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ON SOME COMPLETENESSES OF THE BERGMAN KERNEL AND THE RUDIN KERNEL

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Dedicated to Professor Toshio Umezawa on his 60-th birthday

Let $L_2(G)$ denote the Hilbert space of analytic functions f which are regular in a region G and have finite norms: $\left(\iint_G |f(z)|^2\,dxdy\right)^{1/2} < \infty$. It is well-known that the set $\{K(z,\bar{z}_1) \mid z_1 \in G\}$ of the Bergman kernels for the class $L_2(G)$ is complete in $L_2(G)$. In this paper, for regular regions G in the plane, it is shown that the set $\{K(z,\bar{z}_1) \mid z_1 \in G\}$ is also complete in the Hilbert space of analytic functions f which are regular in G and finite norms: $\left(\int_{\partial G} |f(z)|^2 ds\right)^{1/2} < \infty$.

The object of this paper is to discuss some problems of this type.

1. Introduction. Let S be a compact bordered Riemann surface with contours m and of genus n. Let $\{C_\nu\}_{\nu=1}^{2n+m-1}$ denote a canonical homology basis and $\{C_\nu\}_{\nu=2n+1}^{2n+m}$ denote the boundary components. Let M denote the Hilbert space of analytic differentials $f(z)dz$ which are regular in S and have finite norms: $\left(\iint_S |f(z)|^2 dxdy\right)^{1/2} < \infty (z = x + yi)$. Let $F = F(C_{j_1}, C_{j_2}, \cdots, C_{j_a})$ be the closed subspace of M composed of differentials $f(z)dz$ such that

$$\int_{C_{j_\lambda}} f(z)dz = 0, \quad \lambda = 1, 2, \cdots, a. \tag{1.1}$$

In terms of local parameters z and z_1, let $K_F(z,\bar{z}_1)dz$ denote the Bergman kernel for the class F which is characterized by the following reproducing property:

$$f(z_1) = \iint_S f(z)\overline{K_F(z,\bar{z}_1)}\,dx\,dy \quad \text{for all} \quad f(z)dz \in F.$$

On the other hand, we consider the Hilbert space H_2^D of analytic differentials $f(z)dz$ which are regular in S and finite norms: $\left(\frac{1}{2\pi}\int_{\partial S} |f(z)dz|^2/idW(z,t)\right)^{1/2} < \infty$. Here $W(z,t)$ denotes $g(z,t) + ig^*$

(z, t), g is the Green function of S with pole at t and g^* is the conjugate harmonic function of g. In this paper, for simplicity, we shall use the same notation for a point on $\bar{S}$ and a fixed local parameter around there. Let H_2^{DF} denote the closed subspace of H_2^D satisfying the condition (1.1). In terms of local parameters z and z_1, let $\hat{R}_t^F(z, z_1)dz$ denote the conjugate Rudin kernel for the class H_2^{DF} which is characterized by the following reproducing property (cf. [2]):

$$(1.2) \qquad f(z_1) = \frac{1}{2\pi} \int_{\partial S} \frac{f(z)dz\overline{\hat{R}_t^F(z, z_1)dz}}{idW(z, t)} \text{ for all } f(z)dz \in H_2^{DF}.$$

Let S_0 denote any point set $\{P_j\}$ of S such that $\lim_{j\to\infty} P_j = P$, for some $P \in S$. Then as we see from the reproducing property, the sets of kernel functions $\{K_F(z, \bar{z}_1)dz \mid z_1 \in S_0\}$ and $\{\hat{R}_t^F(z, z_1)dz \mid z_1 \in S_0\}$ are complete (or equivalently closed) in the Hilbert spaces F and H_2^{DF}, respectively. In the present paper, we shall show that the sets $\{K_F(z, \bar{z}_1)dz \mid z_1 \in S_0\}$ and $\{\hat{R}_t^F(z, z_1)dz \mid z_1 \in S_0\}$ are also complete in H_2^{DF} and F, respectively, and further we refer to some completenesses of the Rudin kernel functions. These results will be obtained from some fundamental properties of the Bergman kernel and the Rudin kernel.

2. Completeness of $\{K_F(z, \bar{z}_1)dz \mid z_1 \in S_0\}$. Let

$$L_F(z, z_1)dz$$

denote the adjoint L-kernel of $K_F(z, \bar{z}_1)dz$. $L_F(z, z_1)dz$ is an analytic differential on $\bar{S}$ except for z_1 where it has a double pole:

$$(2.1) \qquad L_F(z, z_1)dz = \left(\frac{1}{\pi} \frac{1}{(z - z_1)^2} + \text{regular terms}\right) dz.$$

Further $L_F(z, z_1)dz$ has the following properties:

$$(2.2) \qquad \int_{C_{1_\lambda}} L_F(z, z_1)dz = 0, \quad \lambda = 1, 2, \cdots, a.$$

$$(2.3) \qquad \iint_S f(z)\, \overline{L_F(z, z_1)}\, dx\, dy = 0 \text{ for all } f(z)dz \in F.$$

$$(2.4) \qquad -K_F(z, \bar{z}_1)dz = \overline{L_F(z_1, z)dz} \text{ along } \partial S\ (z \in \partial S).$$

