PACIFIC JOURNAL OF MATHEMATICS
Vol. 56, No. 2, 1975

WEAK-UNICOHERENCE

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1. Introduction. A connected topological space X is said to be *weakly-unicoherent* if whenever $X = A \cup B$, where A is compact and connected and B is closed and connected, then $A \cap B$ is connected. A brief review of the origin of unicoherence and weak-unicoherence is given and then followed by several new characterizations of weak unicoherence for locally connected generalized continua.

A connected topological space X is said to be unicoherent if whenever $X = A \cup B$, where A and B are closed connected subsets of X, then $A \cap B$ is connected. All Euclidean spaces, spheres of dimension greater than one, all closed cells, and locally connected continua which do not separate the plane are examples of unicoherent spaces.

Unicoherence for continua was defined by K. Kuratowski in a paper, [5], in which the property played an important part in a topological characterization of a sphere. The property itself had been used earlier by K. Kuratowski in [4]. L. Vietoris in [6] referred to the property as continua "ohne Henkel".

For the purpose of generalizing basic theorems on light-open mappings on two manifolds to light-open mappings on higher dimensional manifolds or at least obtaining analogues of those results, G. T. Whyburn in [8] introduced a new version of unicoherence for generalized continua. A generalized continuum X is said to be unicoherent if whenever $X = A \cup B$, where A is compact and connected and B is closed and connected, then $A \cap B$ is connected. For locally connected generalized continua Whyburn obtained the following characterizations: (A) A locally connected generalized continuum X is unicoherent if and only if every compact set in X which separates a containing region also separates X. (B) A locally connected generalized continuum X is unicoherent if and only if the boundary of every conditionally compact component of a closed and connected set is a continuum.

In [2] R. F. Dickman, Jr. called spaces which satisfies the newer version of unicoherence *weakly-unicoherent* and gave a characterization of locally connected generalized continua which are weakly-unicoherent in terms of the Complementation Property. In [1] M. H. Clapp and R. F. Dickman, Jr. gave two new characterizations of weakly-unicoherent locally connected generalized continua X by using the Freudenthal compactification of X. In this paper we obtain several new characterizations of weak-unicoherence which in turn gives a means for obtaining a direct proof of one of the characterizations in [1] and thus answers a query of that paper.

2. Notation. A generalized continuum is a locally compact, connected, separable metric space and a continuum is a compact connected metric space. For a subset A of a space X, the closure of A in X will be denoted by $\bar{A}$ or $CL_X A$ and the boundary or frontier of A in X will be denoted by $\mathrm{Fr}A$ or $\mathrm{Fr}_X A$. A region is an open connected set and an open compact set U is an open set which has a compact closure. By a mapping will be meant a continuous function.

3. Characterizations. Definition: Let a space X be separated by a set F. A subset H of F is said to be essential if there exists a separation $X - F = A_H \cup B_H$ such that $H \cap \bar{A}_H \neq \phi$ and $H \cap \bar{B}_H \neq \phi$.

THEOREM 1. *Let X be a locally connected generalized continuum. The space X is weakly-unicoherent if and only if for every closed set F which separates X each open set containing an essential compact component of F contains a compact open subset of F which separates X.*

Proof. Suppose X is weakly-unicoherent, H is an essential compact component of F, and U is an open set containing H. There is a compact region W such that $H \subset W \subset \bar{W} \subset U$ and $\mathrm{Fr}W \cap F = \phi$. Let $K = \bar{W} \cap F$ and suppose K does not separate X. Let $X - F = A_H \cup B_H$ be a separation such that $\bar{A}_H \cap H \neq \phi$ and $\bar{B}_H \cap H \neq \phi$. Choose a region R such that $K \subset R \subset \bar{R} \subset W$, $\mathrm{Fr}R \cap A_H \neq \phi$, $\mathrm{Fr}R \cap B_H \neq \phi$ and $X - R$ is connected. By weak-unicoherence $\mathrm{Fr}R$ is a continuum and $\mathrm{Fr}R = (\mathrm{Fr}R \cap A_H) \cup (\mathrm{Fr}R \cap B_H)$ is a separation, hence the condition of the theorem holds.

