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PRODUCTS OF TERMINATING ${}_3F_2(1)$ SERIES

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It is shown that a well-known formula of Bailey for the product of two hypergeometric functions in terms of an F_4 Appell function has a discrete analogue of the form

$$
(1)\quad \begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, -x; \\ c, d \end{matrix}\right] {}_3F_2\left[\begin{matrix} a, b, -y; \\ a+b-c+1, e \end{matrix}\right] \\
&= F\left[\begin{matrix} a, b : -x, y+e; -y, x+d; \\ d, e : \quad c; a+b-c+1; \end{matrix}\right],
\end{aligned}
$$

where $x, y = 0, 1, \cdots$ and the F-function on the right-hand side is a double hypergeometric series. Additional formulas are derived, including a discrete analogue of an important transformation formula of Watson, and discrete analogues of some more general formulas due to Burchnall and Chaundy.

In **[10]** it was shown that Watson's well-known formula **[13]** for the product of two terminating hypergeometric functions in terms of an F_4 Appell function

$$
(2)\quad \begin{aligned}
&{}_2F_1[-n, n+a; c; z]\, {}_2F_1[-n, n+a; c; Z] \\
&= \frac{(-1)^n (a-c+1)_n}{(c)_n} F_4[-n, n+a; c, a-c+1; zZ, (1-z)(1-Z)],
\end{aligned}
$$

admits a generalization of the form

$$
(3)\quad \begin{aligned}
&{}_3F_2\left[\begin{matrix} -n, n+a, b; \\ c, d \end{matrix}\right] {}_3F_2\left[\begin{matrix} -n, n+a, e; \\ c, f \end{matrix}\right] \\
&= \frac{(-1)^n (a-c+1)_n}{(c)_n} F\left[\begin{matrix} -n, n+a : b, e; d-b, f-e; \\ d, f : \quad c; \quad a-c+1; \end{matrix}\right],
\end{aligned}
$$

where the F-function is defined as in Burchnall and Chaundy **[8]** by

$$
F\left[\begin{matrix} a_1, \cdots, a_j : b_1, \cdots, b_k; c_1, \cdots, c_m; \\ d_1, \cdots, d_n : e_1, \cdots, e_p; f_1, \cdots, f_q; \end{matrix}\; z, Z\right]
$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a_1)_{r+s}\cdots(a_j)_{r+s}(b_1)_r\cdots(b_k)_r(c_1)_s\cdots(c_m)_s}{r!\,s!\,(d_1)_{r+s}\cdots(d_n)_{r+s}(e_1)_r\cdots(e_p)_r(f_1)_s\cdots(f_q)_s}z^rZ^s$$

and, as elsewhere, the arguments z and Z are not displayed when they are both equal to 1. To obtain Watson's formula from (3) one needs but set $b = dz$, $e = dZ$, $f = d$ and then let $d \to \infty$.

Just as (2) was the main tool used by Bailey in **[4]** to derive a representation for the Poisson kernel for Jacobi polynomials in terms of a *positive* F_4 function, formula (3) was the main tool used in **[10]** to derive a double series representation for a discrete Poisson kernel for the Hahn polynomials

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left[\begin{matrix}-n, n+\alpha+\beta+1, -x;\\ \alpha+1, -N\end{matrix}\right], \qquad n = 0, 1, \cdots, N$$

(a discrete analogue of Jacobi polynomials) from which the nonnegativity of the discrete Poisson kernel and of some other kernels in **[11]** can easily be established.

Bailey showed in **[6**, §9.6] that (2) follows easily from the case $a = -n$ of his formula (**[1]**, **[2]**)

$$\begin{aligned}&{}_2F_1[a,b;c;z]\,{}_2F_1[a,b;a+b-c+1;Z]\\ &= F_4[a,b;c,a+b-c+1;z(1-Z),Z(1-z)],\end{aligned} \tag{4}$$

which is valid inside simply-connected regions surrounding $z = 0$, $Z = 0$ for which

$$|z(1-Z)|^{1/2} + |Z(1-z)|^{1/2} < 1.$$

Since formulas (2) and (3) were so useful, this suggested that it might be of interest to show that Bailey's formula (4) has a discrete analogue of the form (1). Note that (4) follows from (1) by setting $x = -dz$, $y = -dZ$, $e = d$, and letting $d \to \infty$. Since (4) is a special case of the general formula

$$\begin{aligned}&F_4[a,b;c,c';z(1-Z),Z(1-z)]\\ &= \sum_{r=0}^{\infty}\frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!\,(c)_r(c')_r}z^rZ^r\\ &\times {}_2F_1[r+a,r+b;r+c;z]\,{}_2F_1[r+a,r+b;r+c';Z],\end{aligned} \tag{5}$$

which is due to Burchnall and Chaundy ([7, p. 257]; see also **[5]**), we shall also show that (5) has a discrete analogue of the form

$$F\begin{bmatrix} a,b: -x, y+e; -y, x+d; \\ d,e: \quad c; \quad c'; \end{bmatrix}$$

