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AN OBSTRUCTION TO EXTENDING ISOTOPIES OF PIECEWISE LINEAR MANIFOLDS

EWING L. LUSK

Let $F: M \times I^n \to Q \times I^n$ be an n-isotopy (not necessarily PL) of a compact PL m-manifold M in a PL q-manifold Q, and let $G: Q \times I^n \to Q \times I^n$ be an ambient isotopy of Q which covers F on $Q \times \partial I^n$. If $m \leqq q - 3$ there is in $\pi_n \mathrm{PL}(M, Q)$ an obstruction to finding an ambient isotopy of Q, isotopic to G, which covers F and agrees with G on $Q \times \partial I^n$.

1. Introduction. In the proof of the Hudson-Zeeman covering isotopy theorem [6], one has no control over the homeomorphism of the ambient manifold which one obtains at the end of the isotopy. In general, one might ask for sufficient conditions under which a given n-isotopy $F: M \times I^n \to Q \times I^n$ of one PL manifold in another, fixed on ∂M, can be covered by an ambient n-isotopy $H: Q \times I^n \to Q \times I^n$ fixed on ∂Q, in such a way that $H | Q \times \partial I^n$ is equal to some given level-preserving homeomorphism G of $Q \times \partial I^n$ which covers $F | M \times \partial I^n$. Necessary conditions are that F be level-preservingly locally flat and that G have some extension to $Q \times I^n$ which is fixed on ∂Q. That these conditions are not sufficient can be seen by considering an isotopy $F: S^1 \times I \to I^2 \times I$ of a circle in the interior of I^2 which rotates the circle through 360°. Since F can be chosen PL and locally flat, it follows from the ordinary covering isotopy theorem [6] that F can be covered by an ambient isotopy H of I^2 which is fixed on ∂I^2. But if $G: \partial(I^2 \times I) \to \partial(I^2 \times I)$ is the identity homeomorphism, then H cannot be an extension of G. The difficulty here arises from the fact that the space of embeddings of S^1 into I^2 is not simply connected. The theorem below extends results of Gluck, Husch, and Rushing [3,8]. Let M and Q be PL m- and q-manifolds respectively, with M compact, and let $\mathrm{PL}(M, Q; f)$ denote the semi-simplicial complex of proper PL embeddings of M into Q, with base point f.

THEOREM 1. *Let $F: M \times I^n \to Q \times I^n$ be a proper level-preservingly locally flat n-isotopy* (not necessarily PL) *fixed on ∂M. Let $G: Q \times I^n \to Q \times I^n$ be an ambient n-isotopy of Q, fixed on ∂Q, such that $G \circ (F_0 \times 1) | M \times \partial I^n = F | M \times \partial I^n$. Suppose that $m \leqq q - 3$. Then there is a homeomorphism h of Q such that hF_0 is PL and an obstruction α in π_n $\mathrm{PL}(M, Q; hF_0)$ such that $\alpha = 0$ if and only if there is a level-preserving isotopy K of $Q \times I^n$, fixed on $\partial(Q \times I^n)$, such that $K_1 G \circ (F_0 \times 1) = F$;* i.e. *$K_1 G$ extends $G | Q \times \partial I^n$ and covers F.*

REMARK 1. If F and G are PL, then the local flatness condition on F need not be level-preserving, and K can be taken to be PL. The proof of Theorem 1 in this PL case is like the proof given in [8] for the case $n = 1$ and so is known. In the topological case, Theorem 1 follows straightforwardly from the fact that the inclusion $\mathrm{PL}(M, Q) \subset \mathrm{TOP}(M, Q)$ is dense and a weak homotopy equivalence (See Theorem 2 below).

REMARK 2. Various combinations of dimension and connectivity conditions are sufficient to ensure that $\pi_n \, \mathrm{PL}(M, Q; hF_0) = 0$ and hence that the obstruction vanishes. We list some of them here. (See [7] and [9].)

(a) $\pi_r(Q) = 0$ for $n \leqq r \leqq m + n$ and $2m + n \leqq q - 2$.

(b) M is $(2m - q + n)$-connected, Q is $(2m - q + n + 1)$-connected, $\pi_r(Q) = 0$ for $n \leqq r \leqq m + n$, and $m + n \leqq q - 2$.

(c) $\pi_r(Q) = 0$ for $n \leqq r \leqq m + n$, F_0 is $(2m - q + n + 1)$-connected, and $m + n \leqq q - 2$.