For the converse suppose X is not weakly-unicoherent and let $X = A \cup B$, where A is closed and connected, B is a continuum, and $A \cap B$ is not connected. Let R be a component of $X - A$ or $X - B$ whose frontier is not connected. Let H be a component of $A \cap B$ such that $\mathrm{Fr}R \cap H \neq \phi$. Let W be a compact region containing H and such that $\mathrm{Fr}W \cap (A \cap B) = \phi$ and $(X - \bar{W}) \cap \mathrm{Fr}R \neq \phi$. Let $L = \mathrm{Fr}W \cap R$ and notice that L is a compact subset of R. Let L_0 be a continuum in R which contains L and let K be the continuum in R which contains L_0 and all the components of $R - L_0$ which have their frontiers in L_0. The continuum K is an essential component of a compact separating set of X so applying the condition of the theorem to R and K it follows that K separates X. On the other hand, if $R \subset B$, then $X - K = A$ union the components of $X - A$ except R union the components of $R - L_0$ which meet A. Thus $X - K$ is connected and consequently X is weakly unicoherent.

THEOREM 2. *Let X be a locally connected generalized continuum. The space X is weakly-unicoherent if and only if for every closed and connected set F and any component R of $X - F$ with a compact frontier implies $\mathrm{Fr}R$ is a continuum.*

Proof. Suppose X is weakly-unicoherent and let R be any component of $X - F$ which has a compact frontier. There is the separation $X - \mathrm{Fr}R = R \cup (X - \bar{R})$, when $\bar{R} \neq X$. Let H be a compact component of $\mathrm{Fr}R$. By a modification of the argument in Lemma 3.13 of [3] there exists a region W in $\bar{R}$ such that $H \subset W$, $\mathrm{Fr}_{\bar{R}} W \cap \mathrm{Fr}R = \phi$, $R - \bar{W}$ is connected and $(\bar{R} - W) \cap \mathrm{Fr}R \neq \phi$. Notice that $X - R$ is connected so that $X - \bar{W} = (X - R) \cup (R - \bar{W})$ is connected and hence $\overline{X - \bar{W}}$ is connected. We now have that

$$\bar{W} \cap (\overline{X - \bar{W}}) = (\mathrm{Fr}R \cap W) \cup (\mathrm{Fr}_R W)$$

is a separation which thus contradicts the weak-unicoherence of X. Therefore $\mathrm{Fr}R$ is a continuum.

Assume now that the condition holds and $X = A \cup B$, where A is closed and connected and B is a continuum. Suppose $A \cap B = D \cup E$ is a separation. Let $\mathscr{D}$ be the union of all components C of $X - B$ or $X - A$ which have $\mathrm{Fr}C \subset D$ and let $\mathscr{E}$ be the union of all components C of $X - B$ or $X - A$ which have $\mathrm{Fr}C \subset E$. Then $(\mathscr{E} \cup E) \cup (\mathscr{D} \cup D) = X$ is a separation of X. Since X is connected it must also be weakly-unicoherent.

THEOREM 3. *A locally connected generalized continuum X is weakly-unicoherent if and only if whenever $X = A \cup B$, where A and B are closed and connected sets wtih compact frontiers, then $A \cap B$ is connected.*

Proof. Assume X is weakly-unicoherent and $X = A \cup B$, where A and B are closed and connected sets with compact frontiers. Suppose $A \cap B = D \cup E$ is a separation. Each component C of $X - A$ or $X - B$ has a compact frontier so by Theorem 2 $\mathrm{Fr}C$ is a continuum. As in the proof of Theorem 2 this leads to a separation of X. But X is connected, hence $A \cap B$ is connected.

If the condition holds X is clearly weakly-unicoherent.

COROLLARY 1. *A locally connected generalized continuum is weakly-unicoherent if and only if whenever $X = A \cup B$, where A and B are closed and connected, then $A \cap B$ compact implies $A \cap B$ is a continuum.*

It is an easy observation that any open connected subset of a locally connected unicoherent continuum is weakly-unicoherent. Given a weakly-unicoherent locally connected generalized continuum X we will construct a compactification of X which is a unicoherent locally connected continuum and in which X is a dense region. By the paper of Clapp and Dickman it follows that our compactification is the Freudenthal compactification of X.

