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# COMPLETELY OUTER GALOIS THEORY OF PERFECT RINGS

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**Let $G$ be a finite completely outer group of automorphisms of a perfect ring $R$. Let $\Delta$ be the crossed product of $R$ with $G$. Then $\Delta$ modules which are $R$ projective are $\Delta$ projective and $\Delta$ submodules which are $R$ direct summands are $\Delta$ direct summands.**

Let $R$ be a ring with identity. All modules and homomorphisms are unital. Let $G = \{1, \sigma, \cdots, \tau\}$ be a finite group of automorphisms of $R$.

By $Ru_\sigma$, $\sigma \in G$, we mean a bi-$R$ module where $r(r'u_\sigma) = (rr')u_\sigma$ and $(r'u_\sigma)r = (r'r^\sigma)u_\sigma$ for $r, r' \in R$ and $r^\sigma = \sigma(r)$. We call $G$ a completely outer group of automorphisms of $R$, if for each pair $\sigma \neq \tau$ of automorphisms in $G$, the bi-$R$ modules $Ru_\sigma$ and $Ru_\tau$ have no nonzero isomorphic subquotients. This notion was defined by T. Nakayama in [**3**, p. 203] and Y. Miyashita in [**2**, p. 126].

Let $S$ be the fixed ring of $R$ under $G$, i.e. $\{r \in R \mid \sigma(r) = r$ for all $\sigma \in G\}$. If $G$ is completely outer, then $R$ over $S$ is a $G$-Galois extension and the center of $R$ is the centralizer of $S$ in $R$. See [**2**, Proposition 6.4, p. 127]. Furthermore, if $R$ is a commutative ring and $R$ over $S$ is $G$-Galois, then $G$ is a completely outer group of automorphisms of $R$. See [**2**, Theorem 6.6, p. 128]. If $R$ is a simple ring and $G$ contains no inner automorphisms, then $G$ is a completely outer group of automorphisms of $R$ and conversely. See [**2**, Corollary, p. 128].

The crossed product $\Delta$ of $R$ with $G$ is $\Sigma_{\sigma \in G} \oplus Ru_\sigma$ with $(xu_\sigma) \times (yu_\tau) = xy^\sigma u_{\sigma\tau}$ for any $x$ and $y$ in $R$. We can view $R$ as a left $\Delta$ module by defining $(xu_\sigma)r = xr^\sigma$ for $x$ and $r$ in $R$. Thus $R$ is a bi $\Delta$-$S$ module.

We now assume $R$ is a left perfect ring. These rings were studied by H. Bass [**1**].

Let $J(\Delta)$ (respectively $J(R)$) denote the Jacobson radical of $\Delta$ (respectively $R$).

LEMMA 1. $J(\Delta) = J(R)\Delta = \Delta J(R)$. *Thus for any left $\Delta$ module $M$, $J(R)M$ is a $\Delta$ submodule of $M$.*

*Proof.* Because $\sigma(J(R)) = J(R)$ for all $\sigma \in G$, $J(R)\Delta = \Delta \cdot J(R)$. Thus for any simple, nonzero left $\Delta$ module $M, J(R)M$ is a $\Delta$ submodule. Now $M$ is a finitely generated $R$ module, since $\Delta$ is a finitely generated $R$ module. Nakayama's Lemma then shows

$J(R)M = 0$. Since $J(R)$ annihilates every simple left $\Delta$ module, $J(R) \subseteq J(\Delta)$.

Since $J(\Delta)$ is a bi-$R$ submodule of $\Delta, J(\Delta) = J(\Delta) \cap Ru_1 + J(\Delta) \cap Ru_\sigma + \cdots + J(\Delta) \cap Ru_\tau$. See [2, Proposition 6.1, p. 126]. Let $x = \delta = ru_\sigma \in J(\Delta) \cap Ru_\sigma$ for $\delta$ in $J(\Delta)$ and $r \in R$. Assume $\delta = \sum_{\sigma \in G} y_\sigma u_\sigma$, $y_\sigma \in R$, then $x = y_\sigma u_\sigma = ru_\sigma$, so $ru_1 = y_\sigma u_1 = xu_{\sigma^{-1}} \in J(\Delta) \cap Ru_1$. It follows that $1 - y_\sigma s$ is right invertible in $R$ for all $s$ in $R$; since $u_1 - y_\sigma s u_1$ in $J(\Delta)$ all $s$ in $R$. Thus $y_\sigma \in J(R)$; hence $x = y_\sigma u_\sigma \in J(R)u_\sigma$. Therefore $J(R)\Delta = J(\Delta)$.