(6)
$$= \sum_{r=0}^{\min(x,y)} \frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!(c)_r(c')_r} \frac{(-x)_r(-y)_r}{(d)_r(e)_r}$$

$$\times {}_3F_2\begin{bmatrix} r+a, r+b, r-x; \\ r+c, r+d \end{bmatrix} {}_3F_2\begin{bmatrix} r+a, r+b, r-y; \\ r+c', r+e \end{bmatrix},$$

where $x, y = 0, 1, \cdots$. Clearly (6) reduces to (1) when $c' = a+b-c+1$.

In addition, we shall derive the transformation formula

$$F\begin{bmatrix} a,b: -x, y+e; -y, x+d \\ d,e: \quad c; \quad b; \end{bmatrix}$$

(7)
$$= \frac{(d-a)_x(e-a)_y}{(d)_x(e)_y} F\begin{bmatrix} a, -x: 1+a-c, -y; c-b; \\ c, 1+a-d-x: 1+a-e-y; \text{——}; \end{bmatrix}$$

for $x, y = 0, 1, \cdots$, which is a discrete analogue of the following formula of Bailey [3]

$$F_4\left[a, b; c, b; -\frac{z}{(1-z)(1-Z)}, -\frac{Z}{(1-z)(1-Z)}\right]$$

(8)
$$= (1-z)^a(1-Z)^a F_1[a; c-b, 1+a-c; c; z, zZ].$$

2. Formulas (1) and (6)

Since (1) is a special case of (6) it suffices to prove (6). The simplest way to prove (6) seems to be by means of the following extension of the proof which Bailey gave in [5] for formula (5). First put

$$S = \frac{(d)_x(e)_y}{(d-a)_x(e-b)_y} F\begin{bmatrix} a,b: -x, y+e; -y, x+d; \\ d,e: \quad c; \quad c'; \end{bmatrix}$$

and observe that

$$S = \frac{(d)_x(e)_y}{(d-a)_x(e-b)_y} \sum_{r,s \geqq 0} \frac{(a)_{r+s}(b)_{r+s}(-x)_r(y+e)_r(-y)_s(x+d)_s}{r!s!(d)_{r+s}(e)_{r+s}(c)_r(c')_s}$$

$$= \sum_{r,s \geqq 0} \frac{(a)_{r+s}(b)_{r+s}(-x)_r(-y)_s(-1)^{r+s}}{r!s!(c)_r(c')_s(1+a-d-x)_r(1+b-e-y)_s}$$

$$\times {}_2F_1\begin{bmatrix} r-x, a+r+s; \\ r+1+a-d-x \end{bmatrix} {}_2F_1\begin{bmatrix} s-y, b+r+s; \\ s+1+b-e, -y \end{bmatrix}$$

$$= \sum_{r,s,j,k \geqq 0} \frac{(a)_{r+s}(b)_{r+s}(-x)_{r+j}(-y)_{s+k}(a+r+s)_j(b+r+s)_k(-1)^{r+s}}{r!\,s!\,j!\,k!\,(c)_r(c')_s(1+a-d-x)_{r+j}(1+b-e-y)_{s+k}}$$

$$= \sum_{m=0}^{x} \sum_{n=0}^{y} \frac{(-x)_m(-y)_n}{(1+a-d-x)_m(1+b-e-y)_n}$$

$$\times \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(a)_{r+s}(b)_{r+s}(a+r+s)_{m-r}(b+r+s)_{n-s}(-1)^{r+s}}{r!\,s!\,(m-r)!\,(n-s)!\,(c)_r(c')_s}$$

where in the second line Vandermonde's theorem was employed to write

$$\frac{(d)_x(x+d)_s}{(d-a)_x(d)_{r+s}} = \frac{(-1)^r}{(1+a-d-x)_r} \, {}_2F_1\left[\begin{matrix} r-x, a+r+s; \\ r+1+a-d-x \end{matrix}\right]$$

and this expression with a, d, x, r, s replaced by b, e, y, s, r, respectively. From [**5**, p. 13] we have