2. Definitions. Let I^n be the n-fold product of the unit interval $[0,1]$. The point $(0,0,\cdots 0)$ in I^n will be denoted by 0, and the subset $I^{n-1} \times 0 \cup \partial I^{n-1} \times I$ of $I^n = I^{n-1} \times I$ will be denoted by J^{n-1}. An *n-isotopy* of M in Q is an embedding $F\colon M \times I^n \to Q \times I^n$ which is level-preserving ($p \circ F = p$ where p is projection onto I^n). It is *proper* if $F^{-1}(\partial Q \times I^n) = \partial M \times I^n$. An embedding $F_t\colon M \to Q$ is defined for each $t \in I^n$ by $F(x,t) = (F_t(x), t)$. A 1-isotopy is called an *isotopy*, and F_0 and F_1 are said to be *isotopic*. An n-isotopy F is *fixed on* X if $F | X \times I^n = F_0 \times 1 | X \times I^n$, where 1 denotes the identity map. It is *level-preservingly locally flat* if for each $(x,t) \in M \times I^n$ there is a neighborhood N of t in I^n, a level-preserving embedding H of either $E^m \times N$ or $E^m_+ \times N$ into $M \times N$ (depending on whether x is in int M or ∂M) with $H(0,t) = (x,t)$, and a level preserving embedding G of either $E^q \times N$ or $E^q_+ \times N$ into $Q \times N$ depending on whether $F_t(x)$ is in int Q or ∂Q) with $G(0,t) = F(x,t)$, such that $G^{-1} FH$ is of the form $i \times 1$, where i is the natural inclusion of E^m into E^q or E^m_+ into E^q_+, as the case may be. An *ambient n-isotopy* of Q is a level-preserving homeomorphism H of $Q \times I^n$ such that $H_0 = 1$. If $A \subset X$, an *ε-push* of (X, A) is an ambient isotopy of X which is fixed outside an ε-neighborhood of A.

We make use of the semi-simplicial complexes $\mathrm{Aut}_{\mathrm{PL}}(Q)$ and $\mathrm{PL}(M, Q)$, whose k-simplices are ambient k-isotopies of Q fixed on ∂Q and proper k-isotopies of M in Q fixed on ∂M, respectively. The Hudson covering n-isotopy theorem [5] can be used to prove, as in [4], that if $f\colon M \to Q$ is a given PL embedding then the simplicial map $p\colon \mathrm{Aut}_{\mathrm{PL}}(Q) \to \mathrm{PL}(M, Q)$ given by $p(H) = H \circ (f \times 1)$ is a fibration, i.e., given level-preserving embeddings $K: Q \times J^{n-1} \to Q \times J^{n-1}$ and $L: M \times I^n \to Q \times I^n$ such that $p(K) = L \mid M \times J^{n-1}$, there is an n-isotropy $H: Q \times I^n \to Q \times I^n$ such that $p(H) = L$ and $H \mid Q \times J^{n-1} = K$. An element of π_n PL$(M,Q;f)$ is represented by a level-preserving PL embedding $L: M \times \partial I^{n+1}$ such that $L_0 = f$.

3. Spaces of embeddings. In this section we consider the relationship between PL(M, Q) and TOP(M, Q), the semi-simplicial complex of topological embeddings of M into Q. Recent work of Edwards and Miller **[2, 12]** has relaxed the dimension restrictions on the results in **[10]**. The key lemma is the following.

LEMMA 1. *Let* $H: M \times I^n \to Q \times I^n$ *be a level-preserving embedding. Suppose that* $m \leqq q - 3$ *and* $q \geqq 5$. *Then for any* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon, H) > 0$ *such that if* G_0, G_1: $M \times I^n \to Q \times I^n$ *are level-preserving* PL *embeddings with* $d(G_i, H) < \delta$, *then there is a level-preserving* ε*-push* K *of* $(Q \times I^n,\ H(M \times I^n))$ *such that* $K_1 G_0 = G_1$. *If* G_0 *and* G_1 *agree on* $M \times \partial I^n$, *then* K *can be assumed fixed on* $Q \times \partial I^n$.

Proof. If H is of the form $h \times 1$ for some embedding $h: M \to Q$, then the lemma follows directly from Corollary 2 of **[2]** and Corollary 3 of **[1]**. Generalization to the case in which H is not of this form can be carried out as in the second half of the proof of Theorem 4.2 (m, s) in **[10]**.

REMARK 3. The above "local solvability" result is the basis for Theorems 2.1–2.5 of **[10]** which are stated there with more stringent dimension restrictions. We may now regard those results to be true for $m \leqq q - 3$, $q \geqq 5$. In particular, Theorems 2.1 and 2.4 give us

THEOREM 2. *If* $m \leqq q - 3$ *and* $q \geqq 5$, *then the inclusion* PL$(M, Q) \subset$ TOP(M, Q) *is dense and a weak homotopy equivalence*; i.e., *if* $f: M \to Q$ *is* PL, *then the homomorphism* i_*: π_nPL$(M, Q; f) \to \pi_n$ TOP$(M, Q; f)$ *induced by inclusion is an isomorphism for all* n.

4. Proof of Theorem 1. The following lemma, which is Theorem 2.3 of **[10]** with the new dimension conditions, makes possible the treatment of the non-PL case with PL techniques.

