PACIFIC JOURNAL OF MATHEMATICS
Vol. 56, No. 2, 1975

A CHARACTERISTIC SUBGROUP OF A GROUP OF ODD ORDER

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Let G be a finite solvable group of odd order. Suppose p is a prime, S is a Sylow p-subgroup of G, and $O_{p'}(G) = 1$. Let $J(S)$ be the Thompson subgroup of S. Then, by a result of the second author (Lemma 6), $Z(J(S)) \lhd G$.

The object of this paper is to generalize the above result by replacing the prime p by a set of primes π.

We obtain the following results:

THEOREM 1. *Let G be a finite solvable group of odd order, π be a set of primes, and H be a Hall π-subgroup of G. Assume that $O_{\pi'}(G) = 1$. Then:*

(a) *for every $p \in \pi - \{3\}$ and $A \in \mathscr{A}(H)$, $O_p(A) \subseteq O_p(G)$;*

(b) *the prime divisors of $d(H)$, of $|Z(J(H))|$, and of $|F(G)|$ coincide;*

(c) *$d(G) = d(H)$; and*

(d) *$Z(J(G)) = Z(J(H))$.*

In particular, if $G \neq 1$, then $1 \subset Z(J(H)) \lhd G$.

COROLLARY. *Suppose G is a finite solvable group of odd order, p is a prime, and S is a Sylow p-subgroup of G. Assume that $O_{p'}(G) = 1$. Then $Z(J(S)) = Z(J(G))$. Moreover, if $p \neq 3$, then $J(S) = J(G) = J(F(G))$.*

By the Odd Order Theorem of Feit and Thompson **[1]**, Theorem 1 and its corollary apply to all finite groups of odd order. Since much of our argument requires only that G be π-solvable and have an Abelian Sylow 2-subgroup, we obtain a related result:

THEOREM 2. *Suppose π is a set of primes, G is a finite π-solvable group, and H is a Hall π-subgroup of G. Assume that G has an Abelian Sylow 2-subgroup and that $O_{\pi'}(G) = 1$. Then:*

(a) *$O_2(G) = O_2(Z(J(G)) = O_2(Z(J(H))) = O_2(H)$;*

(b) *if $2 \notin \pi$, then for every $p \in \pi - \{3\}$ and $A \in \mathscr{A}(H)$, $O_p(A) \subseteq O_p(G)$;*

(c) *if $2 \notin \pi$, then $Z(J(H)) \lhd G$; and*

(d) *if $2 \notin \pi$, then the prime divisors of $d(H)$, of $|Z(J(H))|$, and of $|F(G)|$ coincide.*

In particular, if $2 \notin \pi$ and $G \neq 1$, or if $O_2(G) \neq 1$, then there exists a nonidentity characteristic subgroup of H that is a normal subgroup of G.

COROLLARY. *Assume the hypothesis of Theorem 2 and assume that* $2,3 \notin \pi$. *Then* $J(H) = J(F(G))$.

Some related results for groups with a nilpotent Hall π-subgroup were obtained by Schoenwaelder in **[5]**.

All groups in this paper are assumed to be finite. Our notation is standard and taken mainly from **[4]**. In particular, let G be a group. Then $F(G)$ denotes the Fitting subgroup of G and $[A, B, C]$ denotes the triple commutator $[[A, B], C]$ of three subgroups A, B, C of G. Moreover, $d(G)$ is the maximum of the orders of the Abelian subgroups of G. Let $\mathscr{A}(G)$ be the set of all Abelian subgroups of order $d(G)$ in G. (This is denoted by $A'(G)$ in **[4]**.) Then, as in **[4]**, $J(G)$ is the subgroup of G generated by $\mathscr{A}(G)$, that is, the Thompson subgroup of G.

For a prime power q, we will denote the finite field of q elements by $GF(q)$. Let p be a prime. Sometimes we will use Z_p to denote $GF(p)$ considered as a field or as an additive group. We will often use without reference the elementary result that if G is a group, π a set of primes, and H a normal subgroup of G, then $O_\pi(H) \subseteq O_\pi(G)$.

At times we shall assume one of the following hypotheses:

(H) (a) π is a set of primes
  (b) G is a π-solvable group
  (c) H is a Hall π-subgroup of G

(H_2) (a) π, G, and H satisfy (H)
  (b) G has an Abelian Sylow 2-subgroup.