In general, we have $K_F(z, \bar{z}_1) = \overline{K_F(z_1, \bar{z})}$, but $L_F(z, z_1) = L_F(z_1, z)$ if and only if the class F is symmetric. As to the properties of the Bergman kernel for the class F on compact bordered Riemann surfaces, the reader is referred to Schiffer-Spencer **[4]**. Let $\{t_\nu\}_{\nu=1}^{2n+m-1}$ denote the critical points of $g(z,t)$. Let $\{C_{k_1}, C_{k_2}, \cdots, C_{k_b}\}$ denote $\{C_\nu\}_{\nu=1}^{2n+m-1} - \{C_{j_1}, C_{j_2}, \cdots, C_{j_a}\}$. Then we have the following theorem which is a generalized form of Lemma 2.1 in **[2]**:

THEOREM 2.1.

$$\det\begin{bmatrix} \int_{C_{k_1}} L_F(z,t_\nu)dz \\ \vdots \\ \int_{C_{k_b}} L_F(z,t_\nu)dz \\ \int_{C_{j_1}} \left(\int_t^z L_F(\zeta,t_\nu)d\zeta\right) idW(z,t) \\ \vdots \\ \int_{C_{j_a}} \left(\int_t^z L_F(\zeta,t_\nu)d\zeta\right) idW(z,t) \end{bmatrix}^{(2n+m-1)\times(2n+m-1)} \neq 0.$$

Here we assume that $\{t_\nu\}$ are all simple. In the other cases, we obtain modified forms.

Proof. From (2.1), (2.4) and the identity

$$K_F(z_1,\bar{z}) = \frac{1}{2\pi}\int_{\partial S} \frac{K_F(\zeta,\bar{z})d\zeta\overline{\hat{R}_t^F(\zeta,z_1)d\zeta}}{idW(\zeta,t)}$$

we have

$$\overline{K_F(z_1,\bar{z})} = -\frac{1}{2\pi i}\int_{\partial S}\frac{L_F(z,\zeta)\hat{R}_t^F(\zeta,z_1)d\zeta}{W'(\zeta,t)}$$
$$= -\frac{1}{\pi}\left(\frac{\hat{R}_t^F(z,z_1)}{W'(z,t)}\right)' - \sum_{\nu=1}^{2n+m-1}\frac{\hat{R}_t^F(t_\nu,z_1)L_F(z,t_\nu)}{W''(t_\nu,t)}.$$

Hence

$$\frac{1}{\pi}\left(\frac{\hat{R}_t^F(z,z_1)}{W'(z,t)}\right)' = -K_F(z,\bar{z}_1) - \sum_\nu \frac{\hat{R}_t^F(t_\nu,z_1)}{W''(t_\nu,t)}L_F(z,t_\nu). \tag{2.5}$$

Further we get

$$\frac{1}{\pi}\hat{R}_t^F(z, z_1)dz \tag{2.6}$$

$$= -\left\{\int_t^z K_F(\zeta, \bar{z}_1)d\zeta + \sum_\nu \frac{\hat{R}_t^F(t_\nu, z_1)}{W''(t_\nu, t)}\int_t^z L_F(\zeta, t_\nu)d\zeta\right\} dW(z, t).$$

At first from (2.5) we have

$$\sum_\nu \frac{\hat{R}_t^F(t_\nu, z_1)}{W''(t_\nu, t)}\int_{C_{k_\mu}} L_F(z, t_\nu)dz \tag{2.7}$$

$$= -\int_{C_{k_\mu}} K_F(z, \bar{z}_1)dz, \quad \mu = 1, 2, \cdots, b.$$

Next from (2.6), since $\hat{R}_t^F(z, z_1)dz \in H_2^{DF}$, we have

$$\sum_\nu \frac{\hat{R}_t^F(t_\nu, z_1)}{W''(t_\nu, t)}\int_{C_{l_\lambda}} \left(\int_t^z L_F(\zeta, t_\nu)d\zeta\right) idW(z, t) \tag{2.8}$$

$$= -\int_{C_{l_\lambda}} \left(\int_t^z K_F(\zeta, \bar{z}_1)d\zeta\right) idW(z, t), \quad \lambda = 1, 2, \cdots, a.$$

Here we shall see that the coefficients $\{\hat{R}_t^F(t_\nu, z_1)/W''(t_\nu, t)\}_\nu$ in the representation (2.6) of $\hat{R}_t^F(z, z_1)dz$ are determined uniquely as the solution of the equations (2.7) and (2.8).

We take $\{X_\nu\}_{\nu=1}^{2n+m-1}$ as a solution of (2.7) and (2.8) and define

$$\frac{1}{\pi}\tilde{R}_t^F(z, z_1)dz = -\left\{\int_t^z K_F(\zeta, \bar{z}_1)d\zeta + \sum_\nu X_\nu \int_t^z L_F(\zeta, t_\nu)d\zeta\right\} dW(z, t).$$

Then $\tilde{R}_t^F(z, z_1)dz \in H_2^{DF}$ and from (2.7) and (2.2) we see that $(K_F(\zeta, \bar{z}_1) + \Sigma_\nu X_\nu L_F(\zeta, t_\nu))d\zeta$ is exact. For any analytic differential $f(z)dz$ on $\bar{S}$ (in fact, S) such that $f(z)dz \in H_2^{DF}$, we have

$$\frac{1}{2\pi}\int_{\partial S} \frac{f(z)dz\overline{\tilde{R}_t^F(z, z_1)dz}}{idW(z, t)}$$

$$= \frac{1}{2i}\int_{\partial S} f(z)\overline{\left(\int_t^z (K_F(\zeta, \bar{z}_1) + \sum_\nu X_\nu L_F(\zeta, t_\nu))d\zeta\right)} dz,$$

from the Green's formula, as usual,

$$= \iint_S f(z)\overline{\left(K_F(z, \bar{z}_1) + \sum_\nu X_\nu L_F(z, t_\nu)\right)} dx\, dy$$

from (2.3),

$$= f(z_1),$$

which implies that $\tilde{R}_t^F(z, z_1) \equiv \hat{R}_t^F(z, z_1)$. Since $\left\{\int_t^z L_F(\zeta, t_\nu)d\zeta\right\}$ is linearly independent, we have the desired result.