REMARK. Lemma 1 is true even if $R$ is not left perfect.

LEMMA 2. *As a right $S$ module, $S$ is a direct summand of $R_S$. Let* $J(S)$ (respectively $J(R)$) *denote the Jacobson radical of $S$* (respectively of $R$), *then* $J(S) = S \cap J(R)$ *and* $J(R) = J(S)R = RJ(S)$. *Thus if $M$ is a left $\Delta(R)$ module $J(S)M$ is a left $\Delta(R)$ submodule.*

*Proof.* Let $\bar{\Delta} = \Delta/J(\Delta)$ and $\bar{R} = R/J(R)$. Because $R$ is perfect, $\bar{R}$ is a semisimple, Artinian ring, which makes $\bar{\Delta}$ a semisimple, Artinian ring. Thus $\bar{R}$ is a finitely generated, projective $\bar{\Delta}$ module. By the Dual Basis Lemma, there exists $f_1, \cdots, f_n \in \operatorname{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta})$ and $\bar{x}_1, \cdots, \bar{x}_n \in \bar{R}$ such that $\bar{x} = \sum_{i=1}^n f_i(\bar{x})\bar{x}_i$. Since $\operatorname{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta}) \subseteq \operatorname{Hom}_{\bar{R}}(\bar{R}, \bar{\Delta}) = \bar{\Delta}$ we conclude $\operatorname{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta}) = \sum_{\sigma \in G} u_\sigma \bar{R}$. Thus each $f_i$, $i = 1, \cdots, n$, is of the form $\sum_{\sigma \in G} u_\sigma \bar{r}_i$, for some suitable $\bar{r}_i \in R$. Let $\bar{x} \in \bar{R}$, then $\bar{x} = \sum_{i=1}^n f_i(\bar{x})\bar{x}_i = \sum_{i=1}^n (\sum_{\sigma \in G} \bar{x} u_\sigma \bar{r}_i)\bar{x}_i = \bar{x} \sum_{i=1}^n \sum_{\sigma \in G} (\bar{r}_i \bar{x}_i)^\sigma$. Thus $\bar{1} = \sum_{\sigma \in G} \sum_{i=1}^n (\bar{r}_i \bar{x}_i)^\sigma$. Let $\bar{d} = \sum_{i=1}^n \overline{r_i x_i}$, then tr $\bar{d} = \bar{1}$; hence tr $d - 1 \in J(R) \cap S$.

Now $J(R) \cap S \subseteq J(S)$, for let $x = j = s$, $j \in J(R)$ and $s \in S$, then $1 - sy$ is right invertible in $R$ for any $y$ in $S$. Assume $(1 - sy)z = 1$, $z \in R$, then $(1 - sy)z^\sigma = 1$ for all $\sigma \in G$; hence $z \in S$. So $1 - sy$ is right invertible in $S$ for all $y$ in $S$, thus $x \in J(S)$ or $J(R) \cap S \subseteq J(S)$.

Thus tr $R + J(S) = S$, so by Nakayama's Lemma tr $R = S$. Thus there is a $c$ in $R$ such that tr $c = 1$. Hence tr $: R_S \to S_S$ is onto and so splits. Thus $S_S$ is a direct summand of $R_S$.

The conclusion concerning the Jacobson radical of $S$ follows from [2, Theorem 7.10, p. 132].

