PACIFIC JOURNAL OF MATHEMATICS
Vol. 56, No. 1, 1975

ON THE EXTREMAL ELEMENTS OF THE CONVEX CONE OF SUPERADDITIVE n-HOMOGENEOUS FUNCTIONS

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Let P_n be the collection of finite-valued functions defined on the nonnegative orthant, $E^+_{n^2}$, of euclidean n^2-space such that for $p \in P_n$ it follows that $p: E^+_{n^2} \to E^+_1$ and in addition

(a) p is continuous,
(b) $p(\alpha x) = \alpha^n p(x), \alpha \geqq 0$,
(c) $p(x+y) \geqq p(x) + p(y)$.

It follows readily that P_n is closed with respect to addition and nonnegative scalar multiplication. Therefore, P_n is a convex cone, whose vertex is the zero function, in the linear space of real functions defined on $E^+_{n^2}$. The purpose of this paper is to investigate the extremal elements of P_n.

1. Introduction. One well known member of P_n is the permanent function. Recently functions that generalize the permanent function have been studied by Rothaus [9] and new representations for the permanent function have been sought (for example see [2] by Marcus and Newman). The interest in determining the extremal elements of P_n comes from the fact that under certain circumstances it is possible to give an integral representation for any $p \in P_n$ in terms of the extremal elements of P_n [1] (examples of similar studies may be found in papers by McLachlan [4], [5], [6] and Rakestraw [7]). In this paper it is shown that for $a \in E^+_{n^2} \setminus 0$, the functions $p_a(x) = \sup\{\lambda^n : x \geqq \lambda a\}$ are extremal elements of P_n. Replacing condition (b) by

$$\text{(b)} \quad p(\alpha x) = \alpha p(x), \ \alpha \geqq 0,$$

gives the collection of *monotone concave gauges*, denoted by P'_n, defined on $E^+_{n^2}$ [8]. If for all $i = 1, \cdots, n$, $A_i \in P'_n$, then the function A defined as $A(x) = \Pi^n_{i=1} A_i(x)$ is an element of P_n. If S_n denotes all those $p \in P_n$ which are finite nonnegative linear combination of functions of this type, then clearly S_n is a subcone of P_n and S_n contains the permanent function. It is shown here that for a function $p \in S_n$ to be an extremal element of P_n, then p must be of the form $p(x) = [A(x)]^n$, where $A(x)$ is an extremal element of P'_n.

In the material to follow define $[p:\alpha]=\{x:p(x)=\alpha\}$, where $p \in P_n$. It follows that $\alpha[p:1]=[p:\alpha^n]$ for all $\alpha \geqq 0$. Also, use will be made of the fact that for $p \in P_n'$ or $p \in P_n$, then $x \geqq y$ (or $x > y$) implies that $p(x) \geqq p(y)$ (or $p(x) > p(y)$). Further if $x \in \operatorname{int} E_{n^2}^+$ and $p \neq 0$, then $p(x) > 0$.

2. Extremal elements of P_n. The first theorem of this section gives some of the extremal elements of P_n. It is conjectured that this set includes all the extremal elements of P_n. The following lemmas will be needed.

LEMMA 1.1. *If $p, q \in P_n$. Define*

$$(p \wedge q)(x) = \min\{p(x), q(x)\}.$$

Then $p \wedge q \in P_n$.

Proof. It follows readily from the definitions that $p \wedge q$ is continuous and homogeneous. Also,

$$\begin{aligned}(p \wedge q)(x+y) &= \min\{p(x+y), q(x+y)\}\\ &\geqq \min\{p(x)+p(y), q(x)+q(y)\}\\ &\geqq \min\{p(x), q(x)\} + \min\{p(y), q(y)\}\\ &= (p \wedge q)(x) + (p \wedge q)(y).\end{aligned}$$

For all $k = 1, \cdots, n^2$, let $p_k(x) = x_k^n$, $x = (x_1, \cdots, x_{n^2}) \in E_{n^2}^+$. Then $p_k \in P_n$. With this in mind consider the following:

LEMMA 1.2. *Let $a = (a_1, \cdots, a_{n^2}) \in E_{n^2}^+ \setminus \{0\}$. Define p_a as follows:*

$$p_a(x) = \sup\{\lambda^n : x \geqq \lambda a, \lambda \geqq 0\}.$$

Then $p_a \in P_n$.