To begin the construction write $X = \bigcup_{n=1}^{\infty} A_n$, where for each positive interger n, A_n is a continuum, $A_n \subset \operatorname{int} A_{n+1}$, and each component of $X - A_n$ is not conditionally compact. For the positive integer $n, X - A_n$ has only finitely many components, say $C_{n,1}, C_{n,2}, \cdots, C_{n,m_n}$. With each decreasing sequence of components, $C_{1,i_1} \supset C_{2,i_2} \supset \cdots \supset C_{r,i_r} \supset \cdots$ associate a point $x_{i_1,i_2,\cdots,i_r,\cdots}$ not in X and let K be the set of all such points. On the set $X \cup K$ take as a basis for a topology a countable basis of connected open sets for the topology of X and for any point $x_{i_1,i_2,\cdots} \in K$ take the basic open sets containing $x_{i_1,i_2,\cdots}$ to be the components C_{r,i_r} of the defining sequence which determines the point and the point. Clearly this gives a topology for $X \cup K = \hat{X}$ which has a countable basis and which is Hausdorff. To show $\hat{X}$ is compact let $\{U_\alpha\}$ be some open cover of $\hat{X}$ and we may as well suppose it is a countable collection $\{U_i\}$ and that each U_i is a basis element. There is some positive integer n such that each component $C_{n,i}$ of $X - A_n$ is contained in some U_j of the covering for if not, then there exists a decreasing sequence $C_{i,i_1} \supset C_{2,i_2} \supset \cdots$ of such components. The decreasing sequence determines a point $x_{i_1,i_2,\cdots}$ of K and there is an element U_j of the covering with $x_{i_1,i_2,\cdots} \in U_j$. But $U_j = C_{r,i_r}$ for some subscript i_r of $x_{i_1,i_2,\cdots}$, hence we can suppose that there is an integer N such that each component $C_{N,i}$ of $X - A_N$ is contained in some U_j of the covering. Since there are only finitely many components of $X - A_N$, and A_N is a compact subset of X, it follows that some finite subcollection of $\{U_j\}$ covers $\hat{X}$.

Since $\hat{X}$ is compact Hausdorff and has a countable basis it can be considered as a compact metric space. Considering X as a subspace of $\hat{X}$ it is homeomorphic to the space X and is therefore connected. Considering X as a subset of $\hat{X}$ it is open and dense in $\hat{X}$ so that $\hat{X}$ is connected. The basis chosen for $\hat{X}$ is made up of open connected sets so that $\hat{X}$ is a locally connected continuum. The set K is a compact totally disconnected set in the relative topology.

To show $\hat{X}$ is unicoherent suppose $\hat{X} = A \cup B$, where A and B are continua and $A \cap B = D \cup E$ is a separation. There exist open sets U and V with disjoint closures such that $D \subset U$, $E \subset V$, $\operatorname{Fr}(U \cup V)$ is compact and $\operatorname{Fr}(U \cup V) \subset X$. Let A_1 be the component of $A \cup U \cup V$ containing A and let B_1 be the component of $B \cup U \cup V$ containing B. Notice that $\operatorname{Fr} A_1 \subset \operatorname{Fr}(U \cup V)$ and therefore is compact and similarly for $\mathrm{Fr}B_1$. It is also true that $A_1 - K$ and $B_1 - K$ are open and connected and $\mathrm{CL}_X(A_1 - K) \cap \mathrm{CL}_X(B_1 - K) \subset \bar{U} \cup \bar{V}$ contradicts the weak-unicoherence of X.

THEOREM 4. *A locally connected generalized continuum X is weakly-unicoherent if and only if X is homeomorphic to a region in a locally connected unicoherent continuum.*

4. Application. As an application we show that a compact quasi-monotone mapping preserves weak-unicoherence. A mapping $f: X \to Y$ is compact if the inverse image of each compact set is itself compact. A mapping $f(X) = Y$ is quasi-monotone [7] if whenever Q is a continuum with a nonvoid interior in Y, then each component of $f^{-1}(Q)$ maps onto Q.

THEOREM 5. *If $f(X) = Y$ is a compact quasi-monotone mapping on locally connected generalized continua and X is weakly-unicoherent, then so is Y.*

Proof. Let $Y = A \cup B$, where A and B are closed connected sets and $A \cap B$ is compact. If either A or B is Y, then the intersection is connected so we may as well suppose A and B are proper subsets. Since f is quasi-monotone and compact and A has interior points, $f^{-1}(A)$ has only finitely many components, $A_1, \cdots, A_n$, and $f(A_i) = A$ for $i = 1, 2, \cdots, n$. Similarly $f^{-1}(B) = B_1 \cup B_2 \cup \cdots \cup B_m$, where each B_i is a component of $f^{-1}(B)$ and $f(B_i) = B$ for $i = 1, \cdots, m$. If $n = 1$ and $m = 1$, then $X = A_1 \cup B_1$, $A_1 \cap B_1$ is a continuum and $f(A_1 \cap B_1) = A \cap B$ so that $A \cap B$ is a continuum. Suppose then that at least one of m and n is greater than one so that X is the union of a finite number (> 2) of connected sets. There is at least one A_i or one B_j such that the union of the remaining sets is connected. Let B_j be such that if H is the union of all the A_i's and all the B_i's except B_j, then H is connected. Now it follows that $X = H \cup B_j$, H and B_j are closed and connected and $H \cap B_j$ is compact so that $H \cap B_j$ is a continuum. Furthermore, $f(H \cap B_j) = A \cap B$ and hence $A \cap B$ is a continuum.

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Received March 27, 1974 and in revised form June 26, 1974.

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