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(a)_{r+s}(b)_{r+s}(a+r+s)_{m-r}(b+r+s)_{n-s}(-1)^{r+s}}{r!\,s!\,(m-r)!\,(n-s)!\,(c)_r(c')_s}$$

$$\text{(9)} \qquad = \frac{(a)_m(b)_n(c'-a-m)_n(c-b-n)_m}{m!\,n!\,(c')_n(c)_m}$$

$$= \frac{(a)_m(b)_n(c'-a)_n(c-b)_m}{m!\,n!\,(c')_n(c)_m} \, {}_3F_2\left[\begin{matrix} a+b-c-c'+1, -m, -n; \\ 1-c+b-m, 1-c'+a-n \end{matrix}\right].$$

Hence

$$S = \sum_{m=0}^{x} \sum_{n=0}^{y} \frac{(a)_m(b)_n(c-b)_m(c'-a)_n(-x)_m(-y)_n}{m!\,n!\,(c)_m(c')_n(1+a-d-x)_m(1+b-e-y)_n}$$

$$\times \sum_{r=0}^{\min(m,n)} \frac{(a+b-c-c'+1)_r(-m)_r(-n)_r}{r!\,(1-c+b-m)_r(1-c'+a-n)_r},$$

and so, putting $m = r+s$, $n = r+t$ and changing the order of summation, we get

$$S = \sum_{r=0}^{\min(x,y)} \frac{(a)_r(b)_r(a+b-c-c'+1)_r(-x)_r(-y)_r}{r!\,(c)_r(c')_r(1+a-d-x)_r(1+b-e-y)_r}$$

$$\times {}_3F_2\left[\begin{matrix} r+a, c-b, r-x; \\ r+c, r+1+a-d-x \end{matrix}\right] {}_3F_2\left[\begin{matrix} r+b, c'-a, r-y; \\ r+c', r+1+b-e-y \end{matrix}\right],$$

which, on using the relation

$$\text{(10)} \qquad {}_3F_2\left[\begin{matrix} a, b, -n; \\ c, d \end{matrix}\right] = \frac{(d-a)_n}{(d)_n} \, {}_3F_2\left[\begin{matrix} a, c-b, -n; \\ c, 1+a-d-n \end{matrix}\right],$$

gives (6). Formula (10) is the realtion between $Fp(0)$ and $Fn(4)$ in [**6**, p. 22].

It should be observed that (1) can be used to prove (3). From (1) we have

$$(11)\qquad {}_3F_2\left[\begin{matrix}-n, n+a, -x;\\ c, d\end{matrix}\right]{}_3F_2\left[\begin{matrix}-n, n+a, -y;\\ a-c+1, f\end{matrix}\right] = F\left[\begin{matrix}-n, n+a: -x, y+f; -y, x+d;\\ d, f: \quad c; a-c+1;\end{matrix}\right]$$

for $n, x, y = 0, 1, \cdots$. Since both sides of this identity are polynomials in x and in y, (11) holds for all complex values of x and y; and so setting $x = -b$, $y = e-f$, and using (10) to see that

$${}_3F_2\left[\begin{matrix}-n, n+a, f-e;\\ a-c+1, f\end{matrix}\right] = \frac{(-1)^n(c)_n}{(a-c+1)_n}\,{}_3F_2\left[\begin{matrix}-n, n+a, e;\\ c, f\end{matrix}\right],$$

we get (3). Conversely, (3) can be used to prove (1) by reversing the above steps and noticing that if x and y are nonnegative integers then both sides of (11) are polynomials in the variable n.