Proof. Without loss of generality, assume the nonzero coordinates of a are $a_1, \cdots, a_k$, $k \leqq n^2$. Let

$$p(x) = \left(\frac{1}{a_1^n} p_1 \wedge \cdots \wedge \frac{1}{a_k^n} p_k\right)(x). \tag{1.1}$$

Lemma 1.1 implies that $p \in P_n$. Now for any given $x \in E_{n^2}^+$ suppose that

$$p(x) = \frac{x_l^n}{a_l^n},$$

$1 \leqq l \leqq k$. Then it follows readily that for each i

$$x_i \geqq \frac{x_l}{a_l} a_i$$

with equality when $i = l$. Further there does not exist $\lambda > x_l / a_l$ such that $x \geqq \lambda a$ since otherwise

$$x_l > \lambda a_l > \frac{x_l}{a_l} a_l = x_l.$$

Hence, $p_a(x) = p(x)$ for every $x \in E^+_{n^2}$, which implies that $p_a = p$.

Notice that if a_i is a nonzero coordinate of a and $x \in E^+_{n^2}$ such that $x_i = 0$, then $x \geqq \lambda a$ implies that $\lambda = 0$. Thus, $p_a(x) = 0$. Also, if $a = e_k$, where e_k is that vector having all zero coordinates except the kth coordinate which is 1, then $p_a = p_k$.

In general, if $p \in P_n$ the set $[p: 1]$ is difficult to characterize. For example a complete characterization of the set $[p: n!/n^n]$ (and hence $[p: 1]$) where p is the permanent function is not known **[3]**. However, if $p = p_a$ for some $a \neq 0$, then a characterization is possible. Let $a = (a_1, \cdots, a_{n^2}) \in E^+_{n^2}$, $a \neq 0$. For every $i \in \{1, \cdots n^2\}$, define

$$R(a_i) = \{(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_{n^2}) : x_j \geqq a_j \quad \text{for} \quad j \neq i\}.$$

LEMMA 1.3. *If $a \in E^+_{n^2} \setminus \{0\}$, then*

$$[p_a: 1] = \cup \{R(a_i): a_i \neq 0\}.$$

Proof. Let $y \in R(a_i)$, where $a_i \neq 0$. Clearly, $y \geqq a$. Notice there does not exist $\lambda > 1$ such that $y \geqq \lambda a$, for otherwise $a_i \geqq \lambda a_i > a_i$. Hence, by definition $p_a(y) = 1$. This implies that

$$\cup \{R(a_i): a_i \neq 0\} \subset [p_a: 1].$$

Now suppose $y \in [p_a: 1]$. Considering (1.1), there exists $k \in \{1, \cdots, n^2\}$ such that $a_k > 0$ and $(y_k / a_k)^n = 1$. This implies that $y_k^n = a_k^n$, which implies that $y_k = a_k$. For all other $i \in \{1, \cdots, n^2\}$ such that $a_i > 0$, $(y_i / a_i)^n \geqq 1$ and hence $y_i \geqq a_i$. It follows that $y \in R(a_k)$ and the proof is complete.

Using this result it is possible to show that $p_a = p_b$ if and only if $a = b$. Next using Lemma 1.3, p_a is shown to be an extremal element of P_n.

THEOREM 1.1. *The function p_a is an extremal element of P_n.*

Proof. Suppose $p_a = f + g$. Let $y \in R(a_i)$, where $a_i \neq 0$ and $i \in \{1, \cdots, n^2\}$, then

$$P_a(a) = P_a(y) = f(y) + g(y) \geqq f(a) + g(a) = P_a(a).$$

This implies $f(y) = f(a)$ and $g(y) = g(a)$, since $f(y) \geqq f(a)$ and $g(y) \geqq g(a)$. Also, $p_a(a) = f(a) + g(a)$ implies $p_a(a) \geqq f(a)$ and $p_a(a) \geqq g(a)$. Therefore, there exists $\alpha \geqq 0$ and $\beta \geqq 0$ such that $\alpha p_a(a) = f(a)$ and $\beta p_a(a) = g(a)$.