(The concept of a π-solvable group is defined in §6.3 of **[4]**, in which it is proved that every π-solvable group possesses a Hall π-subgroup.)

2. Preliminary results.

LEMMA 1. *Suppose p is a prime, V is a finite nonidentity elementary Abelian additive p-group, and A is an Abelian group of automorphisms of V. Regard V as a vector space over* Z_p. *Assume that A acts irreducibly on V and that A preserves some nondegenerate alternating bilinear form on V into* Z_p. *Let F be the ring of endomorphisms of V generated by the elements of A.*

Then:

(a) *There exists a positive integer k such that* $|V| = p^{2k}$, $F \cong GF(p^{2k})$, *and* $|A|$ *divides* $1 + p^k$.

(b) *Let E be the unique subfield of F that is isomorphic to $GF(p^k)$. Take $v_0 \in V - \{0\}$ and let $W = v_0E$. Then for every nondegenerate alternating bilinear form f on V that is preserved by A,*

$$f(w, w') = 0 \quad \textit{for} \quad \textit{all} \quad w, w' \in W.$$

Proof. Let F_0 be the set (ring) of all endomorphisms of V that commute with every element of A. We regard Z_p as a subfield of F_0. As is well known, F_0 is a division algebra ([**4**], page 76) and, since it is finite, F_0 is a field. Clearly, F is a subfield of F_0. Hence the multiplicative group $F - \{0\}$ is cyclic. As A is a subgroup of $F - \{0\}$, A is cyclic. Let $p^m = |V|$. We may regard V as a vector space over F; then V is a direct sum of 1-dimensional subspaces over F. As $A \subseteq F - \{0\}$ and A acts irreducibly on V, V is 1-dimensional over F. Therefore, $|F| = |V| = p^m$.

Let N be the set of all nondegenerate alternating bilinear forms on V into Z_p that are preserved by A. By hypothesis, N is not empty. Hence m is even. Choose a generator α of A. Define $g(x)$ to be the minimal polynomial of α over Z_p. Then $g(x)$ can be expressed as

$$g(x) = \sum_{0 \leqq i \leqq m} a_i x^i,$$

where $a_0, \cdots, a_m \in Z_p$ and $a_m = 1$. By the elementary theory of fields, the roots of $g(x)$ over F are distinct and are precisely $\alpha, \alpha^p, \cdots, \alpha^{p^{m-1}}$.

Take some $f \in N$ and some $v \in V - \{0\}$. Let $v' = vg(\alpha^{-1})$. Then, for all $w \in V$,

$$f(v', w) = \sum_i a_i f(v\alpha^{-i}, w) = \sum_i a_i f(v, w\alpha^i)$$
$$= f(v, wg(\alpha)) = 0.$$

Since f is not degenerate, $v' = 0$. As v was chosen arbitrarily, $g(\alpha^{-1}) = 0$. Hence, $\alpha^{-1} = \alpha^{p^i}$ for some i such that $0 \leqq i \leqq m - 1$. If $i = 0$, then $\alpha^2 = 1$, contrary to the fact that $m \geqq 2$ and $\alpha \neq \alpha^p$. Therefore, $1 \leqq i \leqq m - 1$. Now

$$\alpha = (\alpha^{-1})^{-1} = (\alpha^{p^i})^{-1} = (\alpha^{-1})^{p^i} = \alpha^{p^{2i}}.$$

Since α generates F and $F \cong GF(p^m)$, $2i$ is a multiple of m. Consequently, $i = \frac{1}{2}m$. Let $k = \frac{1}{2}m$. Then $\alpha^{-1} = \alpha^{p^k}$, and $\alpha^{1+p^k} = 1$. This proves (a).

Let $\delta = \alpha + \alpha^{-1}$. Since

$$\delta^{p^k} = \alpha^{p^k} + \alpha^{p^{2k}} = \alpha + \alpha^{p^k} = \delta,$$

$\delta \in E$. Since α generates F over Z_p, it follows that $\alpha, \alpha^p, \cdots, \alpha^{p^{2k-1}}$ form a basis of F over Z_p. Hence $\delta, \delta^p, \cdots, \delta^{p^{k-1}}$ are distinct. So, δ generates E over Z_p and $\delta, \delta^p, \cdots, \delta^{p^{k-1}}$ form a basis of E over Z_p, that is,

$$\alpha + \alpha^{-1},\ \alpha^p + \alpha^{-p}, \cdots, \alpha^{p^{k-1}} + \alpha^{-p^{k-1}}$$

is a basis of E over Z_p.