3. Formula (7). To prove (7) we proceed as in §2 and use the first identity in (9) to obtain

$$\begin{aligned}
&\frac{(d)_x(e)_y}{(d-a)_x(e-b)_y}\,F\left[\begin{matrix}a, b: -x, y+e; -y, x+d;\\ d, e: \quad c; \quad b;\end{matrix}\right]\\
&= \sum_{m=0}^{x}\sum_{n=0}^{y}\frac{(-x)_m(-y)_n}{(1+a-d-x)_m(1+b-e-y)_n}\\
&\quad\times\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(a)_{r+s}(b)_{r+s}(a+r+s)_{m-r}(b+r+s)_{n-s}(-1)^{r+s}}{r!\,s!\,(m-r)!\,(n-s)!\,(c)_r(b)_s}\\
&= \sum_{m=0}^{x}\sum_{n=0}^{y}\frac{(a)_m(b-a-m)_n(c-b-n)_m(-x)_m(-y)_n}{m!\,n!\,(c)_m(1+a-d-x)_m(1+b-e-y)_n}\\
&= \sum_{m=0}^{x}\frac{(a)_m(c-b)_m(-x)_m}{m!\,(c)_m(1+a-d-x)_m}\,{}_3F_2\left[\begin{matrix}b-a-m, 1+b-c, -y;\\ 1+b-c-m, 1+b-e-y\end{matrix}\right]\\
&= \sum_{m=0}^{x}\frac{(a)_m(c-b)_m(-x)_m(e-a)_y}{m!\,(c)_m(1+a-d-x)_m(e-b)_y}\,{}_3F_2\left[\begin{matrix}1+a-c, -m, -y;\\ 1+b-c-m, 1+a-e-y\end{matrix}\right]\\
&= \frac{(e-a)_y}{(e-b)_y}\sum_{m=0}^{x}\sum_{r=0}^{m}\frac{(a)_m(c-b)_m(-m)_r(1+a-c)_r(-x)_m(-y)_r}{m!\,r!\,(c)_m(1+b-c-m)_r(1+a-d-x)_m(1+a-e-y)_r}
\end{aligned}$$

on using Example 7 of [6, p. 98]. Setting $m = r + s$, the above sum becomes

$$\frac{(e-a)_y}{(e-b)_y}\sum_{r=0}^{y}\sum_{s=0}^{x}\frac{(a)_{r+s}(-x)_{r+s}(1+a-c)_r(-y)_r(c-b)_s}{r!\,s!\,(c)_{r+s}(1+a-d-x)_{r+s}(1+a-e-y)_r}$$

$$=\frac{(e-a)_y}{(e-b)_y}F\left[\begin{matrix} a,-x: 1+a-c,-y; c-b; \\ c, 1+a-d-x: \quad 1+a-e-y; \text{——}; \end{matrix}\right],$$

which gives (7).

Note that as special cases of (7) we have, for $x, y = 0, 1, \cdots$,

$$F\left[\begin{matrix} a,b: -x, y+e; -y, x+d; \\ d,e: \quad b; \quad b; \end{matrix}\right]$$

$$=\frac{(d-a)_x(e-a)_y}{(d)_x(e)_y}\,{}_4F_3\left[\begin{matrix} a, 1+a-b, -x, -y; \\ b, 1+a-d-x, 1+a-e-y \end{matrix}\right]$$

and

$$F\left[\begin{matrix} a,b: -x, y+e; -y, x+d; \\ d,e: \quad a; \quad b; \end{matrix}\right] \tag{12}$$

$$=\frac{(d-a)_x(e-a)_y}{(d)_x(e)_y}F\left[\begin{matrix} -x: \quad 1,-y; a-b; \\ 1+a-d-x: 1+a-e-y; \text{——}; \end{matrix}\right]$$

$$=\frac{(d-b)_x(e-a)_y}{(d)_x(e)_y}\sum_{r=0}^{\min(x,y)}\frac{(-x)_r(-y)_r}{(1+b-d-x)_r(1+a-e-y)_r},$$

which are discrete analogues of the formulas

$$F_4\left[a,b;b,b;-\frac{z}{(1-z)(1-Z)},-\frac{Z}{(1-z)(1-Z)}\right]$$

$$=(1-z)^a(1-Z)^a\,{}_2F_1[a,1+a-b;b;zZ]$$

and

$$\begin{aligned} &F_4\left[a,b;a,b;-\frac{z}{(1-z)(1-Z)},-\frac{Z}{(1-z)(1-Z)}\right] \\ &=(1-z)^b(1-Z)^a(1-zZ)^{-1}, \end{aligned} \tag{13}$$

respectively (see [3, p. 42] or [6, p. 102]). Equation (1) on page 14 of [6] may be used to transform the last sum in (12) into a multiple of a

Saalschützian ${}_3F_2(1)$ series which can then be summed when $y+e-a$ (or $x+d-b$ or $d+e-a-b-1$) is a nonnegative integer to give

$$F\begin{bmatrix} a,b: -x, y+e; -y, x+d; \\ d,e: \quad a; \quad b; \end{bmatrix} \tag{14}$$

$$= \frac{(-1)^x y!(-x)_{y+e-a}\,\Gamma(1+b-d)\Gamma(d-b)}{(d)_x(e)_y\,\Gamma(1+b+y-d-x)\Gamma(d+e-a-b)}.$$