Again, without loss of generality, suppose the nonzero coordinates of a are $a_1, \cdots, a_k$. Let $x \in E^+_{n^2}$ such that $x_1 > 0, \cdots, x_k > 0$. Then for every $i \in \{1, \cdots, k\}$ there exists $\lambda_i > 0$ such that $a_i = \lambda_i x_i$. Let $\lambda = \max\{\lambda_i : i \in \{1, \cdots, k\}\}$. Notice there exists a $j \in \{1, \cdots, k\}$ such that $\lambda = \lambda_j$. Hence, $\lambda x_i \geqq a_i$ with equality when $i = j$. Clearly, if $i \in \{1, \cdots, n^2\} \setminus \{1, \cdots k\}$, then $\lambda x_i \geqq a_i$. Therefore, $\lambda x \in R(a_j)$. Setting $y = \lambda x$, it follows that

$$\begin{aligned} f(x) &= f\left(\frac{1}{\lambda} y\right) = \frac{1}{\lambda^n} f(y) \\ &= \frac{1}{\lambda^n} f(a) = \frac{1}{\lambda_n} \alpha p_a(a) \\ &= \frac{1}{\lambda^n} \alpha p_a(y) = \alpha p_a\left(\frac{1}{\lambda} y\right) \\ &= \alpha p_a(x). \end{aligned}$$

Clearly, if $x \in E^+_{n^2}$ such that $x_i = 0$ for some $i \in \{1, \cdots, k\}$, then $0 = p_a(x)$. This implies $f(x) = 0$, which in turn implies that $f(x) = \alpha p_a(x)$. In either case $f(x) = \alpha p_a(x)$. Hence, $f = \alpha p_a$. Likewise, $g = \beta p_a$. Therefore, p_a is an extremal element of P_n.

By a somewhat similar proof it can be shown that the function p_a is minimal in the set of all elements of P_n which agree with $p_a(a)$ at a. Also, for $\alpha > 0$ the sets $[p_a : 1]$ have the property that $\alpha[p_a : 1] = [p_a : \alpha^n] = [p_{\alpha a} : 1]$.

Recall that $S_n \subset P_n$ is the set of finite nonnegative linear combinations of products of n functions of P'_n. For $a \in E^+_{n^2} \setminus \{0\}$, let $q_a(x) = \sup\{\lambda : x \geqq \lambda a\}$. Then, as in the case for P_n, q_a is an extremal element of P'_n. Also, $p_a(x) = [q_a(x)]^n$, which implies that $p_a \in S_n$. Since S_n is a subcone of P_n then p_a is an extremal element of S_n. It is conjectured that $\{p_a : a \in E^+_{n^2}\{0\}\}$ represents all the extremal elements of S_n.

LEMMA 1.4. *If $p \neq 0$ and $p(x) = \Pi_{i=1}^{n} A_i(x)$, where $A_i \in P'_n$, is an extremal element of S_n, then each A_i is an extremal element of P'_n.*

Proof. Suppose there exists a $k = 1, \cdots, n$ such that A_k is not extremal in P'_n. Then there exists $f, g \in P'_n$ such that $A_k = f + g$ and neither f or g is proportional to A_k. Hence,

$$p(x) = \prod_{i=1}^{n} A_i(x) = A_1(x) \cdots (f(x) + g(x)) \cdots A_n(x)$$
$$= A_1(x) \cdots f(x) \cdots A_n(x) + A_1(x) \cdots g(x) \cdots A_n(x).$$