Take $f \in N$ and $w, w' \in W$ as in (b). If $w = 0$, then $f(w, w') = 0$, as desired. Assume that $w \neq 0$. Then there exists $\beta \in E$ such that $w' = w\beta$. Take $b_0, b_1, \cdots, b_{k-1} \in E$ such that

$$\sum_{0 \leq i \leq k-1} b_i(\alpha^{p^i} + \alpha^{-p^i}) = \beta.$$

For $i = 0, \cdots, k-1$,

$$\begin{aligned} f(w, w(\alpha^{p^i} + \alpha^{-p^i})) &= f(w, w\alpha^{p^i}) + f(w, w\alpha^{-p^i}) \\ &= f(w, w\alpha^{p^i}) + f(w\alpha^{p^i}, w) = 0, \end{aligned}$$

since f is an alternating form. Hence,

$$f(w, w') = f(w, w\beta) = \sum_{0 \leq i \leq k-1} b_i f(w, w(\alpha^{p^i} + \alpha^{-p^i})) = 0,$$

as desired. This completes the proof of (b) and thus of Lemma 1.

LEMMA 2. *Suppose p is a prime, B is a finite, non-Abelian p-group, and A is an Abelian group of automorphisms of B. Assume that A acts irreducibly on $B/\Phi(B)$ and that $O_{p'}(A)$ acts trivially on $\Phi(B)$.*

Then:

(a) *there exists a positive integer k such that $|B/\Phi(B)| = p^{2k}$*;

(b) *$|A|$ divides $1 + p^k$; and*

(c) *B contains an Abelian subgroup B_0 such that $B_0 \supseteq \Phi(B)$ and $|B_0/\Phi(B)| = p^k$.*

Proof. For convenience in notation, we embed A and B in the natural manner in their semi-direct product AB.

Let $A_p = O_p(A)$, $A^* = O_{p'}(A)$, and $V = B/\Phi(B)$. Since A acts irreducibly on V, $A/C_A(V)$ acts faithfully and irreducibly on V. We may regard V as a vector space over Z_p. By **[4]**, Theorem 3.1.3, page 62,

$$A_pC_A(V)/C_A(V) = O_p(A/C_A(V)) = 1.$$

Hence

(1) $$A_p \subseteq C_A(V) \text{ and } A^* \text{ acts irreducibly on } V.$$

Since B is not Abelian, B is not cyclic. Therefore, $|V| = |B/\Phi(B)| \geqq p^2$. It follows that $1 \neq [V, A^*]$ and therefore that

(2) $$[V, A^*] = V.$$

Consequently, $B = [B, A^*]\Phi(B)$. By **[4]**, page 173.

(3) $$B = [B, A^*].$$

By (1) and the hypothesis of this lemma,

$$[A_p, B, A^*] \subseteq [\Phi(B), A^*] = 1 \quad \text{and} \quad [A^*, A_p, B] = [1, B] = 1.$$

Therefore, by (3) and the Three Subgroups Lemma (**[4]**, page 19),

$$1 = [B, A^*, A_p] = [B, A_p].$$

As $A_p \subseteq \operatorname{Aut} B$, $A_p = 1$. Hence A is a p'-group and $A = A^*$. By a theorem of Burnside (**[4]**, page 174),

(4) $$A \text{ acts faithfully on } V.$$

Since $C_{AB}(\Phi(B))$ is a normal subgroup of AB that contains A, (3) yields that $C_{AB}(\Phi(B))$ contains B. Therefore, $\Phi(B) \subseteq Z(B)$. Since B is not Abelian and $B' \subseteq \Phi(B) \subseteq Z(B)$, B has nilpotence class two. By an easy calculation, $[x, y]^p = [x^p, y] = 1$ for all $x, y \in B$. Thus