Thus from the uniqueness of the solution of the equations (2.7) and (2.8), we have the assertion of the theorem. In the cases of which all the t_ν are not simple, we can modify the above arguments slightly and we have modified forms, as usual.

In Theorem 2.1, if $F = F(0) = M$, then from the identities

$$\text{(2.9)} \quad L_F(z, z_1) = -\frac{2}{\pi}\frac{\partial^2 g(z, z_1)}{\partial z \partial z_1} \quad \text{and} \quad Z'_\nu(z) = -\int_{C_\nu} L_F(\zeta, z)d\zeta$$

(cf. **[4]**), we have $\det[Z'_\nu(t_j)]^{(2n+m-1)\times(2n+m-1)} \neq 0$, which is the result of Lemma 2.1 in **[2]**. Here $\{dZ_\nu\}$ is a basis of analytic differentials on $\bar{S}$ which are real along ∂S.

Next let G be a regular region in the plane with contours $\{C_\nu\}_{\nu=1}^m$. If $F = F(1, 2, \cdots, m-1)$, from the identities

$$K_F(z, \bar{z}_1) = \frac{1}{\pi} M'(z, z_1) \quad \text{and} \quad L_F(z, z_1) = -\frac{1}{\pi} N'(z, z_1)$$

(cf. **[1]**, pp. 361–376), we have the following:

COROLLARY 2.1.

$$\det\left[\frac{1}{2\pi}\int_{C_i} N(z, t_\nu)idW(z, t) - N(t, t_\nu)\omega_j(t)\right]^{(m-1)\times m-1} \neq 0.$$

Here ω_j is the harmonic measure of C_j and we assume that $\{t_\nu\}$ are all simple. In the other cases, we have modified forms.

Now we have the first desired result:

THEOREM 2.2. *The set of the Bergman kernels $\{K_F(z, \bar{z}_1)dz \mid z_1 \in S_0\}$ is complete in H_2^{DF}.*

Proof. We assume that for any $f(\zeta)d\zeta \in H_2^{DF}$,

$$\int_{\partial S} \frac{f(\zeta)d\zeta\,\overline{K_F(\zeta, \bar{z})d\zeta}}{idW(\zeta, t)} = 0 \text{ for all } z \in S_0.$$

From (2.1) and (2.4), we have

$$(2.10)\qquad \frac{1}{\pi}\left(\frac{f(z)}{W'(z,t)}\right)' + \sum_{\nu=1}^{2n+m-1} \frac{f(t_\nu) L_F(z,t_\nu)}{W''(t_\nu,t)} = 0 \text{ for all } z \in S_0,$$

and hence for all $z \in S$. Here we assume that $\{t_\nu\}$ are all simple. At first from (2.10), we have

$$(2.11)\qquad \sum_\nu \frac{f(t_\nu)}{W''(t_\nu,t)} \int_{C_{k_\mu}} L_F(z,t_\nu)dz = 0, \quad \mu = 1,2,\cdots,b.$$

Next from (2.10), we have

$$(2.12)\qquad \frac{1}{\pi}\frac{f(z)}{W'(z,t)} + \sum_\nu \frac{f(t_\nu)}{W''(t_\nu,t)} \int_t^z L_F(\zeta, t_\nu)d\zeta = 0.$$

Hence from $f(z)dz \in H_2^{DF}$, we get

$$(2.13)\qquad \sum_\nu \frac{f(t_\nu)}{W''(t_\nu,t)} \int_{C_{l_\lambda}} \left(\int_t^z L_F(\zeta, t_\nu)d\zeta\right) idW(z,t) = 0,$$

$$\lambda = 1,2,\cdots,a.$$

Hence from (2.11), (2.13) and Theorem 2.1, we have $f(t_\nu)=0$, $\nu = 1,2,\cdots,2n+m-1$. Thus $(f(z)/W'(z,t))' \equiv 0$ and $f(z) \equiv 0$. It implies the desired result.

In the cases of which all the t_ν are not simple, by making use of modified forms of Theorem 2.1, we have the desired result, again.

3. Completeness of $\{\hat{R}_t^F(z,z_1)dz \mid z_1 \in S_0\}$. Let $N(z;z_1,t)$ be a Neumann's function on S with poles at z_1 and t, where $N(z;z_1,t)+\log|z-z_1|$ and $N(z;z_1,t)-\log|z-t|$ are harmonic, respectively and $\partial N/\partial \nu = 0$ on ∂S. We set $V(z;z_1,t) = N(z;z_1,t) + iN^*(z;z_1,t)$ and define meromorphic differentials as follows:

$$(3.1)\qquad \begin{aligned} dP(z;z_1,t) &= \tfrac{1}{2}\,[dV(z;z_1,t) - dW(z,z_1) + dW(z,t)] \\ d\tilde{P}(z;z_1,t) &= \tfrac{1}{2}\,[dV(z;z_1,t) - dW(z,z_1) - dW(z,t)] \\ dQ(z;z_1,t) &= \tfrac{1}{2}\,[-dV(z;z_1,t) - dW(z,z_1) + dW(z,t)] \\ d\tilde{Q}(z;z_1,t) &= \tfrac{1}{2}\,[-dV(z;z_1,t) - dW(z,z_1) - dW(z,t)]. \end{aligned}$$