Since p is extremal in S_n, there exists $\alpha \geqq 0$ and $\beta \geqq 0$ such that $A_1(x) \cdots f(x) \cdots A_n(x) = \alpha p(x)$ and $A_1(x) \cdots g(x) \cdots A_n(x) = \beta p(x)$. Let $x \in \operatorname{int} E^+_{n^2}$. Then $p(x) > 0$. Also, it can be shown that each $A_i(x) > 0$, $f(x) > 0$ and $g(x) > 0$. Therefore,

$$\alpha A_1(x) \cdots A_k(x) \cdots A_n(x) = \alpha p(x) = A_1(x) \cdots f(x) \cdots A_n(x),$$

which implies that $\alpha A_k(x) = f(x)$, for all $x \in \operatorname{int} E^+_{n^2}$. It follows from continuity that $\alpha A_k(x) = f(x)$ for all $x \in E^+_{n^2}$. This is a contradiction. Therefore, A_k is an extremal element of P'_n for each k.

In any convex cone, if the sum of two nonzero elements is an extremal element, then the two elements are proportional. Hence, the only possible extremal elements of S_n are those elements of the form

$$p(x) = \prod A_i^{l(i)}(x), \tag{1.2}$$

where $l(i)$ is a nonnegative integer and $\Sigma l(i) = n$. Moreover, Lemma 1.4 implies that the A_i must be extremal elements of P'_n. The Lemma 1.4 and these comments give conditions that are necessary when p is an extremal element in S_n. These conditions are not sufficient as will be seen in Proposition 1.1.

Attention will now be given to considering the extremal elements of P_n.

THEOREM 1.2. *Let p be defined as in (1.2). Let k be the number of i for which $l(i) > 0$. If $k > 1$, then p is not an extremal element of P_n.*

Proof. Assume

$$p(x) = \prod_{i=1}^{k} A_i^{l(i)}(x),$$

where each $l(i)$ is a positive integer, $\Sigma_{i=1}^{k} l(i) = n$ and the A_i are distinct (pairwise nonproportional) extremal elements of P'_n. For each $i \in \{1,2\}$ define

$$f_i(x) = \begin{cases} \dfrac{A_i(x)}{A_1(x) + A_2(x)} p(x), & A_1(x) + A_2(x) > 0 \\ 0 & A_1(x) + A_2(x) = 0. \end{cases}$$

It follows easily that $p = f_1 + f_2$. It will now be shown that each $f_i \in P_n$.

n-Homogeneity: Let $\alpha \geqq 0$ and $x \in E^+_{n^2}$. If $\alpha = 0$, then $A_1(\alpha x) = A_2(\alpha x) = 0$ and hence $f_i(\alpha x) = 0 = \alpha^n f_i(x)$. Suppose $\alpha > 0$. If $0 = A_1(\alpha x) + A_2(\alpha x) = \alpha(A_1(x) + A_2(x))$, then $A_1(x) + A_2(x) = 0$ and hence $f_i(\alpha x) = \alpha^n f_i(x)$. Suppose $\alpha > 0$ and

$$\alpha(A_1(x) + A_2(x)) = A_1(\alpha x) + A_2(\alpha x) > 0,$$

then $A_1(x) + A_2(x) > 0$. Therefore,

$$f_i(\alpha x) = \frac{A_i(\alpha x)}{A_1(\alpha x) + A_2(\alpha x)} p(\alpha x) = \alpha^n \frac{A_i(x)}{A_1(x) + A_2(x)} p(x) = \alpha^n f_i(x).$$

So for all $\alpha \geqq 0$ and $x \in E^+_{n^2}$, $f_i(\alpha x) = \alpha^n f_i(x)$.

Superadditivity: Let $x, y \in E^+_{n^2}$.

Case I. If $A_1(x + y) + A_2(x + y) = 0$, then

$$0 = A_1(x + y) + A_2(x + y) \geqq A_1(x) + A_1(y) + A_2(x) + A_2(y) \geqq 0$$

which implies that $A_1(x) + A_2(x) = 0$ and $A_1(y) + A_2(y) = 0$. Therefore, $f_i(x + y) = f_i(x) + f_i(y)$.

Case II. Suppose that $A_1(x + y) + A_2(x + y) > 0$, $A_1(x) + A_2(x) = 0$ and $A_1(y) + A_2(y) = 0$. Clearly, $f_i(x + y) \geqq f_i(x) + f_i(y)$.