(5) $$B' \text{ is an elementary Abelian group.}$$

Take any subgroup C of index p in B'. Let ϕ be an isomorphism of B'/C onto the additive group of Z_p. Since $\Phi(B) \subseteq Z(B)$, the mapping $f\colon V \times V \to Z_p$ given by

$$f(x\Phi(B), y\Phi(B)) = \phi([x, y]C)$$

is a well-defined, nonzero, alternating bilinear form on V into Z_p. As A acts trivially on B', A preserves f. Therefore, A preserves the radical of f, that is, the group $R/\Phi(B)$, where

$$R \supseteq \Phi(B) \supset C \quad \text{and} \quad R/C = Z(B/C).$$

As $R/\Phi(B) \subset V$ and A acts irreducibly on V, $R/\Phi(B) = 1$. Consequently, f is a nondegenerate form. By (4) and Lemma 1, there exists a positive integer k such that $|V| = p^{2k}$ and $|A|$ divides $1 + p^k$. This yields (a) and (b).

Take E and W as in Lemma 1(b). Define a subgroup B_0 of B such that $B_0 \supseteq \Phi(B)$ and $B_0/\Phi(B) = W$. Then

$$|B_0/\Phi(B)| = |W| = |E| = p^k.$$

Suppose $B_0' \neq 1$. Then, by (5), there exists a subgroup C^* of index p in B' such that $B_0' \not\subseteq C^*$. For convenience in notation, we will assume that C^* is the group C chosen above. Take a form f as above. Take $x, y \in B_0$ such that $[x, y] \not\in C$. Then

$$f(x\Phi(B), y\Phi(B)) = \phi([x, y]C) \neq 0,$$

contrary to Lemma 1(b). This contradiction proves that $B_0' = 1$ and hence completes the proof of (c) and of Lemma 2.

LEMMA 3. *Assume* (H) *and assume that* $O_{\pi'}(G) = 1$. *Then*:

(a) $C_G(F(G)) \subseteq F(G)$, *and*

(b) *if A is a subgroup of Aut G that fixes every element of $F(G)$ and if $|A|$ and $|G|$ are relatively prime, then $A = 1$.*

Proof. (a) Let $N = O_\pi(G)$ and $C = C_G(F(G))$. Then N is a solvable group. Clearly, $F(N) = F(G)$. By **[4]**, Theorem 6.3.2, $C_G(N) \subseteq N$.

Suppose x is a π'-element in C. Let $L = \langle N, x\rangle$. Then

$$N = O_\pi(L) \quad \text{and} \quad [N, O_{\pi'}(L)] \subseteq N \cap O_{\pi'}(L) = 1.$$

Since $C_G(N) \subseteq N$, it follows that $O_{\pi'}(L) = 1$. Hence $F(N) = F(L)$. Since L is solvable,

$$x \in C \cap L = C_L(F(L)) \subseteq F(L) = F(N),$$

by **[4]**, page 218. Therefore, $x = 1$.

Thus, C is a π-group. Since $C \triangleleft G$, $C \subseteq O_\pi(G) = N$. By [4], page 218 again, $C = C_N(F(N)) \subseteq F(N)$.

(b) Embed A and G in their semi-direct product AG. Let $B = O_{\pi'}(AG)$. Since $B \cap G \subsetneq O_{\pi'}(G) = 1$, $|B|$ divides $|AG/G|$, that is, $|B|$ divides $|A|$. Since $|A|$ and $|G|$ are relatively prime and

$$|A/(A \cap B)|$$

divides $|AG/B|$, $B \subseteq A$. However,

$$[G, B] \subseteq [G, O_{\pi'}(AG)] \subseteq O_{\pi'}(G) = 1.$$

As B is a group of automorphisms of G, $B = 1$. Hence $F(AG) = F(G)$. By (a), $A \subseteq F(G)$. Therefore, $A = 1$.

LEMMA 4. *Assume* (H). *Suppose* $p \in \pi$, $O_{\pi'}(G) = 1$, *and* T *is a* p-*subgroup of* $O_{p',p}(G)$ *that centralizes* $F(O_{p'}(G))$. *Then* $T \subseteq O_p(G)$.

