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A NOTE ON QUASIDIAGONAL AND QUASITRIANGULAR OPERATORS

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Let H be a fixed separable Hilbert space of infinite dimension. Let $\mathcal{N}, \mathcal{K}$ $\mathcal{D}$ and $\mathcal{T}$ be all normal, compact, quasidiagonal and quasitriangular operators on H, respectively. In this note it is shown that $\mathcal{N}+\mathcal{K}$ is arcwise connected, nowhere dense subset of $\mathcal{D}$ and $\mathcal{D}$ is an arcwise connected, nowhere dense subset of $\mathcal{T}$.

Introduction. Quasitriangular operators have recieved considerable attention in the recent literature **[1, 2, 3, 4, 7, 8, 9, 10]**. Quasidiagonal operators have not received as much attention [**8**, p. 901], so it is the purpose of this note to prove some of the properties of this class of operators and to put these results in perspective with known results on quasitriangular operators. Hopefully, this will lead to a better understanding of both classes of operators.

A continuous linear transformation from H into H will be called an "operator." If T is an operator on H, then T is *quasitriangular* (*quasidiagonal*) if there exists a sequence $\{P_n\}$ of finite rank orthogonal projections on H converging strongly to I such that $\|TP_n - P_nTP_n\| \to 0$ ($\|TP_n - P_nT\| \to 0$).

Since the diagonal operators are dense in $\mathcal{N}$ and since $\mathcal{D}$ is closed [**8**, p. 902–903], $\mathcal{N} \subseteq \mathcal{D}$. Let K be a compact operator, then by [**8**, p. 902–903] $T+K \in \mathcal{D}$ for all $T \in \mathcal{D}$. Therefore $\mathcal{N} \subseteq \mathcal{N}+\mathcal{K} \subseteq \mathcal{D}$. In the following theorem, it is shown that $\mathcal{N}$ is a "thin" subset of $\mathcal{N}+\mathcal{K}$ and $\mathcal{N}+\mathcal{K}$ is a "thin" subset of $\mathcal{D}$.

THEOREM 1. *$\mathcal{N}, \mathcal{N}+\mathcal{K}$ is an arcwise connected, closed, nowhere dense subset of $\mathcal{N}+\mathcal{K}, \mathcal{D}$; respectively* (norm topology).

Proof. Since the ray from 0 to T is contained in $\mathcal{N}$ if $T \in \mathcal{N}, \mathcal{N}$ is arcwise connected. Similarly $\mathcal{N}+\mathcal{K}$ is arcwise connected. The fact that $\mathcal{N}$ is closed is obvious. Peter Fillmore [**5**] has recently proved that $\mathcal{N}+\mathcal{K}$ is closed. Therefore to complete the proof of the theorem, it suffices to show, then $\mathcal{N}$ has empty interior in $\mathcal{N}+\mathcal{K}$ and $\mathcal{N}+\mathcal{K}$ has empty interior in $\mathcal{D}$.

Let $N = \text{diag}\,(a_1, a_2, \cdots)$ and $\epsilon > 0$. Define $T = \text{diag}(A_1, A_2, \cdots)$ where $A_n = \begin{pmatrix} a_{2n-1} & \epsilon/n \\ 0 & a_{2n} \end{pmatrix}$. Since each A_n is not normal, T is not normal. Notice that $T - N$ is the adjoint of the weighted unilateral shift with weights $\epsilon, 0, \epsilon/2, 0, \epsilon/3, \cdots$. Therefore $T - N$ is compact and $\|T - N\| = \epsilon$. Consequently N cannot be contained in the interior of $\mathcal{N}$ in $\mathcal{N} + \mathcal{K}$. Therefore, since the diagonal operators are dense in $\mathcal{N}, \mathcal{N}$ has empty interior in $\mathcal{N} + \mathcal{K}$.