PROPOSITION 1. *A left $\Delta$ module is completely reducible as a $\Delta$ module if and only if it is completely reducible as an $R$ module. Moreover, a module is completely reducible as a left $R$-module if and only if it is completely reducible as an $S$ module.*

*Proof.* A $\Delta$ module is annihilated by $J(\Delta)$ if and only if it is annihilated by $J(R)$ if and only if it is annihilated by $J(S)$.

PROPOSITION 2. *Let $R$ be a left perfect ring and $G$ a completely outer group of automorphisms acting on $R$. Then $S$, the fixed ring of $R$ under $G$, and $\Delta$, the crossed product of $R$ with $G$ are left perfect.*

*Proof.* Since $R$ is left perfect its Jacobson radical $J(R)$, is left $T$ nilpotent. So the Jacobson radical of $S, J(S)$, which is contained in $J(R)$ (by Lemma 2) is left $T$ nilpotent.

Also $S/J(S)$ is an Artinian ring since $S_S$ is a direct summand of $R_S$. See Lemma 2 and [**2**, Proposition 7.3, p. 130]. Thus $S$ is a left perfect; hence, $S$ is semiperfect.

Now $R$ as a right $S$ module is finitely generated and projective; moreover, $\Delta$ is isomorphic to End $R_S$ [**2**, p. 116]. Since $S$ is a direct summand of $R$, as a right $S$ module (Lemma 2) $R$ is an $S$ generator.

Let $e_1, \cdots, e_n$ be completely primitive idempotents orthogonal idempotents of $S$ such that $1 = e_1 + \cdots + e_n$. Furthermore, let $e_1, \cdots, e_k$ be a maximal family of mutually nonisomorphic idempotents. Then $\bar{R} = R/J(S)R$ is isomorphic, as a right $S$ module, to $\Sigma_{i=1}^{k} + (\bar{e}_i \cdot \bar{S})^{m_i}$, where $\bar{e}_i = e_i + J(S)$, $\bar{S} = S/J(S)$ and $m_i < \infty$, since $\bar{R}$ is finitely generated right $S$ module. Thus $R$ as a right $S$ module, is isomorphic to $\Sigma_{i=1}^{k} \bigoplus \Sigma_{i=1}^{m_i} \bigoplus P_{ij}$, where $P_{ij}$ is right $S$ isomorphic to $e_iS$, since idempotents can be lifted.

Let $f_{ij}$ be the projection of $R$ onto $P_{ij}$, then $f_{ij} \in$ End $R_S = \Delta$ and the $f_{ij}$'s are orthogonal idempotents such that

$$1 = \sum_{i=1}^{k} \sum_{j=1}^{m_i} f_{ij}. \quad \text{Also } e_iSe_i = \text{End}_S(e_iS) = \text{End}_S(P_{ij}) = f_{ij}\,\Delta f_{ij}.$$

Since $e_iSe_i$ is a local ring, $f_{ij}\,\Delta f_{ij}$ is a local ring. Hence $f_{ij}$ is a completely primitive idempotent; therefore $\Delta$ is semiperfect.

We know that $\Delta$ modulo its Jacobson radical, $J(\Delta)$, is semisimple and idempotents can be lifted modulo $J(\Delta)$. Let $M$ be a left $\Delta$ module, by [**1**, Lemma 2.6, p. 473] in order that $M$ have a projective cover it suffices that for any left $\Delta$ module $B$ requiring no more generators than $M, B = J(\Delta)B$ implies $B = 0$. But $B = J(\Delta)B = J(R)B$ and $R$ being left perfect implies $B = 0$. Thus every left $\Delta$ module $M$ has a projective cover and $\Delta$ is then left perfect.

Let $T$ be an arbitrary left perfect ring and let $J(T)$ denote the Jacobson radical of $T$. Then for any nonzero left $T$ module $M, J(T)M$ is a proper submodule. See [**1**, p. 473]. Hence the natural map $\pi : M \to M/J(T)M$ is a minimal epimorphism.

Let $M$ and $N$ be left $T$ modules and $f$ a left $T$ epimorphism from $M$ to $N$. By $\bar{f}$, we mean the induced map from $M/J(T)M \to N/J(T)N$ given by $\bar{f}(m + J(T)M) = f(m) + J(T)N$ for $m \in M$.