Case III. Suppose $A_1(x + y) + A_2(x + y) > 0$, $A_1(x) + A_2(x) > 0$ and $A_1(y) + A_2(y) = 0$. Then

$$f_i(x + y) = \frac{A_i(x + y)}{A_1(x + y) + A_2(x + y)} p(x + y),$$

$$f_i(x) = \frac{A_i(x)}{A_1(x) + A_2(x)} p(x)$$

and $f_i(y) = 0$. It must be shown that

$$\frac{A_i(x+y)}{A_1(x+y)+A_2(x+y)}p(x+y) \geqq \frac{A_i(x)}{A_1(x)+A_2(x)}p(x).$$

This is true if and only if

$$(1.3) \qquad \begin{aligned} &(A_1(x)+A_2(x))\, A_i(x+y) \prod_{j=1}^{k} A_j^{l(j)}(x+y) \\ &\geqq (A_1(x+y)+A_2(x+y))\, A_i(x) \prod_{j=1}^{k} A_j^{l(j)}(x). \end{aligned}$$

It suffices to show that each term on the right hand side of (1.3) is less than or equal to the corresponding term on the left hand side of (1.3). Now for $m = 1$ (or $m = 2$)

$$\begin{aligned} &A_m(x)A_i(x+y) \prod_{j=1}^{k} A_j^{l(j)}(x+y) \\ &= A_m(x+y)A_i(x+y)\, (A_m(x)A_m^{l(m)-1}(x+y) \cdots A_k^{l(k)}(x+y)) \\ &\geqq A_m(x+y)A_i(x) \prod_{j=1}^{k} A_j^{l(j)}(x). \end{aligned}$$

It follows that (1.3) is true.

Case IV. Suppose $A_1(x+y)+A_2(x+y)>0$, $A_1(x)+A_2(x)>0$ and $A_1(y)+A_2(y)>0$. Then

$$f_i(x+y) = \frac{A_i(x+y)}{A_1(x+y)+A_2(x+y)}\, p(x+y),$$

$$f_i(x) = \frac{A_i(x)}{A_1(x)+A_2(x)}p(x),$$

and

$$f_i(y) = \frac{A_i(y)}{A_1(y)+A_2(y)}p(y).$$

It must be shown that

$$\text{(a)} \qquad \frac{A_i(x+y)}{A_1(x+y)+A_2(x+y)} \prod_{j=1}^{k} A_j^{l(j)}(x+y)$$

$$\text{(b)} \qquad \geqq \frac{A_i(x)}{A_1(x)+A_2(x)} \prod_{j=1}^{k} A_j^{l(j)}(x) + \frac{A_i(y)}{A_1(y)+A_2(y)} \prod_{j=1}^{k} A_j^{l(j)}(y).$$

This will be true if and only if

$$\text{(c)} \qquad A_i(x+y)(A_1(x)+A_2(x))(A_1(y)+A_2(y)) \prod_{j=1}^{k} A_j^{l(j)}(x+y)$$

$$\text{(d)} \qquad \geqq A_i(x)(A_1(y)+A_2(y))(A_1(x+y)+A_2(x+y)) \prod_{j=1}^{k} A_j^{l(j)}(x)$$

$$\text{(e)} \qquad + A_i(y)(A_1(x)+A_2(x))(A_1(x+y)+A_2(x+y)) \prod_{j=1}^{k} A_j^{l(j)}(y).$$

Since

$$\text{(f)} \qquad A_i(x+y)(A_1(x)+A_2(x))(A_1(y)+A_2(y)) \prod_{j=1}^{k} A_j^{l(j)}(x+y)$$

$$\text{(g)} \qquad \geqq A_i(x)(A_1(x)+A_2(x))(A_1(y)+A_2(y)) \prod_{j=1}^{k} A_j^{l(j)}(x+y)$$

$$\text{(h)} \qquad + A_i(y)(A_1(x)+A_2(x))(A_1(y)+A_2(y)) \prod_{j=1}^{k} A_j^{l(j)}(x+y),$$

it is sufficient to show that (g) $\geqq$ (d) and (h) $\geqq$ (e). An argument similar to the one in Case III shows that each term of (g) or (h) is greater than or equal to the corresponding term of (d) or (e). Thus, (c) $\geqq$ (d) + (e) and hence (a) $\geqq$ (b). Therefore, each f_i is superadditive.

