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ON BURNSIDE'S OTHER $p^a q^b$ THEOREM

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Suppose G is a finite group whose order is divisible by only two primes. Burnside's famous theorem asserts that G must be solvable. In a less famous theorem, Burnside gave sufficient conditions for G to have a nontrivial normal p-subgroup for a particular prime p. However, this theorem does not apply in certain cases when G has even order. In this paper, we prove an analogue of this theorem which applies to all cases.

1. Introduction.

The "less famous" theorem **[2]**, as opposed to the "famous" theorem (**[5]**, page 131), states the following.

THEOREM (Burnside). *Suppose $|G| = p^a q^b$ for distinct primes p, q and nonnegative integers a, b. Assume that $p^a > q^b$. Then $O_p(G) \neq 1$, except possibly in the following cases:*
(1) *$p = 2$ and q is a Fermat prime;*
(2) *$q = 2$ and p is a Mersenne prime.*

Burnside gave examples to show that the cases (1) and (2) must be excluded. (See also §5 of the present paper.) In this paper we prove an analogue of Burnside's result that covers all cases. To do this, we require a definition. For each finite group G, let $e(G)$ be the maximum of the orders of the nilpotent subgroups of G having nilpotence class at most two. Then we obtain the following result.

THEOREM A. *Suppose p and q are distinct primes, G is a finite group, and $|G| = p^a q^b$ for some nonnegative integers a, b. Let S be a Sylow p-subgroup of G and T be a Sylow q-subgroup of G. Assume that $e(S) > e(T)$. Then $O_p(G) \neq 1$.*

A typical application of Burnside's Theorem is the statement that if p and q are odd and $|G| = |G^*| = p^a q^b \neq 1$, then we cannot have $O_p(G) = 1$ and $O_q(G^*) = 1$. Theorem A yields a similar corollary.

COROLLARY 1. *Suppose p and q are distinct primes, a and b are nonnegative integers, and G and G^* are finite groups of order $p^a q^b$. Assume that G and G^* have isomorphic Sylow p-subgroups and isomorphic Sylow q-subgroups. Suppose $G \neq 1$ and $O_q(G) = 1$. Then $O_p(G^*) \neq 1$.*

Theorem A follows from the following results.

THEOREM B. *Suppose G is a finite group, A is a nilpotent subgroup of G of nilpotence class at most two, and $|A| = e(G)$. Assume that A normalizes a nilpotent subgroup B of G. Then AB is nilpotent.*

COROLLARY 2. *Suppose π is a set of primes, G is a finite π-solvable group, and $O_{\pi'}(G) = 1$. Then π contains every prime divisor of $e(G)$.*

COROLLARY 3. *Suppose p is a prime, G is a finite p-solvable group, and $O_{p'}(G) = 1$. Then $e(G)$ is a power of p.*

COROLLARY 4. *Suppose G is a finite solvable group. Then the prime divisors of $e(G)$ are the same as the prime divisors of $|F(G)|$.*

Theorem B is an analogue of the following result (Proposition 1) of **[1]**: *Suppose G is a finite group, A is an Abelian subgroup of G, B is a nilpotent subgroup of G having an Abelian Sylow 2-subgroup, and A normalizes B. Assume that $|A|$ is the maximum of the orders of the Abelian subgroups of G. Assume also that either $|A|$ is odd or B is Abelian. Then AB is nilpotent.*

Burnside's Theorem was applied in §21 of **[3]** in order to construct a Hall π-subgroup in a group possessing a Hall $\{p, q\}$-subgroup for every pair of primes $p, q \in \pi$. A similar application appears in §7 of **[4]**. In §4 we state an analogue (Corollary 5) of an argument used in these applications.

All groups in this paper are assumed to be finite. Most of our notation is standard and is taken from **[5]**. In addition, for a group G we define $e(G)$ as above and define $\mathcal{B}(G)$ to be the set of all nilpotent subgroups of G of order $e(G)$ that have nilpotence class at most two. We also write "$H \ntriangleleft G$" to indicate that H is a subgroup, but not a normal subgroup, of G.

2. Nilpotent automorphism groups.

PROPOSITION 1. *Let p be a prime and V be a nonidentity elementary Abelian p-group. Suppose A is a nilpotent p'-group of automorphisms of V having nilpotence class at most two. Then $|A| < |V|$.*

Proof. Use induction on $|V|$.