Here we note that

$$(3.2)\qquad \overline{dP(z;z_1,t)} = -dQ(z;z_1,t) \quad \text{along} \quad \partial S,$$

$$(3.3)\qquad \overline{d\tilde{P}(z;z_1,t)} = -d\tilde{Q}(z;z_1,t) \quad \text{along} \quad \partial S.$$

Then we have the following representation of the kernel $\hat{R}_t(z,z_1)dz$ for the class $H_2^D[2]$:

$$(3.4)\qquad \hat{R}_t(z,z_1) = -\overline{W'(z_1,t)}\, P'(z;z_1,t) + \sum_{\nu=1}^{2n+m-1} \overline{\beta_\nu(z_1,t)}\, Z'_\nu(z).$$

Here $\{\beta_\nu(z_1,t)\}$ are constants which depend on z_1 and t and determined uniquely. At first, we note the following fact:

LEMMA 3.1.

$$(3.5)\qquad \det[\beta_\nu(t_\mu,t)]^{(2n+m-1)\times(2n+m-1)} \neq 0.$$

Here we assume that $\{t_\mu\}$ *are all simple. On the other cases, we have modified forms.*

Proof. We assume that the determinant (3.5) is zero. Hence we can take complex numbers $\{X_\mu\}$ such that all X_μ are not zero and

$$(3.6)\qquad \sum_{\mu=1}^{2n+m-1} X_\mu \beta_\nu(t_\mu,t) = 0, \quad \nu = 1,2,\cdots,2n+m-1.$$

On the other hand, from (3.4) we have

$$(3.7)\qquad \hat{R}_t(z,t_\mu) = \sum_{\nu=1}^{2n+m-1} \overline{\beta_\nu(t_\mu,t)}\, Z'_\nu(z), \quad \mu = 1,2,\cdots,2n+m-1.$$

Hence from (3.6) and (3.7), we get

$$\sum_\mu \bar{X}_\mu \hat{R}_t(z,t_\mu) \equiv 0.$$

As we see from the general theory of kernel functions, since $\det[\hat{R}_t(t_\nu,t_\mu)] \neq 0$, we have $X_\mu = 0$ for all μ and hence we arrive at a contradiction.

Now we shall have the following theorem:

THEOREM 3.1.

$$(3.8)\qquad \det\left[\int_{C_\lambda}\left(\int_{C_\mu} \hat{R}_t(z,z_1)dz\right)\overline{dz_1}\right]^{(2n+m-1)\times(2n+m-1)} > 0.$$

Proof. We assume that the determinant (3.8) is zero. Then by making use of the representation of $\hat{R}_t(z, z_1)$ by a complete orthonormal system, we see that $\left\{\int_{C_\lambda} \hat{R}_t(z, z_1)dz\right\}_\lambda$ is linearly dependent for any $z_1 \in S$. Hence there exist complex numbers $\{X_\lambda\}$ such that all X_λ are not zero and $\Sigma_\lambda X_\lambda \int_{C_\lambda} \hat{R}_t(z, z_1)dz \equiv 0$. As to this fact, the reader is referred to the proof of Theorem 2.1 in **[3]**. Hence from (3.4) we have

$$\sum_\lambda X_\lambda \int_{C_\lambda} \left(-\overline{W'(z_1, t)}\, P'(z; z_1, t) + \sum_\nu \overline{\beta_\nu(z_1, t)}\, Z'_\nu(z) \right) dz \equiv 0, \quad z_1 \in S.$$

By setting $z_1 = t_\mu$, we have

$$\sum_\lambda X_\lambda \left(\sum_\nu \overline{\beta_\nu(t_\mu, t)} \int_{C_\lambda} dZ_\nu \right) = 0, \quad \mu = 1, 2, \cdots, 2n + m - 1.$$

Hence from Lemma 3.1 (or from modified forms of (3.5) if all the t_ν are not simple), we have $\Sigma_\lambda X_\lambda \int_{C_\lambda} dZ_\nu = 0$, $\nu = 1, 2, \cdots, 2n + m - 1$, which implies that all the X_λ are zero, because the matrix $\left\| \int_{C_\lambda} dZ_\nu \right\|$ is nonsingular. Thus we have a contradiction.

Next we consider a representation of $\hat{R}^F_t(z, z_1)dz$ by the kernel $\hat{R}_t(z, z_1)dz$. From Theorem 3.1, we can take constants $\{\hat{A}_{j_\lambda}(z_1)\}_{\lambda=1}^a$ which are analytic functions of z_1 and determined uniquely as follows:

$$\hat{R}_t(z, z_1)dz - \frac{1}{2\pi i} \sum_{\lambda=1}^{a} \overline{\hat{A}_{j_\lambda}(z_1)} \int_{C_{j_\lambda}} \overline{\hat{R}_t(\zeta, z)d\zeta}\, dz \in H_2^{DF}. \tag{3.9}$$

As we see by the simple computations, since the differential (3.9) has the reproducing property (1.2), we see that this is the kernel $\hat{R}^F_t(z, z_1)dz$.