Let $T = N + K$ where $N = \operatorname{diag}(a_1, a_2, \cdots)$ and K is compact. Let $\epsilon > 0$ and define $N_\epsilon = \operatorname{diag}(A_1, A_2, A_3, \cdots)$ where $A_n = \begin{pmatrix} a_{2n-1} & \epsilon \\ 0 & a_{2n} \end{pmatrix}$. If $N_\epsilon \in \mathcal{N} + \mathcal{K}$, then there exists a compact operator C such that $N_\epsilon + C$ is normal, i.e. $(N_\epsilon + C)(N_\epsilon + C)^* = (N_\epsilon + C)^*(N_\epsilon + C)$. Multiplying this out, we obtain $N_\epsilon N_\epsilon^* - N_\epsilon^* N_\epsilon = N_\epsilon^* C + C^* N_\epsilon + C^* C - N_\epsilon C^* - C N_\epsilon^* - CC^*$. Therefore $N_\epsilon N_\epsilon^* - N_\epsilon^* N_\epsilon$ is a compact operator. Since $N_\epsilon N_\epsilon^* - N_\epsilon^* N_\epsilon = \operatorname{diag}(A_1 A_1^* - A_1^* A_1, \cdots)$ is compact, $\|A_n A_n^* - A_n^* A_n\| \to 0$. However, $\|A_n A_n^* - A_n^* A_n\| = \left\| \begin{pmatrix} \epsilon^2 & \epsilon(\bar{a}_{2n} - \bar{a}_{2n-1}) \\ \epsilon(a_{2n} - a_{2n-1}) & \epsilon^2 \end{pmatrix} \right\| \geqq \epsilon^2$.

Contradiction. Therefore $N_\epsilon \notin \mathcal{N} + \mathcal{K}$ so that $N_\epsilon + K \notin \mathcal{N} + \mathcal{K}$. Since $N_\epsilon + K \in \mathcal{D}$ and since $\|T - (N_\epsilon + K)\| = \|(N + K) - (N_\epsilon + K)\| = \|N - N_\epsilon\| = \epsilon, T$ is not in the interior of $\mathcal{N} + \mathcal{K}$ in $\mathcal{D}$. Since the diagonal operators are dense in $\mathcal{N}$, the interior of $\mathcal{N} + \mathcal{K}$ in $\mathcal{D}$ is empty. The proof of theorem 1 is now complete.

In [**8**, p. 903] Halmos shows that every quasidiagonal operator T can be written as $D + C$ where D is block diagonal and C is a compact operator. More can be said about this decomposition:

REMARK 1. (Halmos) If T is quasidiagonal and $\epsilon > 0$, then there exists a block diagonal operator D and a compact operator C such that $\|C\| < \epsilon$ and $T = D + C$.

The proof of this remark is essentially the same as Halmos' proof found in [**10**, Theorem 2.2] where he shows that a quasitriangular T can be written as $D + C$ where D is triangular and C is compact with $\|C\| < \epsilon$. Hence the proof is omitted from this paper. As an obvious corollary to Remark 1, we have

REMARK 2. (a) The set of all block diagonal operators is norm dense in $\mathcal{D}$.

(b) If T is invertible and quasidiagonal, then T^{-1} is quasidiagonal.

Proof. Part (a) is obvious. Since T is invertible and quasidiagonal, there exists invertible block diagonal operators $\{D_n\}$ such that $D_n \to T$. Therefore $D_n^{-1} \to T^{-1}$. Since D_n^{-1} is block diagonal and since $\mathcal{D}$ is closed, T^{-1} is quasidiagonal.

Let U be the unilateral shift. From [**6**, problem 109], if $\|U-T\|<1$, then T is not invertible. Hence it follows that the set of all invertible operators is not (norm) dense in $B(H)$, the set of all operators on H. U^* is obviously quasitriangular (in fact, U^* is upper triangular). By the above result $\|U^*-T\|=\|U-T^*\|<1$ implies T is not invertible. Consequently, the set of all invertible operators is not dense in $\mathcal{T}$.

THEOREM 2. *The set of all invertible block diagonal operators is dense in* $\mathcal{D}$.