We regard V as a vector space over $GF(p)$ and A as a group of linear transformations of V over $GF(p)$. Suppose first that V is reducible under A, say, $V = V_1 \oplus V_2$. Let $A_i = A/C_A(V_i)$. Then, by induction, $|A_i| < |V_i|$ for each i. So,

$$|A| \leqq |A_1|\,|A_2| < |V_1|\,|V_2| = |V|.$$

Now assume that V is irreducible under A. Let C be the centralizer of A in the endomorphism ring of V and let F be the subring of C generated by the elements of $Z(A)$. By Schur's Lemma (Theorem 3.5.2, page 76, of **[5]**), C is a division ring. As F is commutative and contains $1, F$ is a finite integral domain. So F is a field. Let us regard V as a vector space over F and A as a group of linear transformations of V over F. Define

$$q = |F| \quad \text{and} \quad d = \dim_F V.$$

Since $Z(A)$ is a subgroup of the multiplicative group $F - \{0\}$,

$$(2.1) \qquad Z(A) \text{ is cyclic and } |Z(A)| \leqq q - 1.$$

Assume first that, for each prime r, every Abelian subgroup of $O_r(A)$ is cyclic. By Theorem 5.4.10, page 199, of **[5]**, $O_r(A)$ is a cyclic group or a generalized quaternion group for each prime r. As A has nilpotence class at most two, $O_2(A)$ is a cyclic group or a quaternion group of order eight. Hence $A = Z(A)$ or $|A/Z(A)| = 4$. If $A = Z(A)$, then (2.1) yields that $|A| < q \leqq |V|$. If $|A/Z(A)| = 4$, then V is not one-dimensional and, by (2.1),

$$q \geqq |Z(A)| + 1 \geqq 3 \quad \text{and} \quad |A| = 4|Z(A)| \leqq (q+1)(q-1) < q^2 \leqq q^d = |V|.$$

Now assume that $O_r(A)$ has a noncyclic Abelian subgroup for some prime r. Since $O_r(Z(A)) = Z(O_r(A))$,

$$(2.2) \qquad r \text{ divides } |Z(A)|.$$

By (2.1), $O_r(A)$ contains an element g of order r that lies outside $Z(A)$. Let $B_1 = \langle g, Z(A)\rangle$. Then

$$(2.3) \qquad B_1 \text{ is Abelian but not cyclic.}$$

Since $B_1 \supseteq A'$, $B_1 \lhd A$. Let U_1 be an irreducible B_1-submodule of V over F and let V_1 be the sum of all the B_1-submodules of V over F that are isomorphic to U_1. By (2.3),

$$C_{B_1}(V_1) = C_{B_1}(U_1) \neq 1.$$

As A acts faithfully on V by hypothesis, $V \neq V_1$. Since $B_1 \lhd A$ and A acts irreducibly on V over F, Clifford's Theorem ([5], page 70) asserts that there exist some natural number n and some B_1-submodules $V_2, \cdots, V_n$ over F such that

$$V = V_1 \oplus \cdots \oplus V_n,$$

A permutes $V_1, \cdots, V_n$ transitively, and $C_A(B_1)$ fixes $V_1, \cdots, V_n$.

Let $B = C_A(B_1)$ and $K = C_B(V_1)$. Since $\Omega_1(O_r(B_1)) \simeq Z_r \times Z_r$, $B_1 = \Omega_1(O_r(B_1))Z(A)$, and A is nilpotent, it follows that $A/B \cong Z_r$. Hence $n = r$ and A/B acts regularly (and faithfully) on $V_1, \cdots, V_r$. Take $x \in A - B$. Then $\langle x \rangle$ permutes the subspaces V_i transitively. So $C_K(x)$ acts trivially on all of them, and

$$C_K(x) = 1. \tag{2.4}$$

Since $K \lhd B$, (2.4) yields that

$$[B, K] \subseteq K \cap A' \subseteq K \cap Z(A) = 1 \textit{ and } K \subseteq Z(B). \tag{2.5}$$

Take any $y \in K$. Let $z = [y, x]$. Then

$$y^x = yz, \quad y^{x^2} = yz^2, \cdots, y^{x^r} = yz^r.$$

By (2.5), $y = y^{x^r} = yz^r$. Thus

$$\textit{for each } y \in K, \ [y, x] \in \Omega_1(O_r(Z(A))). \tag{2.6}$$

Now define a mapping $\phi: K \to \Omega_1(O_r(Z(A)))$ by $\phi(y) = [y, x]$. An easy calculation shows that ϕ is a homomorphism. By (2.4), ϕ is one-to-one. Hence

$$|K| \leqq |\Omega_1(O_r(Z(A)))| \leqq r.$$

Let $c = \dim_F V_1$. Then $d = cr$. As B/K acts faithfully on V_1, induction yields that