Now we shall give the following theorem:

THEOREM 3.2. *For $\{\beta_{k_\mu}(z_1, t)\}_{\mu=1}^b$, we have*

$$\det \begin{bmatrix} \beta_{k_1}(t_\nu, t) \\ \vdots \\ \beta_{k_b}(t_\nu, t) \\ \hat{A}_{j_1}(t_\nu) \\ \vdots \\ \hat{A}_{j_a}(t_\nu) \end{bmatrix}^{(2n+m-1)\times(2n+m-1)} \neq 0.$$

Here we assume that $\{t_\nu\}$ are all simple. In the other cases, we obtain modified forms.

Proof. We assume that the above determinant is zero and hence we can take $\{Y_\nu\}$ such that all Y_ν are not zero and

$$(3.10)\qquad \begin{aligned} &\sum_{\nu=1}^{2n+m-1} Y_\nu \beta_{k_\mu}(t_\nu, t) = 0, \quad \mu = 1, 2, \cdots, b, \quad \text{and} \\ &\sum_{\nu=1}^{2n+m-1} Y_\nu \hat{A}_{j_\lambda}(t_\nu) = 0, \quad \lambda = 1, 2, \cdots, a. \end{aligned}$$

On the other hand, from (3.4) and (3.9) we have

$$(3.11)\qquad \begin{aligned} \hat{R}_t^F(z, z_1) = &- \overline{W'(z_1, t)}\, P'(z; z_1, t) + \sum_\gamma \overline{\beta_\gamma(z_1, t)}\, Z'_\gamma(z) \\ &- \frac{1}{2\pi i} \sum_\lambda \overline{\hat{A}_{j_\lambda}(z_1)} \int_{C_{j_\lambda}} \overline{\hat{R}_t(\zeta, z) d\zeta}. \end{aligned}$$

Hence we have, by setting $z_1 = t_\nu$,

$$(3.12)\quad \hat{R}_t^F(z, t_\nu) = \sum_\gamma \overline{\beta_\gamma(t_\nu, t)} Z'_\gamma(z) - \frac{1}{2\pi i} \sum_\lambda \overline{\hat{A}_{j_\lambda}(t_\nu)} \int_{C_{j_\lambda}} \overline{\hat{R}_t(\zeta, z) d\zeta}.$$

From (3.10) and (3.12), we get

$$\sum_\nu \bar{Y}_\nu \hat{R}_t^F(z, t_\nu) = \sum_{\lambda=1}^{a} \Bigl(\sum_\nu \bar{Y}_\nu \overline{\beta_{j_\lambda}(t_\nu, t)} \Bigr) Z'_{j_\lambda}(z),$$

and hence from $\hat{R}_t^F(z, t_\nu) dz \in H_2^{DF}$,

$$\sum_{\lambda=1}^{a} \Bigl(\sum_\nu \bar{Y}_\nu \overline{\beta_{j_\lambda}(t_\nu, t)} \Bigr) \int_{C_{\lambda'}} dZ_{j_\lambda} = 0, \quad \lambda' = 1, 2, \cdots, a.$$

Since $\det\Bigl[\int_{C_{j_{\lambda'}}} dZ_{j_\lambda} \Bigr] \neq 0$, we have

$$\sum_\nu \bar{Y}_\nu \overline{\beta_{j_\lambda}(t_\nu, t)} = 0, \quad \lambda = 1, 2, \cdots, a.$$

and hence

$$\sum_{\nu} \bar{Y}_\nu \hat{R}_t^F(z, t_\nu) \equiv 0,$$

which implies that all the Y_ν are zero. Hence we have a contradiction.

Especially, in Theorem 3.2, from the case of the subspace of H_2^D such that $f(z)dz \in H_2^D$ are exact, we have the following:

COROLLARY 3.1.

$$\det[\hat{A}_\nu(t_\mu)]^{(2n+m-1)\times(2n+m-1)} \neq 0.$$

Here we assume that $\{t_\mu\}$ *are all simple. On the other cases, we have modified forms.*

Now we can give the second desired result:

THEOREM 3.3. *The set of the conjugate Rudin kernels* $\{\hat{R}_t^F(z, z_1)dz \mid z_1 \in S_0\}$ *is complete in* F.

Proof. We assume that for any $f(z)dz \in F$,

$$\iint_S f(z)\overline{\hat{R}_t^F(z, z_1)}\, dx\, dy = 0 \text{ for all } z_1 \in S_0.$$

From (3.4) and (3.9), we have

$$\iint_S f(z)\left[-W'(z_1, t)\,\overline{P'(z; z_1, t)} + \sum_\nu \beta_\nu(z_1, t)\,\overline{Z'_\nu(z)}\right] dx\, dy$$
$$+\frac{1}{2\pi i}\sum_\lambda \hat{A}_{j_\lambda}(z_1)\iint_S f(z)\left(\int_{C_{j_\lambda}} \hat{R}_t(\zeta, z)d\zeta\right) dx\, dy \equiv 0.$$

Here since

$$\text{(3.13)} \qquad \iint_S f(z)\,\overline{Z'_\nu(z)}\, dx\, dy = -\int_{C_\nu} f(z)dz \text{ (cf. [4])},$$

from $f(z)dz \in F$, we have

$$\text{(3.14)} \qquad -W'(z_1, t)\iint_S f(z)\,\overline{P'(z; z_1, t)}\, dx\, dy$$
$$+\sum_{\mu=1}^{b} \beta_{k_\mu}(z_1, t)\left(-\int_{C_{k_\mu}} f(z)dz\right)$$
$$+\frac{1}{2\pi i}\sum_{\lambda=1}^{a} \hat{A}_{j_\lambda}(z_1)\iint_S f(z)\left(\int_{C_{j_\lambda}} \hat{R}_t(\zeta, z)d\zeta\right) dx\, dy \equiv 0.$$