Proof. Let $T\in\mathcal{D}$. Since the set of all block diagonal operators is dense in $\mathcal{D}$ (Remark 2), we may assume that T is block diagonal relative to $H=M_1\oplus M_2\oplus\cdots$ ($\dim M_n<\infty$ for all n) and $T=T_1\oplus T_2\oplus\cdots$. Since $\dim M_n<\infty$, $T_n=U_nP_n$ where U_n is unitary and $P_n=(T_n^*T_n)^{1/2}\geqq 0$. We may assume that P_n is a diagonal operator. For each positive integer m, let $D_{n,m}$ be the diagonal operator obtained from P_n by replacing each diagonal entry which is less than $1/m$ by $1/m$. Let $S_{n,m}=U_nD_{n,m}$ and let $S_m=S_{1,m}\oplus S_{2,m}\oplus\cdots$. It follows immediately that S_m is an invertible block diagonal operator such that $\|T-S_m\|\leqq 1/m$, and the theorem is proved.

THEOREM 3. $\mathcal{D}$ *is an arcwise connected, closed, nowhere dense subset of* $\mathcal{T}$ (norm topology).

Proof. Since the ray from the zero operator to T is contained in $\mathcal{D}$ when $T\in\mathcal{D}$, $\mathcal{D}$ is trivially arcwise connected. Since $\mathcal{D}$ is a closed subset of $B(H)$, to complete the proof we need only show that $\mathcal{D}$ has empty interior in $\mathcal{T}$.

Let $A\in\mathcal{D}$. Since the block diagonal operators are dense in $\mathcal{D}$ we may assume that $A=A_1\oplus A_2\oplus\cdots$ is block diagonal relative to $H=M_1\oplus M_2\oplus\cdots$ (dimension of each M_n is finite). Since every finite square matrix is unitarily equivalent to an upper triangular matrix, we may assume each A_n is in upper triangular form with entries $a_{ij}(n)$. Since $a_{11}(n)$ is an eigenvalue of A_n (and hence of A), there exists a unit vector $x_n\in M_n$ such that $Ax_n=A_nx_n=a_{11}(n)x_n$. Observe that the x_n's form an orthonormal sequence. Since $\{a_{11}(n)\}_n^\infty=1$ is a subset of the spectrum of A, it must have a convergent subsequence; $a_{11}(n_j)\to\lambda$ as $j\to+\infty$. Let $\epsilon>0$. Then for all j's sufficiently large $(j\geqq N)$, $|a_{11}(n_j)-\lambda|<\epsilon$. For these n_j's replace $a_{11}(n_j)$ by λ and call the new operator, so obtained, A_0. Note that $\|A-A_0\|\leqq\epsilon$, A_0 is block diagonal relative to $M_1\oplus M_2\oplus\cdots$, and $A_0x_{n_j}=\lambda x_{n_j}$ for all $j\geqq N$.

Let M be the closed subspace spanned by all of the x_{n_j}'s, $j \geqq N$. Then relative to $H = M \oplus M^{\perp}$ ($\dim M = \infty$)

$$A_0 - \lambda I = \begin{pmatrix} 0 & E \\ 0 & F \end{pmatrix}$$

where F is block diagonal. Theorem 2 plays a crucial role at this point. Since the invertible block diagonal operators are dense in $\mathcal{D}$, there exists an invertible block diagonal operator F_0 such that $\|F - F_0\| < \epsilon$. Let

$$B = \begin{pmatrix} \epsilon U^* & E \\ 0 & F_0 \end{pmatrix}$$

where U is the unilateral shift. Then

$$\|(A_0 - \lambda I) - B\| = \left\| \begin{pmatrix} \epsilon U^* & 0 \\ 0 & F - F_0 \end{pmatrix} \right\| = \epsilon.$$

Clearly B is upper triangular. Since ϵU and F_0^* are bounded below, $B^* = \begin{pmatrix} \epsilon U & 0 \\ E^* & F_0^* \end{pmatrix}$ is bounded below. Since the kernel of $\epsilon U^* \neq (0)$, the kernel of $B \neq (0)$. Since B^* is bounded below and the kernel of $B \neq (0)$, B^* is not quasitriangular [**4**, Lemma 2.1]. Therefore B^* is not quasidiagonal. Hence B is not quasidiagonal, so that $B + \lambda I$ is not quasidiagonal. Since

$$\begin{aligned} \|A - (B + \lambda I)\| &\leqq \|A - A_0\| + \|A_0 - (B + \lambda I)\| \\ &\leqq \epsilon + \|(A_0 - \lambda I) - B\| \leqq \epsilon + \epsilon = 2\epsilon, \end{aligned}$$

