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THE BOUNDARY OF A SEMILATTICE ON AN n-CELL

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This paper presents a complete solution to the following problem: if S is a topological semilattice on an n-cell ($n \geqq 2$) and B is the boundary, then $B^2 = S$. Other problems of semilattices are solved also.

A topological semilattice S is a Hausdorff space equipped with an associative binary operation which is jointly continuous and satisfies the equations $xy = yx$ and $x^2 = x$ for all x, y. It is easy to see that $x \leqq y$ defined by $xy = x$ is a partial order which is closed in $S \times S$, and we shall use $L(x) = \{y \in S \mid y \leqq x\}$ and $M(x) = \{y \in S \mid y \geqq x\}$. If $M(x)$ is connected for each $x \in S$, then S is called an *M-semilattice.* The boundary of an n-cell in n-space is denoted by $B(I^n)$ or just B if there is no confusion, and I is the unit interval $[0, 1]$.

Before we proceed to the theorems, we need some preliminary notions. If $f: [0, 1]^n \to X$ is a continuous function into a space X such that $f(B(I^n)) = p$, then f goes homotopically to p (denoted by $f \xrightarrow{X} p$) if there exists a continuous function $H: I^{n+1} \to X$ such that $H(x, 1) = f(x)$ and $H(x, 0) = p$ for all $x \in I^n$ and $H(B(I^n) \times I) = p$. The first lemma could be found in [2].

LEMMA 1. *If S is a compact connected subsemilattice of T where T is a semilattice on an n-cell and $B \subseteq S$, then $S = T$.*

LEMMA 2. *If T is an $(n + 1)$-cell $(n \geqq 1)$ and $B(T) \subseteq X \subseteq T$ and there exists $p \in B$ such that $f \xrightarrow{X} p$ for each continuous $f: I^n \to B$ with $f(B(I^n)) = p$, then $X = T$.*

Proof. Let $f: I^n \to B$ be a continuous surjective function such that $f(B(I^n)) = p$ and f is one-to-one on $I^n \setminus B(I^n)$ into $B \setminus p$. Then there exists continuous $H: I^n \times I$ such that $H(x, 1) = f(x)$ and $H(x, 0) = p$ for all $x \in I^n$ and $H(B(I^n) \times I) = p$. We can put an equivalence relation R on I^{n+1} by identifying all the points on $I^n \times \{0\} \cup B(I^n) \times I$. Then H induces a continuous function $H^*: I^{n+1}/R \to X$ such that I^{n+1}/R is topologically an $(n + 1)$-cell and H^* restricted to the boundary of I^{n+1}/R is a homeomorphism onto B.

Suppose $X \neq T$. Then there exists a retraction $r: X \to B$. Then $r \circ H^*: I^{n+1}/R \to B$ is an r-map[1] which leads to a contradiction. Hence $X = T$.

LEMMA 3. *If T is an $(n+1)$-cell $(n \geqq 1)$ and A is an arc contained in T with end-points p and q such that $A \cap B = \{q\}$ and $A \cup B \subseteq X \subseteq T$ such that $f \xrightarrow{X} p$ for each continuous $f: I^n \to A \cup B$ with $f(B(I^n)) = p$, then $X = T$.*

Proof. For the sake of notation, we consider the n-cell I^n to be $\{x \in R^n \mid \|x\| \leqq 1\}$. Let k be a continuous function from $D = \{x \in R^n \mid \|x\| \leqq \frac{1}{2}\}$ onto B such that $k(B(D)) = q$ and k is $1-1$ on $D \setminus B(D)$ into $B \setminus q$. Since A is an arc, we let h be a homeomorphism form $[\frac{1}{2}, 1]$ onto A such that $h(\frac{1}{2}) = q$ and $h(1) = p$.

$$\text{Let } f(x) = \begin{cases} k(x) & \text{if } \ x \in D \\ h(\|x\|) & \text{if } \ \|x\| \geqq \frac{1}{2}. \end{cases}$$