Since (13) can be employed to quickly derive [12, §140] the following useful generating function for Jacobi polynomials

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(z)t^n = 2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{-\beta}(1-t+\rho)^{-\alpha}, \tag{15}$$

where $\rho=(1-2zt+t^2)^{1/2}$, one would expect to be able to use (14) to derive an appropriate discrete analogue of (15) for Hahn polynomials, but the change of variables needed seems to preclude this.

4. Additional formulas. Formula (6) has an inverse expansion of the form

$${}_3F_2\begin{bmatrix} a,b,-x; \\ c,d \end{bmatrix} {}_3F_2\begin{bmatrix} a,b,-y; \\ c',e \end{bmatrix}$$

$$= \sum_{r=0}^{\min(x,y)} \frac{(a)_r(b)_r(c+c'-a-b-1)_r(-x)_r(-y)_r}{r!(c)_r(c')_r(d)_r(e)_r} \tag{16}$$

$$\times F\begin{bmatrix} r+a, r+b: r-x, y+e; r-y, x+d; \\ r+d, r+e: \quad r+c; \quad r+c'; \end{bmatrix},$$

where $x,y=0,1,\cdots,$ This is a discrete analogue of

$${}_2F_1[a,b;c;z]{}_2F_1[a,b;c';z]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c+c'-a-b-1)_r}{r!(c)_r(c')_r} z^r Z^r \tag{17}$$

$$\times F_4[r+a, r+b; r+c, r+c'; z(1-Z), Z(1-z)],$$

which is the inverse expansion of (5) due to Burchnall and Chaundy [7, p. 257]. Just as in the proof of (17) given in [7, p. 260], (16) can be proved by substituting for the F-function on the right from (6), summing diagonally, and then using Vandermonde's theorem to obtain the left side of (16).

In [**6**, p. 100] Bailey noted that (4) and [**6**, §1.4 (1)] can be used to derive the formula

$$
\begin{aligned}
(18) \qquad & {}_2F_1[a, b; c; z]\,{}_2F_1[a, b; c; Z] \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F_4[a, b;, c, a+b-c+1; zZ, (1-z) \\
&\qquad\qquad \times (1-Z)] \\
&+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \{(1-z)(1-Z)\}^{c-a-b} \\
&\times F_4[c-b, c-a; c, c-a-b+1; zZ, (1-z)(1-Z)],
\end{aligned}
$$

which is a generalization of (2) that had been discovered by Watson [**13**, p. 194] by means of contour integrals of Barnes' types. To obtain the analogue of (18) for terminating ${}_3F_2(1)$ series, we first observe that the formula [**6**, §1.4 (1)] used by Bailey is a limiting case of [**6**, §3.8 (1)]; so that using the latter formula, Ex. 7 on p. 98 of [**6**], and then using (1) we find that the required analogue of (18) is

$$
\begin{aligned}
& {}_3F_2\left[\begin{matrix} a, b, -x; \\ c, d \end{matrix}\right] {}_3F_2\left[\begin{matrix} a, b, -y; \\ c, e \end{matrix}\right] \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F\left[\begin{matrix} a, b: -x, -y; y+e, x+d; \\ d, e: \quad c; a+b-c+1; \end{matrix}\right] \\
(19) \qquad & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \frac{(c+d-a-b)_x(c+e-a-b)_y}{(d)_x(e)_y} \\
&\times F\left[\begin{matrix} c-a, c-b: -x, -y; y+c+e-a-b, x+c+d-a-b; \\ c+d-a-b, c+e-a-b: \quad c; \quad c-a-b+1; \end{matrix}\right],
\end{aligned}
$$

where it is assumed that $x, y, -y-e$, and $a+b-y-c-e$ are non-negative integers. This formula can easily be used to derive a discrete analogue of the formula Watson gave in [**13**, p. 193] connecting four F_4 series with two products of hypergeometric functions.

It should also be pointed out that by using the known [**6**] cases in which ${}_3F_2(1)$ series may be summed (e.g., if Sallschützian) one can sum the double series in (1) and (3) for serveral special cases; thus supplie- menting the double sum formulas in Carlitz [**9**].

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