Here we assume that $\{t_\nu\}$ are all simple and we set $z_1 = t_\nu$, in (3.14). Then from Theorem 3.2, we see that

$$\iint_S f(z)\overline{P'(z;z_1,t)}\,dx\,dy = 0 \text{ for all } z_1 \in S,$$

and $f(z)dz$ is exact. We set $\tilde{f}'(z) = f(z)$ and from the Green's formula, we have

$$\int_{\partial S} \tilde{f}(z)\overline{P'(z;z_1,t)\,dz} \equiv 0.$$

From (3.3), we have $\tilde{f}(z_1) \equiv \tilde{f}(t)$, which implies the desired result.

In the cases of which all the t_ν are not simple, by making use of modified forms of Theorem 3.2, we have the desired result, again.

4. Completeness of the Rudin kernel functions. Let H_2 denote the (analytic) Hardy class on S. Let $R_t(z,z_1)$ denote the Rudin kernel for the class H_2 which is characterized by the following reproducing property:

$$f(z_1) = \frac{1}{2\pi}\int_{\partial S} f(z)\overline{R_t(z,z_1)}idW(z,t) \text{ for all } f \in H_2.$$

We shall consider the completenesses of the sets of differentials of $\{R_t(z,z_1)idW(z,t) \mid z_1 \in S_0\}$-type in F. Here we should consider the kernel $R_t^{F_0}(z,z_1)$ for the closed subspace $H_2^{F_0}$ of H_2 such that $f(z)idW(z,t) \in H_2^{DF}$. We note that $R_t^{F_0}(z,z_1)$ is analytic on $\bar{S}$, as we see easily. At first we have the following fact:

THEOREM 4.1. *The set of kernel functions* $\{R_t^{F_0}(z,z_1) \mid z_1 \in S_0\}$ *is complete in* $H_2^{F_0}$. *The set of analytic differentials*

$$\{R_t^{F_0}(z,z_1)idW(z,t) \mid z_1 \in S_0\}$$

is complete in F if and only if S is simply-connected.

Proof. The first part is evident, by the reproducing property. Next we assume that S is not simply-connected. Then there exists at least one critical point t^* of $g(z,t)$. We take $K_F(z,\bar{t}^*)$ and we have, by the reproducing property of $K_F(z,\bar{t}^*)$,

$$\iint_S K_F(z,\bar{t}^*)\overline{R_t^{F_0}(z,z_1)iW'(z,t)}\,dx\,dy \equiv 0 \text{ for all } z_1 \in S.$$

Hence $\{R_t^{F_0}(z, z_1) idW(z,t) \mid z_1 \in S_0\}$ is not complete in F.

If S is simply-connected, then we have the desired result, from the assertion of the next Theorem 4.2.

On the other hand, we consider the Rudin kernel $R_t^F(z, z_1)$ (with poles, in general) for the class H_2^F of meromorphic functions f such that $f(z) idW(z,t) \in H_2^{DF}$. Then we have the following identity, as we see by the simple computations,

$$(4.1) \qquad R_t^F(z,z_1) idW(z,t)\, \overline{idW(z_1,t)} \equiv \hat{R}_t^F(z,z_1) dz\, \overline{dz_1}.$$

Thus from (4.1) and Theorem 3.3, and from Theorem 2.2, we have the following theorem:

THEOREM 4.2. *The set of differentials* $\{R_t^F(z,z_1) idW(z,t) \mid z_1 \in S_0\}$ *is complete in* F. *The set of meromorphic functions* $\{K_F(z, \bar{z}_1) dz / idW(z,t) \mid z_1 \in S_0\}$ *is complete in* H_2^F.

In the last part, we shall give a representation of $R_t^F(z,z_1)$ by the kernel $R_t(z,z_1)$. At first we shall give the following theorem:

THEOREM 4.3.

$$\det \left[\int_{C_\lambda} \left(\int_{C_\mu} R_t(z,z_1) idW(z,t) \right) \overline{idW(z,t)} \right]^{(2n+m)\times(2n+m)} > 0.$$

Proof. As we have pointed out in Theorem 3.1, it is sufficient to show that $\left\{ \int_{C_\lambda} R_t(z,z_1) idW(z,t) \right\}_{\lambda=1}^{2n+m}$ is linearly independent. Suppose that

$$(4.2) \qquad \sum_\lambda X_\lambda \int_{C_\lambda} R_t(z,z_1) idW(z,t) \equiv 0, \quad z_1 \in S.$$

Here we use the following representation of $R_t(z,z_1) idW(z,t)$ [2]:

$$(4.3) \quad R_t(z,z_1) idW(z,t) = \left[-i\tilde{P}'(z; z_1, t) + \sum_{\nu=1}^{2n+m-1} \overline{\alpha_\nu(z_1,t)}\, Z'_\nu(z) \right] dz.$$

Here $\{\alpha_\nu(z_1,t)\}$ are constants which depend on z_1 and t and determined uniquely. From (4.2) and (4.3), we get

$$(4.4) \quad \sum_\lambda X_\lambda \int_{C_\lambda} \left[-i \frac{\partial \tilde{P}'(z; z_1, t)}{\partial \bar{z}_1} + \sum_\nu \left(\overline{\frac{\partial \alpha_\nu(z_1,t)}{\partial z_1}} \right) Z'_\nu(z) \right] dz \equiv 0.$$