Then $f \xrightarrow{X} p$. Hence there exists $H: I^n \times I \to X$ such that $H(x, 1) = f(x)$ and $H(x, 0) = p$ for all $x \in I^n$ and $H(B(I^n) \times I) = p$. Define an equivalence relation R on $I^n \times I$ by $(x, y)R(a, b)$ iff $(x, y) = (a, b)$ or $\{(x, y), (a, b)\} \subseteq I^n \times \{0\} \cup B(I^n) \times I$ or $y = b = 1$ and $\|x\| = \|a\| \geqq \frac{1}{2}$. Then H again induces a continuous $H^*: I^{n+1}/R \to X$ such that I^{n+1}/R is topologically an $(n+1)$-cell and H^* restricted to its boundary is a homeomorphism onto B. By a similar argument to that in Lemma 2, we conclude that $X = T$.

We can now proceed to prove Theorems A and B which answer Problem 44 in **[1]**.

THEOREM A. *If S is a topological semilattice on an $(n+1)$-cell $(n \geqq 1)$, then $B^2 = S$.*

Proof. Let 0 be the zero of S.

Case I. Suppose $0 \in B$. Let $f: I^n \to B$ be a continuous function such that $f(B(I^n)) = 0$. Then one can define $H: I^{n+1} \to B \cdot B$ by $H(x, y) = f(x)f(xy)$ where $xy = (x_1, \cdots, x_n)y = (x_1 y, \cdots, x_m y)$. Then H is the appropriate function to make $f \xrightarrow{B^2} 0$. Since $B \subseteq B^2$, then by Lemma 2, $B^2 = S$.

Case II. Suppose $0 \notin B$. Let $a \in B$. Then there exists an arc chain K from a to 0. Let $q = \inf(K \cap B)$. Then $A = L(q) \cap K$ is an

[1] A discussion of r-map could be found in K. Borsuk's "Theory of Retracts".

arc chain from q to 0 such that $A \cap B = \{q\}$. If $f : I^n \to A \cup B$ is a continuous function such that $f(B(I^n)) = 0$, then $H : I^{n+1} \to (A \cup B) \cdot (A \cup B)$ defined by $H(x, y) = f(x)f(xy)$ is again the appropriate function. By Lemma 3, $(A \cup B)^2 = S$. But $(A \cup B)^2 \subseteq L(q) \cup B^2 \subseteq L(a) \cup B^2$. Hence for each $a \in B$, we have $S \subseteq L(a) \cup B^2$.

Suppose $x \in S$ and $x \neq 0$. If $B \subseteq M(x)$, then the compact connected subsemilattice generated by B, $\bigcup_{n \geqq 1} B^n$, is contained in $M(x)$. By Lemma 1, that would make $S \subseteq M(x)$, which implies that $x = 0$. So it must be that there exists $a \in B$ such that $a \notin M(x)$. In other words, $x \notin L(a)$. But $S \subseteq L(a) \cup B^2$. Hence $x \in B^2$. If each nonzero x belongs to B^2, then $S = B^2$ since B^2 is closed and 0 is a limit point of nonzero elements.

THEOREM B. *There exists a topological semilattice on a two-cell such that there is an element $x \in B$ and if $y \in B$, then $xy \neq 0$.*

Proof. Let A be a topological semilattice on an arc such that A has zero as a cutpoint and an identity as an endpoint (e.g., the subsemilattice of $I \times I$ given by $\{(x, y) \mid x = 0 \text{ or } y = 0 \text{ or } x = 1\}$). Then $A \times A$ is a semilattice on a two-cell with coordinate multiplication. Consider $(1, 1) \in B$ and if $(a, b) \in B$, then $(1, 1)(a, b) = (a, b) \neq (0, 0)$, since $(0, 0)$ is not on the boundary. Hence $(1, 1)$ has no zero-divisor on the boundary.

Theorem C (its corollary) and D are related to questions raised in [2] on M-semilattices, namely, a converse of Lemma 2 and a generalization of Lemma 1 in [2].