$$|B/K| \leqq |V_1| - 1 = q^c - 1.$$

Therefore,

$$|A| = |A/B|\,|B/K|\,|K| \leqq r^2(q^c - 1). \tag{2.7}$$

By (2.2) and (2.1),

$$(2.8) \qquad r \leqq q - 1 \leqq q^c - 1.$$

If $r = 2$, then (2.7) and (2.8) yield that

$$|A| \leqq 4(q^c - 1) \leqq (q^c + 1)(q^c - 1) < q^{2c} = q^d = |V|.$$

If $r > 2$, then (2.7) and (2.8) yield that

$$|A| \leqq (q^c - 1)^3 < q^{3c} \leqq q^d = |V|.$$

This completes the proof of Proposition 1.

3. Proof of Theorem B. To prove Theorem B, we assume it is false and derive a contradiction. Assume that G, A and B are chosen to violate Theorem B in such a way that $|G| + |B|$ is as small as possible. Most of our proof follows the proof of Proposition 1 of **[1]**.

Clearly, $G = AB$. Take a prime p such that $O_p(A) \not\subseteq F(G)$. Then $O_p(A)O_p(B) \ntrianglelefteq G$. As A normalizes $O_p(A)O_p(B)$, B does not. Take a prime q such that $O_q(B)$ does not normalize $O_p(A)O_p(B)$. Then $[O_p(A), O_q(B)] \neq 1$. Therefore, $AO_q(B)$ is not nilpotent. By the minimal choice of G and $B, B = O_q(B)$. Let $A_q = O_q(A)$, $A^* = O_{q'}(A)$, $V = B/\Phi(B)$.

Now, $V = C_V(A^*) \times [V, A^*]$ (by **[5]**, page 177) and A^* does not centralize B. By Theorem 5.1.4, page 174, of **[5]**, $[V, A^*] \neq 1$. Consequently, the minimal choice of G and B yields that A^* centralizes $\Phi(B)$, that $C_V(A^*) = 1$, and that A acts irreducibly on V. Hence

$$(3.1) \qquad C_B(A^*) = \Phi(B)$$

and, by Theorem 3.1.3, page 62, of **[5]**,

$$(3.2) \qquad A_q \textit{ centralizes } V.$$

Since $V = [V, A^*]$, we have $B = [B, A^*]\Phi(B)$. By a basic property of the Frattini subgroup (**[5]**, page 173), and by (3.1),

$$(3.3) \quad B = [B, A^*] \quad \textit{and} \quad C_G(B') \supseteq \langle A^{*g} \mid g \in G\rangle \supseteq [B, A^*] = B.$$

By (3.1) and (3.2),

$$[[A_q, B], A^*] \subseteq [\Phi(B), A^*] = 1 = [1, B] = [[A^*, A_q], B].$$

By (3.3) and the Three Subgroups Lemma (**[5]**, page 19),

$$(3.4) \qquad 1 = [[B, A^*], A_q] = [B, A_q].$$

Let $\bar{A} = A/C_A(B)$ and $C = BC_A(B)$. Then $\bar{A}$ is a q'-group. By Theorem 5.1.4, page 174, of **[5]**, $\bar{A}$ acts faithfully on V. By Proposition 1,

$$(3.5) \qquad |\bar{A}| < |V|.$$

By (3.3), $B' \subseteq Z(B)$. Since $C' = B'(C_A(B))' \subseteq Z(C)$ and $A \in \mathscr{B}(G)$, $|A| \geqq |C|$. By (3.1) and (3.5),

$$|A| \geqq |C| = |BC_A(B)| = |B/(B \cap C_A(B))|\,|C_A(B)|$$
$$\geqq |B/\Phi(B)|\,|C_A(B)| = |V|\,|C_A(B)| > |\bar{A}|\,|C_A(B)| = |A|,$$

a contradiction. This completes the proof of Theorem B.

4. Proof of remaining results. We now apply Theorem B to obtain the other results mentioned in the introduction.

For Corollary 2, let $H = O_\pi(G)$ and take $A \in \mathscr{B}(G)$. Then A normalizes H and $C_G(H) \subseteq H$, by Lemma 1.2.3 of Hall and Higman (Theorem 6.3.2, page 228, of **[5]**). Therefore, AH is a group. Since H is solvable, AH is a solvable. Since

$$[O_{\pi'}(AH), H] \subseteq O_{\pi'}(AH) \cap H = 1,$$

$O_{\pi'}(AH) = 1$. So $F(AH)$ is a π-group. By Theorem B, $AF(AH)$ is nilpotent. Hence $O_{\pi'}(A)$ centralizes $F(AH)$. By Theorem 6.1.3, page 218, of **[5]**,

$$C_{AH}(F(AH)) \subseteq F(AH).$$

So, $O_{\pi'}(A) = 1$. This proves Corollary 2.

