrices.)

Let $0 \to Z \to X \to D \to 0$ be an exact sequence of abelian groups, where Z is the group of integers and D is torsion-free divisible. The functor $(\ \ ) \otimes X$ has a right adjoint and so $\mathscr{R}^* = \{G \mid G \otimes X \in \mathscr{R}\}$ is a radical class for every radical class $\mathscr{R}$ of abelian groups [5]. The map $G \to G \otimes X$ defined by the isomorphism $G \cong G \otimes Z$ and the given exact sequence defines a natural transformation from the identity to $(\ \ ) \otimes X$. All requirements of §1 are satisfied.

PROPOSITION 3.5. *Every radical class $\mathscr{R}$ of abelian groups satisfies (α) for the functor $(\ \ ) \otimes X$.*

Proof. If $G \subseteqq \mathscr{R}(G \otimes X)$, there is an epimorphism

$$G \otimes D \cong (G \otimes X)/G \longrightarrow (G \otimes X)/\mathscr{R}(G \otimes X) .$$

If $\mathscr{R}$ contains only torsion groups, then G is torsion and so $G \otimes D = 0 \in \mathscr{R}$. If $\mathscr{R}$ contains a nontorsion group, then it contains all divisible groups (see e.g. [3], Corollary 2.3) and so $G \otimes D \in \mathscr{R}$. Hence $(G \otimes X)/\mathscr{R}(G \otimes X) \in \mathscr{R}$ in all cases. Thus $G \otimes X \in \mathscr{R}$.

In all the examples considered so far, the natural transformation involved has arisen from a natural embedding $A \to A^e$. We conclude with a simple example in which the relevant map $A \to A^e$ need not be monic.

Let R_1 and R_2 be associative rings with identity, $R = R_1 \oplus R_2$

(ring direct sum) and let Φ be the functor defined on the category Mod (R) of right unital R-modules by $M \mapsto MR_1$. The classes $\mathscr{R}_i = \{MR_i \mid M \in \text{Mod}\,(R)\}$, $i = 1, 2$, are actually hereditary radical classes and we have the situation analysed in Theorem 2.4 of Jans [6]. It is straightforward to show that Φ is exact and preserves unions of ascending chains and hence (see [5]) that $\mathscr{R}^* = \{M \mid \Phi(M) \in \mathscr{R}\}$ is a radical class for every radical class $\mathscr{R}$ in Mod (R). The projection $M \rightarrow MR_1$ defines a natural transformation with the properties we want. If N is a submodule of $MR_1 = M^e$ then $N^c = N \oplus MR_2$.

PROPOSITION 3.3. *Every radical class $\mathscr{R}$ in* Mod (R) *satisfies* (α) *for the functor defined by the correspondence* $M \mapsto MR_1$.

Proof. If $\mathscr{R}(MR_1) \oplus MR_2 = M = MR_1 \oplus MR_2$, then $MR_1 = \mathscr{R}(MR_1)$.

4. **The question of "going up".** We revert to our general situation to briefly mention a related problem: to determine when $\mathscr{R}(A^e) = \mathscr{R}^*(A)^e$. Since $\mathscr{R}^*(A) \in \mathscr{R}^*$, we always have $\mathscr{R}^*(A)^e \rightarrow A^e \subseteqq \mathscr{R}(A^e) \rightarrow A^e$. The other inclusion seems to be more difficult. Amitsur [1] has given a (highly nontrivial) proof for $A^e = A[x]$ which is valid when $\mathscr{R}$ is strongly hereditary or the Jacobson radical class. On the other hand, when $A^e = A_n$, it is relatively easy to check that $\mathscr{R}(A^e) = \mathscr{R}^*(A)^e$ for all rings A when $\mathscr{R}$ is hereditary.

PROPOSITION 4.1. *Let A be an associative ring, $\mathscr{R}$ a radical class of associative rings. If A has an identity or $\mathscr{R}$ is hereditary, then $\mathscr{R}(A_n) = \mathscr{R}^*(A)_n$.*

Proof. If A has an identity, then $\mathscr{R}(A_n) = I_n$ for some ideal I of A. Since $I_n \in \mathscr{R}$ we have $I \in \mathscr{R}^*$; thus $I \subseteqq \mathscr{R}^*(A)$ and so $\mathscr{R}(A_n) \subseteqq \mathscr{R}^*(A)_n$. But $\mathscr{R}^*(A)_n \in \mathscr{R}$, so $\mathscr{R}(A_n) = \mathscr{R}(A)_n$. If A does not have an identity and $\mathscr{R}$ is hereditary, we make use of the Dorroh extension A^1 of A. Because $\mathscr{R}$ is hereditary (= normally-hereditary), so is $\mathscr{R}^*$, and thus we have $\mathscr{R}(A_n) = A_n \cap \mathscr{R}((A^1)_n) = A_n \cap \mathscr{R}^*(A^1)_n = [A \cap \mathscr{R}^*(A^1)]_n = \mathscr{R}^*(A)_n$.

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