LEMMA 2. *Let* F: $M \times I^n \to Q \times I^n$ *be a level-preservingly locally flat proper n-isotopy which is* PL *on* $\partial(M \times I^n)$. *Suppose* $m \leqq q - 3$ *and* $q \geqq 5$, *and that* $\varepsilon > 0$ *is given. Then there is a level-preserving*

ε-push T of $(Q \times I^n, F(M \times I^n))$, fixed on $\partial(Q \times I^n)$, such that T_1F is PL.

Proof of Theorem 1. By Lemma 2 with $n = 0$ (See **[11]**), there is a small homeomorphism h of Q such that hF_0: $M \to Q$ is PL. Consider the embedding $(h \times 1)G^{-1}F$: $M \times I^n \to Q \times I^n$. Since it is a level-preservingly locally flat n-isotopy and $(h \times 1)$ $G^{-1}F | \partial(M \times I^n) = (hF_0) \times 1$, which is PL, there is by Lemma 2 a level-preserving isotopy T of $Q \times I^n$, fixed on $\partial(Q \times I^n)$, such that $T_1(h \times 1)G^{-1}F$ is PL. Now define $L: M \times \partial I^{n+1} \to Q \times \partial I^{n+1}$ by considering I^{n+1} as $I^n \times I$ and letting L be $T_1(h \times 1)G^{-1}F$ on $M \times I^n \times 1$ and $(hF_0) \times 1$ on $M \times J^n$. Then L is PL and so represents an element α of $\pi_n \mathrm{PL}(M, Q; hF_0)$. To say $\alpha = 0$ in $\pi_n \mathrm{PL}(M, Q; hF_0)$ is to say that there is a PL $(n+1)$-isotopy H': $M \times I^{n+1} \to Q \times I^{n+1}$ such that $H' | M \times \partial I^{n+1} = L$. Therefore we can use the lifting property of the fibration $p: \mathrm{Aut}_{\mathrm{PL}}(Q) \to \mathrm{PL}(M, Q)$ given by $p(K) = K \circ (hF_0 \times 1)$ to find an ambient $(n+1)$-isotopy H'': $Q \times I^{n+1} \to Q \times I^{n+1}$ such that $H'' | Q \times J^n = 1$ *and* $H'' \circ (hF_0 \times 1) = H'$. Now we define

$$K = (G \times 1)(h^{-1} \times 1 \times 1)T^{-1}H''(h \times 1 \times 1)(G^{-1} \times 1):$$
$$(Q \times I^n) \times I \to (Q \times I^n) \times 1.$$

Then K_1G covers F and extends $G | Q \times I^n$, as desired.

Conversely, if K exists with the desired properties, then K': $M \times \partial I^{n+1} \times I \to Q \times \partial I^{n+1} \times I$ defined by $K'_t = T_{1-t}(h \times 1)G'K_{1-t}G(F_0 \times 1)$ on $M \times J^n \times I$ and $hF_0 \times 1$ on $M \times (I^{n-1} \times 1) \times I$ is a level-preserving isotopy taking L to $hF_0 \times 1$. Therefore α is trivial as an element of $\pi_n \mathrm{TOP}(M, Q; hF_0)$, the semi-simplicial complex of embeddings of M into Q. By Theorem 2, α is trivial in $\pi_n \mathrm{PL}(M, Q: hF_0)$.

5. The obstruction α. In the construction above, α appeared to depend on h, T, and G. In this section we show that α can be chosen in such a way that it depends only on F.

In applying Lemma 2 to construct h, above, we may choose h so that hF_0 is within $\delta(F_0, 1)$ of F_0, where δ comes from Lemma 1. Any two such homeomorphisms h and h' will then be such that hF_0 and $h'F_0$ are ambient isotopic. Similarly we choose T to be a $\delta((h \times 1)G^{-1}F, 1)$-push, so that if T' is another push which takes $(h \times 1)G^{-1}F$ to a PL embedding, $T_1(h \times 1)G^{-1}F$ and $T'_1(h \times 1)G^{-1}F$ are PL ambient isotopic, and the α's constructed with them will be homotopic in $\pi_n \mathrm{PL}(M, Q; hF_0)$.

Now suppose that G and G' are level-preserving homeomorphisms of $Q \times I^n$ satisfying the hypotheses of the theorem. Since $G^{-1}F$ and

$G'^{-1}F$ are each isotopic to $F_0 \times 1$, they are isotopic. If we denote by α and α' the obstructions constructed as above from G and G', the isotopy of $G^{-1}F$ to $G'^{-1}F$ will induce a homotopy from α to α' in $\pi_n \mathrm{TOP}(M, Q)$. By Theorem 2, α is homotopic to α' in $\pi_n \mathrm{PL}(M, Q; hF_0)$, and so α does not depend on G.

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NORTHERN ILLINOIS UNIVERSITY