Proof. Let $K = O_{p'}(G)$. Apply Lemma 3 with K in place of G and $T/C_T(K)$ in place of A. We obtain the conclusion that $T/C_T(K) = 1$, in other words, T centralizes K. Let R be a Sylow p-subgroup of $O_{p',p}(G)$ that contains T. Let $T^* = C_R(K)$. Then $O_{p',p}(G) = KR$ and T^* is normalized by K and by R. Hence $T^* \triangleleft KR$ and

$$T \subseteq T^* \subseteq O_p(KR) \subseteq O_p(G).$$

We also use the following result of J. Thompson, whose proof is sketched in the remark on page 164 of [3]:

THEOREM OF THOMPSON. *Suppose* p *is an odd prime,* G *is a* p-*solvable group, and* S *is a Sylow* p-*subgroup of* G. *Assume that* $O_{p'}(G) = 1$. *Assume also that* G *satisfies one of the following conditions*:

(i) $p \geqq 7$;

(ii) $p = 5$ *and* G *has an Abelian Sylow* 2-*subgroup.*

Then $J(S) \subseteq O_p(G)$.

LEMMA 5. *Assume* (H_2). *Suppose* $p \in \pi$, S *is a Sylow* p-*subgroup of* G, *and* $A \in \mathscr{A}(S)$. *Assume that* $p \geqq 5$ *and that* A *centralizes* $F(O_{p'}(G))$. *Then* $A \subseteq O_p(G)$.

Proof. Let $K = O_{p'}(G)$. Note that G *is* p-solvable. By the Theorem of Thompson,

$$AK/K \subseteq O_p(G/K) = O_{p',p}(G)/K.$$

Hence $A \subseteq O_{p',p}(G)$. By Lemma 4, $A \subseteq O_p(G)$, as desired.

LEMMA 6. *Suppose p is an odd prime, G is a p-solvable group, and S is a Sylow p-subgroup of G. If $p = 3$, assume also that G has an Abelian Sylow 2-subgroup. Then*

$$O_{p'}(G)Z(J(S)) \lhd G.$$

Proof. Let $K = O_{p'}(G)$, $G^* = G/K$, and $S^* = SK/K$. Then $O_{p'}(G^*) = 1$ and S^* is a Sylow p-subgroup of G^*. From the hypothesis, G^* must be p-constrained and p-stable. By a theorem of the second author (**[4]**, pages 268–269 and 279, or **[2]**, Theorem A), $Z(J(S^*)) \lhd G^*$. Since

$$Z(J(S^*)) = Z(J(S))K/K,$$

the result follows.

The next result can be easily verified by calculation. It is a special case of Lemma 10.1, page 1131, of **[2]**.

LEMMA 7. *Let K be a group of linear transformations on a finite-dimensional vector space V over a field F. Let V^* be the dual space of V over F and let K act on V^* in the natural manner, i.e.,*

$$f^g(v) = f(v^{g^{-1}}), \quad \textit{for} \quad f \in V^*, g \in K, v \in V.$$

Let T be the set of all ordered triples (v, f, α) for $v \in V$, $f \in V^*$, $\alpha \in F$. Define multiplication on T by the rule

$$(v_1, f_1, \alpha_1)\ (v_2, f_2, \alpha_2) = (v_1 + v_2, f_1 + f_2, \alpha_1 + \alpha_2 - f_1(v_2)).$$

For each $g \in K$, define a mapping $M(g)$ of T into itself by

$$(v, f, \alpha)^{M(g)} = (v^g, f^g, \alpha).$$

Then:

(a) T forms a group under multiplication;

(b) for (v, f, α), (v_1, f_1, α_1) and (v_2, f_2, α_2) in T,

$$(v, f, \alpha)^{-1} = (-v, -f, -f(v) - \alpha)$$

and

$$[(v_1, f_1, \alpha_1), (v_2, f_2, \alpha_2)] = (0, 0, f_2(v_1) - f_1(v_2)); \quad \text{and}$$

(c) M is an isomorphism of K into the automorphism group of T.

3. Some Properties of $\mathscr{A}(G)$.

PROPOSITION 1. *Suppose G is group, $A \in \mathscr{A}(G)$, B is a nilpotent subgroup of G, and A normalizes B. Assume that B has an Abelian Sylow 2-subgroup and that either $|A|$ is odd or B is Abelian. Then AB is nilpotent.*

Proof. Assume that the result is false, that G is a counter-example of minimal order, and that, within G, B has minimal order.

Clearly, $G = AB$ and $G \supset F(G) \supseteq B$. Therefore, $A \not\subseteq F(G)$. For some prime p, $O_p(A) \not\subseteq F(G)$. Let $A_p = O_p(A)$. Then $A_p \not\subseteq O_p(G)$. Hence $A_pB_p \ntrianglelefteq G$. Since A normalizes A_pB_p, B does not. Consequently, there exists a prime q such that $O_q(B)$ does not normalize A_pB_p. Let $B_q = O_q(B)$. Then B_q does not centralize A_pB_p and therefore does not centralize A_p. Thus AB_q is not nilpotent. By the minimal choice of B, $B = B_q$.

