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A NOTE ON DIFFERENTIAL EQUATIONS WITH ALL SOLUTIONS OF INTEGRABLE-SQUARE

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It is shown that if all solutions to $l(y)=\lambda wy$ and $l^+(y)=\lambda wy$ satisfy $\int_a^b |y|^2 w<\infty$ for some complex number λ then so do all solutions for every complex number λ. The result is derived from a corresponding one for first order vector-matrix systems.

We shall be concerned with solutions to

(1) $$l(y)=0 \quad \text{on} \quad (a,b),$$

(2) $$l^+(y)=0 \quad \text{on} \quad (a,b)$$

(3) $$l(y)=\lambda wy \quad \text{on} \quad (a,b), \quad \text{and}$$

(4) $$l^+(y)=\lambda wy \quad \text{on} \quad (a,b)$$

which satisfy

(5) $$\int_a^b |y|^2 w<\infty.$$

In these expressions (a,b) is an interval of the real line ($a=-\infty$ and/or $b=\infty$ is allowed), w is a weight, i.e., a positive valued continuous function on (a,b), λ is a complex number, l is an mth order linear differential operator given by

(6) $$l(y)=\sum_{k=0}^{m} a_k y^{(m-k)}$$

where each a_k is an $m-k$ times continuously differentiable complex valued function defined on (a,b), $a_0(t)\neq 0$ for all $t\in(a,b)$, and l^+ is the formal adjoint of l so that

(7) $$l^+(y)=\sum_{k=1}^{m}(-1)^{m-k}(\bar{a}_k y)^{(m-k)}.$$

In an earlier paper, **[11]**, we defined w to be a compactifying weight for l provided that every function which is a solution either of (1) or of (2) satisfies (5). If follows from Theorem 2–1 of **[11]** that if w is a compactifying weight for l then every function which is a solution either of (3) or of (4) satisfies (5) for every complex number λ.

The deficiency index problem (see for example **[2]** and **[8]**) for formally self-adjoint equations (where $l = l^+$) is concerned with finding the dimension of the linear manifold of solutions to (3) which satisfy (5). One of the results of this theory (**[3]**, **[4]**, **[5]**, **[6]**, **[7]**, **[10]**, and **[12]**) is that if this dimension is m (the order of l) for some complex number λ and $m > 1$ then it is m for every complex number λ.

While much of the theory for the self-adjoint case breaks down when $l \neq l^+$ we wish to show that this result carries over.

THEOREM 1. *Let each of* λ_1 *and* λ_2 *be a complex number* (λ_j *real, even* $\lambda_j = 0$ *is allowed*). *Let* $m > 1$. *If every function which is a solution of either* (3) *or* (4) *satisfies* (5) *when* $\lambda = \lambda_1$ *then every function which is a solution of either* (3) *or* (4) *satisfies* (5) *when* $\lambda = \lambda_2$.

This follows as a corollary to an analogous theorem (Theorem 2 below) for first order vector-matrix equations.

We consider the equations,

$$Jy' = [\lambda A + B]y \quad \text{a.e. on } (a,b), \text{ and} \tag{8}$$

$$Jy' = [\lambda A + B^*]y \quad \text{a.e, on } (a,b) \tag{9}$$

where J is a skew-symmetric ($J^* = -J$, $*$ denoting conjugate transpose) $m \times m$ matrix, each of A and B is a complex $m \times m$ matrix valued function which is Lebesque integrable over each compact subinterval of (a,b), λ is a complex number, and $A(t)$ is nonnegative definate a.e. on (a,b).

It was shown in **[13]** that, given l; J, A, and B may be chosen so that every solution of (3) satisfies (5) if and only if every solution of (8) satisfies

$$\int_a^b y^*Ay < \infty, \tag{10}$$

and every solution of (4) satisfies (5) if and only if every solution of (9) satisfies (10). For the choice of J and A used in **[13]** it is also the case that trace $J^{-1}A \equiv 0$ when $m > 1$.

Thus Theorem 1 above follows from Theorem 2 below.

THEOREM 2. *Let each of* J, A, *and* B *satisfy the conditions imposed above. Let* $m > 1$. *Let each of* λ_1 *and* λ_2 *be a complex number* (λ_j *real, even* $\lambda_j = 0$ *is allowed*). *Let* $\int_a^b |\operatorname{tr} J^{-1}A| < \infty$.

If every vector function which is a solution of either (8) *or* (9) *satisfies* (10) *when* $\lambda = \lambda_1$ *then every vector function which is a solution of either* (8) *or* (9) *satisfies* (10) *when* $\lambda = \lambda_2$.