Continuity: Let $x \in E_{n^2}^+$ and $\{y_j\} \subset E_{n^2}^+$ such that $y_j \to x$. Suppose $A_1(x)+A_2(x) > 0$, then without loss of generality it may be assumed tha $A_1(y_j)+A_2(y_j) > 0$ for each j. In this case

$$f_i(y_j) = \frac{A_i(y_j)}{A_1(y_j)+A_2(y_j)} p(y_j) \to \frac{A_i(x)}{A_1(x)+A_2(x)} p(x) = f_i(x).$$

Suppose that $A_1(x)+A_2(x) = 0$, then $p(x) = f_i(x) = 0$. If there exists $m \in \{1,2\}$ such that $A_m(y_j) = 0$, $f_i(y_j) = 0 = f_i(x)$. Suppose $A_m(y_j) > 0$ for $m = 1,2$. Since each $A_m^{l(m)}(y_j) \to 0$ and the expression

$$\frac{A_i(y_j)}{A_1(y_j)+A_2(y_j)}$$

is obviously bounded by 1, then $f_i(y_j) \to 0$. Therefore, f_i is continuous. Hence, each $f_i \in P_n$.

It remains to be shown that the functions f_i form a nonproportional decomposition of p. Suppose $f_i(x) = \alpha p(x)$, for all $x \in E_{n^2}^+$. Let $x \in E_{n^2}^+$. There exists a sequence $\{y_j\} \subset \operatorname{int} E_{n^2}^+$ such that $y_j \to x$. Since $y_j \in \operatorname{int} E_{n^2}^+$, then $A_1(y_j) > 0$, $A_2(y_j) > 0$ and $p(y_j) > 0$. Hence,

$$\alpha p(y_j) = f_i(y_j) = \frac{A_i(y_j)}{A_1(y_j) + A_2(y_j)} p(y_j)$$

which implies that

$$A_i(y_j) = \alpha(A_1(y_j) + A_2(y_j)).$$

Since $A_i(y_j) \to A_i(x)$ and $\alpha(A_1(y_j) + A_2(y_j)) \to \alpha(A_1(x) + A_2(x))$, then $A_i(x) = \alpha(A_1(x) + A_2(x))$. Since A_1 and A_2 are pairwise nonproportional extremal elements in P'_n, this is a contradiction. Therefore, there does not exists $\alpha \geqq 0$ such that $f_i = \alpha p$. Hence, the decomposition is nonproportional, which implies that p is not an extremal element of P_n.

Two questions immediately arise. First, is $f_i \in S_n$? Secondly, is every extremal element of P'_n of the form q_a, where $a \in E^+_{n^2} \setminus \{0\}$? If both answers are affirmative, then every extremal element of S_n is of the form p_a, where $a \in E^+_{n^2} \setminus 0$. It is entirely possible that the functions f_i do not belong to S_n.

The following is an example of a subcone of P_n that has as extremal elements some functions that are not extremal in P_n.

EXAMPLE 1.1. Let Q_n be the set of all $p: E^+_{n^2} \to E^+_1$ such that

$$p(x) = \sum_{i1,\cdots,in=1}^{n^2} \alpha_{i1,\cdots,in} x_{i1} \cdots x_{in}$$

where $i1 \leqq \cdots \leqq in$, $\alpha_{i1,\cdots,in} \geqq 0$ and $x = (x_1, \cdots, x_{n^2})$. Thus, Q_n is the set of nonnegative superadditive n-forms. Clearly, Q_n is a subcone of $S_n \subset P_n$. Therefore, the functions $p_1, \cdots, p_{n^2}$ are extremal elements of Q_n. However, these are not all of the extremal elements of Q_n. In fact without much difficulty it can be shown that the extremal elements of Q_n are the positive scalar multiples of functions of the form

$$p(x) = x_{k1} \cdots x_{kn},$$

where $kj \in \{1, \cdots, n^2\}$, for $j = 1, \cdots, n$ and $k1 \leqq \cdots \leqq kn$.