Let $A^* = O_{q'}(A)$ and $V = B/B'$. Then A^* does not centralize B. By [4], page 174, A^* does not centralize V. By the minimal choice of B,

$$A^* \text{ centralizes } \Phi(B). \tag{7}$$

From [4], page 177, $V = C_V(A^*) \times [V, A^*]$. By the minimal choice of B,

$$V = [V, A^*] \quad \text{and} \quad C_V(A^*) = 1.$$

Let W be a minimal A-invariant subgroup of V. Then W is elementary Abelian. Since $C_W(A^*) \subseteq C_V(A^*) = 1$, the minimal choice of V yields that $V = W$. Hence $\Phi(B) \subseteq B' \subseteq \Phi(B)$. Consequently,

$$B' = \Phi(B) \text{ and } A \text{ acts irreducibly and nontrivially on } B/B'. \tag{8}$$

Let $C = C_A(B)$ and $n = |A/C|$. Then A/C acts faithfully as a group of automorphisms of B. By (8),

$$C \cap B \subseteq B'. \tag{9}$$

Take $B_1 \in \mathscr{A}(B)$. Since CB_1 is Abelian and $A \in \mathscr{A}(G)$,

$$|A| \geqq |CB_1| = |C|\ |B_1|/|C \cap B_1| \geqq |C|\ |B_1|/|B'|,$$

by (9). Hence

$$n = |A/C| \geqq |B_1|/|B'| = d(B)/|B'|. \tag{10}$$

Suppose first that B is Abelian. Then $B' = 1$ and $d(B) = |B|$. For every $a \in A - C$, $C_B(a) \subset B$ and $C_B(a) \lhd AB$; by (8), $C_B(a) = 1$. Hence every non-identity element of A/C acts in a fixed-point-free manner on B, and

$$|A/C| \leqq |B\text{-}\{1\}| < |B| = d(B)/|B'|.$$

However, this contradicts (10).

Thus B is not Abelian. By hypothesis,

$$q \text{ is an odd prime and } |A| \text{ is odd.} \tag{11}$$

By (7) and (8), A and B satisfy the hypothesis of Lemma 2. Take k and B_0 as in Lemma 2. Then

$$|B/B'| = q^{2k},\ n \text{ divides } 1 + q^k,\ B_0 \text{ is abelian, and } |B_0/B'| = q^k.$$

Therefore, by (10), $n \geqq d(B)/|B'| \geqq |B_0/B'| = q^k$. Since n divides $1 + q^k$, $n = 1 + q^k$. But this is impossible, by (11). This contradiction completes the proof of Proposition 1.

PROPOSITION 2. *Assume* (H_2). *Suppose* $O_{\pi'}(G) = 1$. *Then*

$$O_2(G) = O_2(H) = O_2(Z(J(H))) = O_2(Z(J(G))).$$

Proof. Let $K = O_2(Z(J(H)))$ and $N = O_\pi(G)$. Then N is a solvable group. By (H_2), K centralizes $O_2(G)$. For every odd prime p,

$$O_p(G) \subseteq O_p(H) \subseteq C_G(O_2(H)) \subseteq C_G(K).$$

Hence K centralizes $F(G)$. By Lemma 3, $K \subseteq C_G(F(G)) \subseteq F(G)$. So $K \subseteq O_2(F(G)) = O_2(G)$.

On the other hand, let $A \in \mathscr{A}(H)$ and $B = O_2(G)$. By Proposition 1, AB is nilpotent. Therefore, $O_{2'}(A)$ centralizes B. By (H_2), A centralizes B. Hence $B \subseteq C_H(A) = A$. Thus $B \subseteq Z(J(H))$ and $B \subseteq K$. Consequently, $B = K$, as desired. Since π, H, and H satisfy (H_2), we obtain as a special case that $K = O_2(H)$.

A similar argument with $A \in \mathscr{A}(G)$ and $B = O_2(G) = K$ shows that $K \subseteq Z(J(G))$. Hence

$$K \subseteq O_2(Z(J(G))) \subseteq O_2(G) = K.$$

So $K = O_2(Z(J(G)))$.