LEMMA 3. *The following are equivalent*:

(1) $f: M \to N$ *is a minimal* $T$ *epimorphism.*

(2) $\bar{f}: M/J(T)M \to N/J(T)N$ *is an isomorphism. See* [**4**, Proposition 8, p. 713].

PROPOSITION 3. *Let* $M$ *and* $N$ *be left* $\Delta$ *modules and* $f$ *a minimal* $\Delta$ *epimorphism form* $M$ *to* $N$. *Then* $f$ *is a minimal* $R$ *epimorphism and* $f$ *is a minimal* $S$ *epimorphism.*

*Proof.* Certainly $f$ is an $R$ and an $S$ epimorphism. Since $J(\Delta)M = J(R)M = J(S)M$ and $J(\Delta)N = J(R)N = J(S)N$, then $\bar{f}$, which is a $\Delta$ isomorphism, is an $R$ and an $S$ isomorphism.

PROPOSITION 4. *Let* $M$ *be a left* $\Delta$ *module which is projective as an* $S$ *module, then* $M$ *is projective as a* $\Delta$ *module.*

*Proof.* Let $P$ be the $\Delta$ cover of $M$ and $f: P \to M$ a minimal $\Delta$ epimorphism. Since $M$ is $S$ projective, $f$ splits as an $S$ epimorphism. Thus $P$ as an $S$ module is isomorphic to $\ker f + X$, for some $S$ submodule $X$ of $P$. But $f$ is a minimal $S$ epimorphism (Proposition 3), therefore $\ker f = 0$. So $f$ is a $\Delta$ isomorphism and $M$ is $\Delta$ projective.

PROPOSITION 5. *Let* $M$ *be a left* $\Delta$ *module which is projective as an* $R$ *module. Then* $M$ *is projective as a* $\Delta$ *module.*

*Proof.* Let $P$ be the $\Delta$ projective cover of $M$ and $f$ a minimal $\Delta$ epimorphism. Now $f$ splits as an $R$ map, and $f$ is a minimal $R$ epimorphism, so $\ker f = 0$.

PROPOSITION 6. *Let* $M$ *and* $N$ *be left* $\Delta$ *modules such that* $M/J(\Delta)M$ *and* $N/J(\Delta)N$ *are isomorphic as* $R$ *modules. If* $M$ *is* $R$ *projective there exists a* $\Delta$ *epimorphism* $\phi: M \to N$. *Moreover, if* $N$ *is* $R$ *projective, then* $M$ *and* $N$ *are* $\Delta$ *isomorphic. See* [**3**, Lemma 5, p. 212].

*Proof.* We assume that $M$ and $N$ are completely reducible $\Delta$ modules. Hence they are completely reducible $R$-modules, by Proposition 1.

Now Nakayama in [**3**, p. 214] has shown that if $M$ and $N$ are isomorphic as $R$ modules, then they are isomorphic as $\Delta$ modules.

If $M$ and $N$ are arbitrary left $\Delta$ modules, then $M/J(\Delta)M$ and $N/J(\Delta)N$ are nonzero, completely reducible left $\Delta$ modules. We have assumed they are isomorphic as left $R$ modules; hence by the above argument, they are isomorphic as left $\Delta$ modules. Call the isomorphisms from $M/J(\Delta)M$ to $N/J(\Delta)N, f$.

Let $\pi : M \to M/J(\Delta)M$ and $\pi' : N \to N/J(\Delta)N$ be the natural maps. Since $M$ is $R$ projective, it is $\Delta$ projective. Thus we can find a $\Delta$ homomorphism $g$ from $M$ to $N$ such that $\pi' g = f\pi$. Now $g$ is an epimorphism since $\pi'$ is a minimal $\Delta$ map.

If $N$ is also $R$ projective, it is $\Delta$ projective. Thus $g$ is an isomorphism.