We recall the following identities:

$$\tilde{P}'(z;z_1,t)=\frac{\partial N(z;z_1,t)}{\partial z}-\frac{\partial g(z,z_1)}{\partial z}-\frac{\partial g(z,t)}{\partial z}, \tag{4.5}$$

and

$$\frac{\partial^2 g(z,z_1)}{\partial z\,\partial \bar{z}_1}=\frac{\partial^2 N(z;z_1,t)}{\partial z\,\partial \bar{z}_1}-\frac{\pi}{2}\sum_{\mu,\nu=1}^{2n+m-1} c_{\mu\nu}Z'_\mu(z)\overline{Z'_\nu(z_1)}. \tag{4.6}$$

Here the constants $c_{\mu\nu}$ are real, $c_{\mu\nu}=c_{\nu\mu}$ and the matrix $\|c_{\mu\nu}\|$ is nonsingular (cf. **[4]**, p. 97). On the other hand, from (4.3) we have the following equations:

$$\sum_\nu \overline{\alpha_\nu(z_1,t)}\,Z'_\nu(t_j)=i\tilde{P}'(t_j;z_1,t),\quad j=1,2,\cdots,2n+m-1.$$

Here we assume that $\{t_j\}$ are all simple. On the other cases, we can modify the following arguments, as usual. Then since $\det[Z'_\nu(t_j)]\neq 0$, we get

$$\left(\overline{\frac{\partial \alpha_\nu(z_1,t)}{\partial z_1}}\right)=\left|\left(\frac{i\,\partial \tilde{P}'(t_j;z_1,t)}{\partial \bar{z}_1}\right)\right|^{\overset{\nu}{\downarrow j}}\Big/|Z'_\nu(t_j)|. \tag{4.7}$$

Further we note that

$$\int_{C_\lambda} K(z,\bar{z}_1)dz=-\iint_S K(z,\bar{z}_1)\overline{Z'_\lambda(z)}\,dx\,dy=-\overline{Z'_\lambda(z_1)}, \tag{4.8}$$

and we set $P_{\nu\lambda}=\int_{C_\lambda} dZ_\nu$.

Now from (4.4), (4.5), (4.6), (2.9), (4.7) and (4.8), we get

$$\pi i\sum_\lambda X_\lambda\overline{Z'_\lambda(z_1)}+\frac{\pi i}{2}\sum_\lambda X_\lambda\left(\sum_{\mu,\nu} c_{\mu\nu}P_{\nu\lambda}\overline{Z'_\nu(z_1)}\right)$$
$$+\sum_\lambda X_\lambda\left(\sum_\nu \left|\, i\left(\pi K(t_j,\bar{z}_1)-\frac{\pi}{2}\sum_{\mu,\nu} c_{\mu\nu}Z'_\mu(t_j)\overline{Z'_\nu(z_1)}\right)\right|^{\nu}\frac{P_{\nu\lambda}}{|Z'_\nu(t_j)|}\right)\equiv 0. \tag{4.9}$$

Since in (4.9), each of the coefficients of $K(t_j,\bar{z}_1)$ must be zero, we compute the coefficients. Let $M_{\nu,j}$ denote the cofactor of the (ν,j)-component of the matrix $\|Z'_\nu(t_j)\|$. Then the coefficient of $\pi i K(t_j,\bar{z}_1)/|Z'_\nu(t_j)|$ is given by

$$X_1 P_{1,1} M_{1,j} + X_1 P_{2,1} M_{2,j} + \cdots + X_1 P_{2n+m-1,1} M_{2n+m-1,j}$$
$$+ X_2 P_{1,2} M_{1,j} + X_2 P_{2,2} M_{2,j} + \cdots + X_2 P_{2n+m-1,2} M_{2n+m-1,j}$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$+ X_{2n+m-1} P_{1,2n+m-1} M_{,j} + \cdots$$
$$+ X_{2n+m-1} P_{2n+m-1,2n+m-1} M_{2n+m-1,j}.$$

Hence we have

$$\sum_{k=1}^{2n+m-1} M_{k,j}(X_1 P_{k,1} + X_2 P_{k,2} + \cdots + X_{2n+m-1} P_{k,2n+m-1}) = 0,$$
$$j = 1, 2, \cdots, 2n+m-1.$$

Since the matrix $\|M_{j,k}\|$ is the adjoint matrix of the regular matrix $\|Z'_\nu(t_j)\|$, it is nonsingular. Hence we get $\Sigma_\nu X_\nu P_{k,\nu} = 0$ for $k = 1, 2, \cdots, 2n+m-1$. Hence we have all the X_ν are zero, which implies the desired result.

Next we shall consider the class $H_2^{g_t}$ of meromorphic functions f such that $f(z)idW(z,t)$ is analytic on S except for t and $f \in L_2(\partial S)$. We shall construct the kernel $R_2^{g_t}(z, z_1)$ (with poles, in general) for the class $H_2^{g_t}$. In the following, without loss of generality, we assume that $\{t_\nu\}$ are all simple. Because in the other cases, we can modify the following arguments, slightly.