THEOREM C[2]. *If S is a semilattice on an n-cell such that for each $x \in S$, $M(x) \cap B$ is connected, then S is an M-semilattice.*

Proof. Let K be the component of $M(ab)$ containing $M(ab) \cap B$ where $a, b \in B$. Since K is a subsemilattice of $M(ab)$, then $ab \in K^2 \subseteq K$. Let $x \in M(ab)$. Then $x \leqq y$ for some maximal element y in $M(ab)$. Hence y is also maximal in S. Thus $y \in B \cap M(ab) \subseteq K$. But $x, ab \in Kx$ which is a connected set contained in $M(ab)$. We have $M(ab)$ connected. By Theorem A, each element in S can be written as a product of some $a, b \in B$. Hence S is an M-semilattice.

COROLLARY. *If S is a semilattice on a two-cell such that if $a, b \in B = [a, b] \cup [b, a]$, then $[a, b] \subseteq M(ab)$ or $[b, a] \subseteq M(ab)$, then S is an M-semilattice.*

[2] The author is grateful to the referee for this generalization.

Proof. If $x, y \in B$, then denote the counter-clockwise arc from x to y on B by $[x, y]$. Let $a, b \in B$. We claim that $M(ab) \cap B$ is connected. If it is not connected, then there exist $r, t \in M(ab) \cap B$ such that $r \neq t$ and $[r, t) \cap (M(ab) \cap b) = \{r, t\}$. Note that $M(rt) \subseteq M(ab)$. Since $[r, t] \subseteq M(rt)$, then $[t, r] \subseteq M(rt) \subseteq M(ab)$. We have $M(ab) \cap B = [t, r]$ which is connected.

The proof of Lemma 1 in **[2]** relies on the existence of arc-chains in compact M-semilattice. Theorem D applies to topological M-semilattices.

THEOREM D. *If S is an M-semilattice and f is a continuous homomorphism from S onto a semilattice T, then T is an M-semilattice and f is a monotone function.*

Proof. Let $y \in T$ and $a, b \in M(y)$. Since f is surjective, there exist $c, d \in S$ such that $f(c) = a$, $f(d) = b$. Then $f(cd) = f(c)f(d) = ab \geqq y$. Hence $f(M(cd)) \subseteq M(y)$. But $c, d \in M(cd)$ which is connected. Hence $f(c)$, $f(d)$ belong to a connected set $f(M(cd))$ which is contained in $M(y)$. Thus $M(y)$ is connected.

To show f is monotone, one has to show $f^{-1}(y)$ is connected. Let $a, b \in f^{-1}(y)$ and $a \leqq b$. Since $M(a)$ is connected, then $b \cdot M(a)$ is connected. If $x \in M(a)$, then $a = ab \leqq xb \leqq b$ which yields $nf(a) \leqq f(xb) \leqq f(b)$, i.e., $f(xb) = y$. Hence $b \cdot M(a)$ is contained in $f^{-1}(y)$ and contains a, b. If $a \not\leqq b$, then $ab \leqq a$ and $ab \leqq b$ and $f(ab) = f(a)f(b) = y^2 = y$. In this case, there exists connected sets in $f^{-1}(y)$ which contain $\{ab, a\}$ and $\{ab, b\}$. Hence $f^{-1}(y)$ is connected.

It would be interesting to generalize the concept of boundary (by homotopy or cohomology) to general semilattices (e.g., as in **[4]**) such that $B^2 = S$ still holds. Also, there is no structure theorem concerning semilattices on a two-cell which are not M-semilattices.

REFERENCES

1. D. R. Brown and A. R. Stralka, *Problems on compact semilattices*, Semigroup Forum, **6** No. 3 (1973), 265–270.
2. D. R. Brown, *Topological semilattices on the two-cell*, Pacific J. of Math., **15** (1965), 35–46.
3. R. J. Koch, *Arcs in partially ordered spaces*, Pacific J. of Math., **9** (1959), 723–728.
4. J. D. Lawson and B. Madison, *Peripheral and inner points*, Fund. Math., **LXIX** (1970), 253–266.
5. J. W. Lea, Jr., *The peripherality of irreducible elements of a lattice*, Pacific J. of Math., **45** (1973), 555–560.

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