Corollary 3 is a special case of Corollary 2.

To obtain Corollary 4, let π be the set of all prime divisors of $|F(G)|$ and let σ be the set of all prime divisors of $e(G)$. Then $F(O_{\pi'}(G)) \subseteq O_{\pi'}(F(G)) = 1$. Hence $O_{\pi'}(G) = 1$. By Corollary 2, σ is a subset of π. Take $A \in \mathscr{B}(G)$ and let $Z = Z(O_{\sigma'}(F(G)))$. By Theorem B, AZ is nilpotent. Therefore, $AZ = A \times Z$. By the choice of A, $A \supseteq Z$. Thus $Z = 1$. Consequently, $O_{\sigma'}(F(G)) = 1$ and $\sigma = \pi$.

Corollary 4 yields that, if $O_p(G) = 1$ in Theorem A, then $e(G)$ is a power of q. But then, by Sylow's Theorem and the definition of $e(G)$,

$$e(G) \leqq e(T) < e(S) \leqq e(G),$$

a contradiction. This proves Theorem A.

Theorem A easily yields Corollary 1. However, the following result generalizes Corollary 1. Note that it is not trivial if $k = 1$.

COROLLARY 5. *Suppose p and q are primes, a and b are nonnegative integers, and $G_1, \cdots, G_k$ are nonidentity finite groups of order $p^a q^b$. Assume that $G_1, \cdots, G_k$ have isomorphic Sylow p-subgroups and have isomorphic Sylow q-subgroups. Assume also that the notation is chosen such that*

$$e(O_p(G_1)) = \max\{e(O_r(G_i)) \mid r = p, q\,;\ i = 1, 2, \cdots, k\}.$$

Let S be a Sylow p-subgroup of G_1. Suppose that $1 \leqq i \leqq k$ and that ϕ is an isomorphism of S onto a Sylow p-subgroup of G_i. Then

$$O_p(G_i) \cap \phi(O_p(G_1)) \neq 1.$$

Proof. Set

$$T = O_p(G_1), \quad S^* = \phi(S), \quad T^* = \phi(T), \quad \text{and} \quad Q = O_q(G_i).$$

Then S^* normalizes T^* and T^* normalizes Q. Since $G_1 \neq 1$, $e(T) > 1$. By hypothesis, $e(T) \geqq e(Q)$. Hence

$$e(T^*) = e(T) > e(Q).$$

By Theorem A, $O_p(T^*Q) \neq 1$. Since S^* normalizes T^*Q,

$$1 \neq O_p(T^*Q) \lhd S^*.$$

Let $U = O_p(T^*Q) \cap Z(S^*)$. Then

$$U \neq 1 \quad \textit{and} \quad U \subseteq T^* = \phi(O_p(G_1)). \tag{4.1}$$

Since $[U, Q] \subseteq O_p(T^*Q) \cap Q = 1$,

$$U \textit{ centralizes } Q. \tag{4.2}$$

By Theorem 6.3.3, page 228, of **[5]**, $U \subseteq Z(S^*) \subseteq O_{q,p}(G_i)$. By (4.2),

$$C_G(U) \supseteq Q(O_{q,p}(G_i) \cap S^*) = O_{q,p}(G_i).$$

Therefore, $U \subseteq O_p(O_{q,p}(G_i)) \subseteq O_p(G_i)$. By (4.1), this completes the proof of Corollary 5.

5. Examples. The following examples, suggested by Burnside's examples, show that the Burnside Theorem cannot be extended to cover the excluded cases (1) and (2).

EXAMPLE 1. Let q be a Fermat prime and V be an elementary Abelian group of order q^2. Then Aut V contains a Sylow 2-subgroup A of order $2(q-1)^2$, and

$$|A| = 2(q-1)^2 > q^2 = |V|.$$

This shows that Proposition 1 cannot be extended to allow A to have arbitrary nilpotence class. By letting G be the semi-direct product of V by A, we see that Burnside's Theorem cannot be extended to cover case (1).

EXAMPLE 2. Let p be a Mersenne prime. Let $2^n = p + 1$ and let V be an elementary Abelian group of order 2^{np}. Then a few calculations show that

$$|\mathrm{Aut}\ V|_p \geqq |GL(p, 2^n)|_p = p^{p+1} > |V|.$$

Consequently, we obtain an example analogous to Example 1.

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