Let $L_t(z, z_1)$ and $\hat{L}_t(z, z_1)$ denote the adjoint L-kernels of $R_t(z, z_1)$ and $\hat{R}_t(z, z_1)$, respectively. They are analytic on $\bar{S}$ except for a simple pole at z_1 with residue 1, and the following properties:

$$\overline{R_t(z, z_1)} idW(z,t) = \frac{1}{i} L_t(z, z_1) dz \quad \text{along} \quad \partial S, \quad \text{and} \tag{4.10}$$

$$\overline{\hat{R}_t(z, z_1)} = \frac{1}{i} \hat{L}_t(z, z_1) idW(z,t) \quad \text{along} \quad \partial S. \tag{4.11}$$

respectively.

Further we have $\hat{L}_t(z, z_1) = -L_t(z_1, z)$ and

$$L_t(z,t) = -\hat{L}_t(t,z) = -W'(z,t)[2]. \tag{4.12}$$

As we see by the simple computations, we have the following representation of $R_t^{g_t}(z, z_1)$:

(4.13) $$R_t^{g_t}(z, z_1) \equiv R_t(z, z_1) + \sum_{\nu=1}^{2n+m-1} \overline{Y_\nu(z_1)}\, \hat{L}_t(z, t_\nu).$$

Here $\{Y_\nu(z_1)\}$ are determined as the unique solution of the following equations:

(4.14) $$\sum_{\nu=1}^{2n+m-1} Y_\nu(z_1)\, \hat{R}_t(t_j, t_\nu) = \hat{L}_t(z_1, t_j), \quad j = 1, 2, \cdots, 2n+m-1.$$

Here we shall give the following theorem:

THEOREM 4.4.

$$\det\left[\int_{C_\lambda}\left(\int_{C_\mu} R_t^{g_t}(z, z_1) i dW(z, t)\right) \overline{i dW(z_1, t)}\right]^{(2n+m)\times(2n+m)} > 0.$$

Proof. Suppose that $\Sigma_\lambda X_\lambda \int_{C_\lambda} R_t^{g_t}(z, z_1) i dW(z, t) \equiv 0$ and hence

$$\sum_\lambda X_\lambda \int_{C_\lambda} R_t(z, z_1) i dW(z, t) + \sum_\lambda X_\lambda \left(\sum_\nu \overline{Y_\nu(z_1)} \int_{C_\lambda} \hat{L}_t(z, t_\nu) i dW(z, t)\right) \equiv 0.$$

Since each $Y_\nu(z_1)$ is represented as a linear combination of $\{\hat{L}_t(z_1, t_j)\}_j$, we get

$$\sum_\lambda X_\lambda \int_{C_\lambda} R_t(z, z_1) i dW(z, t) \equiv \sum_\lambda X_\lambda \left(\sum_\nu \overline{Y_\nu(z_1)} \int_{C_\lambda} \hat{L}_t(z, t_\nu) i dW(z, t)\right) \equiv 0.$$

Hence from Theorem 4.3, we have all the X_λ are zero, which implies the desired result.

Now we construct the kernel $R_t^F(z, z_1)$. We set $C_{j_0} = \partial S$. Then from Theorem 4.4, we have

$$\det\left[\int_{C_{j_\lambda}}\left(\int_{C_{j_{\lambda'}}} R_t^{g_t}(z, z_1) i dW(z, t)\right) \overline{i dW(z_1, t)}\right]^{(a+1)\times(a+1)} > 0,$$

$$\lambda, \lambda' = 0, 1, 2, \cdots, a.$$

Hence we can take the unique constants $\{A_{j_\lambda}(z_1)\}_{\lambda=0}^a$ such that

(4.15) $$R_t^{g_t}(z, z_1) - \sum_{\lambda=0}^{a} \overline{A_{j_\lambda}(z_1)} \int_{C_{j_\lambda}} \overline{R_t^{g_t}(\zeta, z)} i dW(\zeta, t) \in H_2^F,$$

which is the kernel $R_t^F(z, z_1)$, as we see from the simple computations.

From (4.13) and (4.15), we have

$$R_t^F(z,z_1) = R_t(z,z_1) + \sum_{\nu=1}^{2n+m-1} \overline{Y_\nu(z_1)}\,\hat{L}_t(z,t_\nu) - \sum_{\lambda=0}^{a} \overline{A_{j\lambda}(z_1)} \int_{C_{j_\lambda}} \left[\overline{R_t(\zeta,z)} + \sum_{\nu=1}^{2n+m-1} Y_\nu(z)\overline{\hat{L}_t(\zeta,t_\nu)}\right] i\,dW(\zeta,t). \tag{4.16}$$

Since $R_t^F(t,z_1) = 0$, $\hat{L}_t(t,t_\nu) = 0$ and $Y_\nu(t) = 0$, as we see from (4.12) and (4.14), we have, by setting $z = t$ in (4.16),

$$1 - \sum_{\lambda=0}^{a} \overline{A_{j\lambda}(z_1)} \int_{C_{j_\lambda}} i\,dW(\zeta,t) = 0.$$

Hence we get (Note that the integral on C_{j_0} is zero.)

$$R_t^F(z,z_1) = (R_t(z,z_1) - 1) + \sum_{\nu=1}^{2n+m-1} \overline{Y_\nu(z_1)}\,\hat{L}_t(z,t_\nu) - \sum_{\lambda=1}^{a} \overline{A_{j\lambda}(z_1)} \int_{C_{j_\lambda}} \left[\overline{(R_t(\zeta,z) - 1)} + \sum_{\nu=1}^{2n+m-1} Y_\nu(z)\,\overline{\hat{L}_t(\zeta,t_\nu)}\right] i\,dW(\zeta,t). \tag{4.17}$$

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