Now for every $x = (x_1, \cdots, x_{n^2}) \in E^+_{n^2}$ define $p(x)$ as

$$p(x) = x_1^{l(1)} \cdots x_{n^2}^{l(n^2)}, \tag{1.4}$$

where (i) is a nonnegative integer and $\sum_{i=1}^{n^2} l(i) = n$. Notice that $l(i) > 0$ for at most n values of $i = 1, \cdots, n^2$. Clearly, $p \in Q_n$. In fact the preceding example shows that p is an extremal element of Q_n. If k is the number of $i \in \{1, \cdots n^2\}$ for which $l(i) > 0$ and $k > 1$, Theorem 1.2

says that p is not an extremal element of P_n. The following proposition shows that p is not an extremal element of S_n.

PROPOSITION 1.1. *Let p be defined as in* (1.4). *If $k > 1$, then p is not an extremal element of S_n.*

Proof. Without loss of generality assume

$$p(x) = x_1^{l(1)} x_2^{l(2)} \cdots x_k^{l(k)}$$

where each $l(k) > 0$. As seen in the proof of Theorem 1.2, $p = f_1 + f_2$ where

$$f_i(x) = \begin{cases} \dfrac{x_i}{x_1 + x_2} x_1^{l(1)} \cdots x_k^{l(k)}, & x_1 + x_2 > 0 \\ 0 & , \quad x_1 + x_2 = 0. \end{cases}$$

Consider f_1. Notice that

$$f_1(x) \begin{cases} \dfrac{x_1 x_2}{x_1 + x_2} x_1^{l(1)} x_2^{l(2)-1} \cdots x_k^{l(k)}, & x_1 + x_2 > 0 \\ 0 & , \quad x_1 + x_2 = 0. \end{cases}$$

Let

$$g(x) = \begin{cases} \dfrac{x_1 x_2}{x_1 + x_2}, & x_1 + x_2 > 0 \\ 0 \quad , & x_1 + x_2 = 0 \end{cases}$$

Then $f_1(x) = g(x) x_1^{l(1)} x_2^{l(2)-1} \cdots x_k^{l(k)}$. Since the objective is to show that $f_1 \in S_n$, it remains to be shown that $g \in P_n'$. As in Theorem 1.2 g is continuous and homogeneous of degree 1. To show supperadditively let $x, y \in E_{n^2}^+$. If $x_1 + x_2 = 0$ or $y_1 + y_2 = 0$, then it follows readily that $g(x + y) = g(x) + g(y)$. Suppose $x_1 + x_2 > 0$ and $y_1 + y_2 > 0$. In this case it must be shown that

$$\frac{(x_1 + y_1)(x_2 + y_2)}{x_1 + y_1 + x_2 + y_2} \geqq \frac{x_1 x_2}{x_1 + x_2} + \frac{y_1 y_2}{y_1 + y_2},$$

which is equivalent to proving that

$$(x_1 + y_1)(x_2 + y_2)(x_1 + x_2)(y_1 + y_2) - [(x_1 x_2)(x_1 + y_1 + x_2 + y_2)(y_1 + y_2) \\ + y_1 y_2 (x_1 + y_1 + x_2 + y_2)(x_1 + x_2)] \geqq 0.$$

By direct calculation the left hand side of the above inequality is equal to $(x_1 y_2 - x_2 y_1)^2$. The computation is tedious but straightforward.

Hence, g is superadditive and $f_1 \in S_n$. Likewise, $f_2 \in S_n$. Hence, p is not an extremal element of S_n.

Actually, it can be shown that if p is defined as in (1.2) and if at least two of the A_i are additive, then p is not an extremal element of S_n.

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Received October 15, 1973 and in revised form May 16, 1974. This paper is part of the author's doctoral thesis which was directed by Professor E. K. McLachlan. The author wishes to express his gratitude to Professor McLachlan and also to acknowledge the many useful suggestions made by the referee.

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