PROPOSITION 7. *Let $M$ and $N$ be left $\Delta$ modules and $f$ a $\Delta$ epimorphism from $M$ to $N$. If $f$ is a minimal epimorphism as an $R$ map, then $f$ is a minimal epimorphism as a $\Delta$ map. Furthermore, if $M$ and $N$ are $R$ projective, then $f$ is an isomorphism.*

*Proof.* We know that $M/J(R)M$ and $N/J(R)N$ are isomorphic as $R$ modules and completely reducible. Hence they are isomorphic as $\Delta$ modules. Thus $f$ is a minimal $\Delta$ map.

If $M$ and $N$ are $R$ projective, then they are projective as $\Delta$ modules. Thus $f$ splits; let $g : N \to M$ be a $\Delta$ map such that $fg$ is the identity on $N$. Since the natural map $\pi : M \to M/J(\Delta)M$ is minimal $g$ is an isomorphism.

PROPOSITION 8. *Let $M, N$ be left $\Delta$ modules such that $M$ is a projective $R$ module. If $N$ is an $R$ direct summand of $M$, then $N$ is a $\Delta$ direct summand of $M$.*

*Proof.* Since $M$ is $R$ projective and $N$ is a direct summand, then $N$ is $R$ projective. Hence $M$ and $N$ are $\Delta$ projective.

If $N$ is an $R$ direct summand of $M$, then $J(R)M \cap N = J(R)N$ so $J(\Delta)M \cap N = J(\Delta)N$. Thus $N/J(\Delta)N$ is a $\Delta$ submodule of the completely reducible $\Delta$ module $M/J(\Delta)M$.

Thus $N/J(\Delta)N$ is a $\Delta$ direct summand of $M/J(\Delta)M$, so there is a $\Delta$ epimorphism $\phi : M \to N/J(\Delta)N$. Let $\pi' : N \to N/J(\Delta)N$, be the natural map. Since $M$ is $\Delta$ projective there exists a $\Delta$ map $\psi : M \to N$ such that $\pi'\psi = \phi$. Now $\psi$ is an epimorphism since $\pi'$ is minimal. Thus $\psi$ splits as a $\Delta$ map and $N$ is then a $\Delta$ direct summand of $M$.

PROPOSITION 9. *Let $M$ and $N$ be left $\Delta$ modules such that $N/J(R)N$ is an $R$ homomorphic image of $M/J(R)M$ and $M$ is projective as an $R$ module. Then $N$ is a $\Delta$ homomorphic image of $M$. Moreover,* *if $N$ is $R$ projective, then $N$ is a $\Delta$ direct summand of $M$.*

*Proof.* Let $f$ be an $R$ epimorphism from $M/J(R)M$ to $N/J(R)N$. Since $R/J(R)$ is a semisimple, Artinian ring, $f$ splits; hence $N/J(R)N$ is a $R$ direct summand of $M/J(R)M$.

Now $G$ is a completely outer group of automorphisms of $R/J(R)$. The crossed product of $R/J(R)$ and $G$ is $\Delta/J(\Delta)$.

Applying Proposition 8 we see that $N/J(R)N$ is a $\Delta$ direct summand of $M/J(R)M$. Thus there is a $\Delta$ map $\phi$ from $M$ to $N/J(R)N$. Let $\pi' : N \to N/J(R)N$ be the natural map. Since $M$ is $R$ projective, there is a $\Delta$ map $g : M \to N$ such that $\pi' g = \phi$. Now $g$ is an epimorphism, since $\pi'$ is a minimal $\Delta$ map.

If $N$ is $R$ projective, then $N$ is $\Delta$ projective. So $g$ splits.

PROPOSITION 10. *Let $g = |G|$, then $R^g$ is $\Delta$ isomorphic to $\Delta$.*

*Proof.* $(R/J(R))^g$ is $R$ isomorphic to $\Delta/J(R)\Delta$.

Thus Proposition 6 implies $R^g$ and $\Delta$ are isomorphic.

PROPOSITION 11. *$R$ has a normal basis.*

*Proof.* Proposition 10 imples $S^g$ is $S$ isomorphic to $R$ so $R$ has a normal basis by [**2**, Theorem 1.7, p. 118].

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