Proof. Let $\boldsymbol{Y}(\lambda)$ and $\boldsymbol{Z}(\lambda)$ be fundamental matrices for (8) and (9) respectively. (We will write $\boldsymbol{Y}(t,\lambda)$ and $\boldsymbol{Z}(t,\lambda)$ to denote the value of these functions at $t \in (a,b)$.) Let $\boldsymbol{U}$ be defined by

$$\boldsymbol{Y}(\lambda_2) = \boldsymbol{Y}(\lambda_1)\boldsymbol{U} \quad \text{on} \quad (a,b). \tag{11}$$

Multiplying on the left by J, differentiating, and using (8) we have,

$$(\lambda_2 A + B)\boldsymbol{Y}(\lambda_2) = (\lambda_1 A + B)\boldsymbol{Y}(\lambda_1)\boldsymbol{U} + J\boldsymbol{Y}(\lambda_1)\boldsymbol{U}' \quad \text{a.e.} \quad \text{on} \quad (a,b).$$

From (11) we have,

$$J\boldsymbol{Y}(\lambda_1)\boldsymbol{U}' = (\lambda_2 - \lambda_1) A\boldsymbol{Y}(\lambda_1)\boldsymbol{U} \quad \text{a.e.} \quad \text{on} \quad (a,b).$$

Multiplying on the left by $\boldsymbol{Z}^*(\lambda_1)$ we have

$$\boldsymbol{Z}^*(\lambda_1)J\boldsymbol{Y}(\lambda_1)\boldsymbol{U}' = (\lambda_2 - \lambda_1)\boldsymbol{Z}^*(\lambda_1)A\boldsymbol{Y}(\lambda_1)\boldsymbol{U} \quad \text{a.e.} \quad \text{on} \quad (a,b). \tag{12}$$

We first note that

$$\int_a^b \|\boldsymbol{Z}^*(t,\lambda_1)Y(t,\lambda_1)\|dt < \infty \tag{13}$$

where $\|\cdot\|$ is any matrix norm. In order that (13) hold it is sufficient that

$$\int_a^b |z_i^*(t,\lambda_1)A(t)y_j(t,\lambda_1)|dt < \infty \tag{14}$$

whenever z_i a column of $\boldsymbol{Z}$ and y_j is a column of $\boldsymbol{Y}$. By the Cauchy-Schwartz inequality we have a.e. on (a,b) (writing z for $z_i(t,\lambda_1)$ and y for $y_j(t,\lambda_1)$) that

$$|z^*Ay| \leqq (z^*Az)^{1/2}(yAy)^{1/2}. \tag{15}$$

From

$$0 \leqq ((z^*Az)^{1/2} - (y^*Ay)^{1/2})^2$$

we have that

$$(z^*Az)^{1/2}\cdot(y^*Ay)^{1/2}\leqq\tfrac{1}{2}(z^*Az+y^*Ay). \tag{16}$$

From (15), (16) and the hypothesis that every solution of (8) or (9) satisfies (10) when $\lambda=\lambda_1$ we see that 14 holds.

Next we establish that

$$(\boldsymbol{Z}^*(\lambda_1)J\boldsymbol{Y}(\lambda_1))^{-1} \tag{17}$$

is bounded on (a,b). Let $\alpha\in(a,b)$ then by Theorem 4 of **[13]** it follows that

$$\boldsymbol{Z}^*(t,\lambda_1)J\boldsymbol{Y}(t,\lambda_1)$$

$$=\boldsymbol{Z}^*(\alpha,\lambda_1)J\boldsymbol{Y}(\alpha,\lambda_1)+(\lambda_1-\bar{\lambda}_1)\int_\alpha^t \boldsymbol{Z}^*(s,\lambda_1)A(s)\boldsymbol{Y}(s,\lambda_1)ds$$

for all $t\in(a,b)$. Thus from (13) we see that

$$\boldsymbol{Z}^*(t,\lambda_1)J\boldsymbol{Y}(t,\lambda_1) \tag{18}$$

has a limit as $t\to a$ and as $t\to b$. In order to show that (17) (which is continuous) is bounded it is then sufficient to show that the limits of (18) at a and at b are nonsingular. From Abel's formula for (8) and (9) (recall that $J^*=-J$, $A^*=A$, and $\operatorname{tr} PQ=\operatorname{tr} QP$ for matrices P and Q) we have that

$$\begin{aligned}&\det(\boldsymbol{Z}^*(t,\lambda_1)J\boldsymbol{Y}(t,\lambda_1))\\&=\det(\boldsymbol{Z}^*(\alpha,\lambda_1)J\boldsymbol{Y}(\alpha,\lambda_1))\\&\cdot\exp\int_\alpha^t \operatorname{tr}((J^{-1}\lambda_1A+J^{-1}B^*)^*+J^{-1}\lambda_1A+J^{-1}B)\\&=\det((\boldsymbol{Z}^*(\alpha,\lambda_1)j\boldsymbol{Y}(\alpha,\lambda_1))\exp\int_\alpha^t(\lambda_1-\bar{\lambda}_1)\operatorname{tr}J^{-1}A.\end{aligned}$$

Since by hypothesis $\int_a^b|\operatorname{tr}J^{-1}A|<\infty$ the limits of (18) must be nonsingular.

It now follows that (12) is equivalent to an equation of the form

$$U'=MU\quad\text{a.e.}\quad\text{on}\quad(a,b) \tag{19}$$

where $\int_a^b\|M(t)\|\,dt<\infty$. It is well known (see, e.g. Theorem 5.4.2 of **[9]**) that all solutions of (19) are bounded.

Returning to (11) we see that every solution of (8) when $\lambda = \lambda_2$ is a bounded multiple of a solution of (8) when $\lambda = \lambda_1$.

The argument to show that every solution of (9) satisfies (10) when $\lambda = \lambda_2$ is similar.

Theorem 2 is a generalization of a result of Atkinson (Theorem 9.11.2 of [**1**]) for the case where $B^* = B$.

Theorem 1 is also valid for the quasidifferential expressions considered in [**13**] where no smoothness conditions on the coefficients of l are required.

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