A can be approximated by quasitriangular operators that are not quasidiagonal. Therefore $\mathcal{D}$ has empty interior in $\mathcal{T}$ and the theorem is proved. In [**1**, Theorem 2.8] C. Apostol, C. Foias, and D. Voiculescu have recently shown that $\mathcal{T}$ is a nowhere dense subset of $B(H)$.

If $A \oplus B$ is quasidiagonal then neither A nor B need be quasidiagonal [**8**, p. 905]. However if $A \oplus 0$ is quasitriangular then A must also be quasitriangular [**4**, Theorem 8]. Using this last result, the following theorem can be easily proved.

THEOREM 4. *If $A \oplus 0$ is quasidiagonal, then A is quasidiagonal.*

Proof. The sequence $\{P_n\}$ of finite rank orthogonal projections converging strongly to I is said to *implement* the quasitriangularity of T if $\|TP_n - P_n TP_n\| \to 0$. Since $A \oplus 0$ is quasidiagonal, $A \oplus 0$ and $A^* \oplus 0$

are quasitriangular implemented by the same sequence of finite rank projections converging strongly to I. By [**4**, Theorem 8] and its proof A and A^* are quasitriangular implemented by the same sequence $\{P_n\}$. Thus $\|AP_n - P_nAP_n\| \to 0$ and $\|A^*P_n - P_nA^*P_n\| = \|P_nA - P_nAP_n\| \to 0$. Combining these statements we have

$$\|AP_n - P_nA\| \leqq \|AP_n - P_nAP_n\| + \|P_nA - P_nAP_n\| \to 0.$$

Consequently A is quasidiagonal and the theorem is proved.

Halmos has shown [**9**] that every operator with countable spectrum is quasitriangular. Douglas and Pearcy [**4**, Theorem 9] have shown that T quasitriangular and S invertible implies STS^{-1} is quasitriangular. Neither of the above theorems is true when "quasitriangular" is replaced by "quasidiagonal". Russel Smucker has found a counterexample to each of these conjectures.

In [**4**, Proposition 5.1], Douglas and Pearcy prove that T is quasitriangular if and only if there exists a sequence $\{K_n\}$ of positive semi-definite compact operators that converges strongly to I and satisfies $\|TK_n - K_nTK_n\| \to 0$. However the analogous statement for quasidiagonal operators is not true (the unilateral shift is not quasidiagonal):

REMARK 3. Let U be unilateral shift. Then there exists a sequence $\{K_n\}$ of positive semi-definite finite rank operators that converges strongly to I and satisfies $\|UK_n - K_nU\| \to 0$.

To see that this is true, suppose that U is the unilateral shift with respect to the orthonormal basis $\{x_1, x_2, \cdots\}$. With respect to this basis define K_n as

(i) if $1 \leqq i \leqq n$, then $K_nx_i = x_i$

(ii) if $n \leqq i \leqq 2n$, then $K_nx_i = \left(\frac{2n - i}{n}\right)x_i$, and

(iii) if $i \geqq 2n$, then $K_nx_i = 0$.

Then each K_n is diagonal, has finite rank, and $K_n \geqq 0$. If we fix n and denote K_n by $\mathrm{diag}(a_1, a_2, a_3, \cdots)$, then

$$\begin{aligned}(UK_n - K_nU)x_i &= U(a_ix_i) - K_n(x_{i+1}) \\ &= a_ix_{i+1} - a_{i+1}x_{i+1} \\ &= (a_i - a_{i+1})x_{i+1}.\end{aligned}$$

Hence $UK_n - K_nU$ is a weighted shift so by [**6**, Problem 78], $\|UK_n - K_nU\| = \sup_i |a_i - a_{i+1}| = \frac{1}{n} \to 0$ as $n \to +\infty$. Thus we have shown the above remark to be true.

Let $\mathcal{P}$ denote the directed set consisting of all finite dimensional orthogonal projections on H under the usual ordering ($P \leqq Q$ if and only if $(Px, x) \leqq (Qx, x)$ for all $x \in H$). For a fixed operator T, the mapping $P \to \| TP - PTP \|$ is a net on $\mathcal{P}$, and Halmos has proved [**7**, Theorem 2] that T is quasitriangular if and only if

$$\liminf_{P \in \mathcal{P}} \| TP - PTP \| = 0.$$

In an analogous fashion one shows that T is quasidiagonal if and only if

$$\liminf_{P \in \mathcal{P}} \| TP - PT \| = 0.$$

An operator T on H is said to be *thin* if T can be written as the sum of a scalar operator and a compact operator. Among the quasitriangular operators are the operators T that satisfy

$$\lim_{p \in \mathcal{P}} \| TP - PTP \| = 0.$$

In [**3**, Theorem 1] Douglas and Pearcy have shown that T satisfies the above relationship if and only if T is thin. Using this result we prove a slightly sharper version of this theorem.

REMARK 4. T is a thin operator on H if and only if

$$\lim_{P \in \mathcal{P}} \| TP - PT \| = 0.$$

Proof. The fact that T thin implies $\lim_{P \in \mathcal{P}} \| TP - PT \| = 0$ is found in [**8**, P. 903]. On the other hand, if $\lim_{P \in \mathcal{P}} \| TP - PT \| = 0$ then, since

$$\| TP - PTP \| = \| (TP - PT)P \| \leqq \| TP - PT \|,$$

$\lim_{P \in \mathcal{P}} \| TP - PTP \| = 0$. So by Douglas and Pearcy's result [**3**, Theorem 1], T is thin and the proof is complete.

The operator T is *uniformly quasitriangular* (*quasidiagonal*) if and only if whenever $\{P_n\}$ is a sequence of finite rank orthogonal projections converging strongly to I, then

$$\| TP_n - P_n TP_n \| \to 0 \qquad (\| TP_n - P_n T \| \to 0).$$

THEOREM 5. *The operator T is uniformly quasitriangular* (quasidiagonal) *if and only if T is thin.*

Proof. Suppose T is a thin operator so that $T = \lambda I + K$ where K is compact. Let P_n be a sequence of finite rank orthogonal projections converging strongly to the identity operator. Since P_n converges strongly to I and since K and K^* are compact, $P_nK \to K$ and $P_nK^* \to K^*$. Thus

$$\|TP_n - P_nT\| = \|KP_n - P_nK\| \leqq \|KP_n - K\| + \|P_nK - K\|$$
$$= \|P_nK^* - K^*\| + \|P_nK - K\| \to 0.$$

Therefore T is uniformly quasidiagonal.

Suppose that there exists a uniformly quasidiagonal T that is not thin. Then by Remark 4 there exists $\epsilon_0 > 0$ such that for every $E_0 \in \mathcal{P}$ there exists $E \geqq E_0$, $E \in \mathcal{P}$ such that $\|TE - ET\| \geqq \epsilon_0$. Let $\{x_1, x_2, x_3, \cdots\}$ be an orthonormal basis for H. Let P_1 be the projection on the space spanned by $\{x_1\}$. Then there exists $E_1 \geqq P_1$, $E_1 \in \mathcal{P}$ such that $\|TE_1 - E_1T\| \geqq \epsilon_0$. Let P_2 be the projection on the space spanned by $E_1(H) + \{x_2\}$. Then there exists $E_2 \geqq P_2$ such that $\|TE_2 - E_2T\| \geqq \epsilon_0$. We continue in this manner obtaining a sequence $\{E_n\}$ of finite rank orthogonal projections converging strongly to I such that $\|TE_n - E_nT\| \geqq \epsilon_0$. However T is assumed to be uniformly quasidiagonal; contradiction. Hence the theorem is proved for the uniformily quasidiagonal case. The uniformily quasitriangular case is proved in a similar manner.

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