PROPOSITION 3. *Assume* (H_2). *Suppose* $p \in \pi$ *and* $A \in \mathscr{A}(H)$. *Assume that* $O_{\pi'}(G) = 1$, $d(H)$ *is odd, and* $p \geqq 5$. *Then* $O_p(A) \subseteq O_p(G)$.

Proof. We use induction on the order of G. Let $A_p = O_p(A)$, $T = O_p(G)$, $K = O_{p,p'}(G)$ and $G^* = AK$, and $H^* = A(H \cap K)$. Then $H \cap K$ is a Hall π-subgroup of K and H^* is a Hall π-subgroup of G^*.

Suppose $G^* \subset G$. Since $A \subseteq H^*$, $d(H^*) = d(H)$. By induction, $A_p \subseteq O_p(G^*)$. Hence

$$[K, A_p] \subseteq K \cap O_p(G^*) \subseteq O_p(K) = T.$$

Therefore, $A_pT/T \subseteq C_{G/T}(K/T)$. By **[4]**, page 228, $C_{G/T}(K/T) \subseteq K/T$. Consequently, $A_p \subseteq K$. So,

$$A_p \subseteq K \cap O_p(G^*) = O_p(K) = T,$$

as desired.

Suppose $G^* = G$. Then A_pT is a Sylow p-subgroup of G. Let $A^* = O_{p'}(A)$. By hypothesis, $|A|$ is odd. By Proposition 1, AT is nilpotent. Therefore, A^* centralizes T and hence A_pT. For every Abelian subgroup B of A_pT, A^*B is Abelian and

$$|A^*|\ |A_p| = |A| \geqq |A^*B| = |A^*|\ |B|.$$

Hence $A_p \in \mathscr{A}(A_pT)$. By Proposition 1, $AF(O_{p'}(G))$ is nilpotent. Then A_p centralizes $F(O_{p'}(G))$. By Lemma 5, $A_p \subseteq O_p(G)$, as desired.

PROPOSITION 4. *Assume* (H_2). *Suppose* π *is a set of odd primes and* $O_{\pi'}(G) = 1$.

Let $K = C_G(O_{3'}(G))$. *For every* $p \in \pi$ *and* $A \in \mathscr{A}(H)$, *let* $A_p = O_p(A)$. *Define* d_3 *to be the maximum of* $|C|$ *for all Abelian* 3-*subgroups* C *of* $H \cap K$ *and define* $\mathscr{A}_3$ *to be the set of all Abelian* 3-*subgroups of order* d_3 *in* $H \cap K$. *Let* S *be any Sylow* 3-*subgroup of* K. *Then*:

(a) $\{A_p \mid A \in \mathscr{A}(H)\} = \mathscr{A}(O_p(G))$, *for every prime* $p \geqq 5$;
(b) $\{A_3 \mid A \in \mathscr{A}(H)\} = \mathscr{A}_3$;
(c) $O_p(Z(J(H))) = Z(J(O_p(G)))$, *for every prime* $p \geqq 5$; *and*
(d) $O_3(Z(J(H))) = Z(J(S)) \lhd G$ *and* $d_3 = d(S)$.

Proof. Note that $d(H)$ is odd.

(a) Assume $p \geqq 5$. Let $A \in \mathscr{A}(H)$. Let $A^* = O_{p'}(A)$ and $M = O_p(G)$. By Proposition 3, $A_p \subseteq M$. By Proposition 1, A^* centralizes M. Hence, for every Abelian subgroup B of M, $A^* \times B$ is Abelian. Therefore, $|A_p| = d(M)$, and $A^* \times B \in \mathscr{A}(H)$ for every $B \in \mathscr{A}(M)$. This proves (a).

(b) Suppose $A \in \mathscr{A}(H)$. By Proposition 1, $AF(G)$ is nilpotent. Hence, A_3 centralizes $F(O_{3'}(G))$. Since

$$O_{\pi'}(O_{3'}(G)) \subseteq O_{\pi'}(G) = 1,$$

A_3 centralizes $O_{3'}(G)$, by Lemma 3. By (a), $O_{3'}(A) \subseteq O_{3'}(G)$. Now (b) follows by an argument similar to that of (a).

(c) This follows immediately from (a).

(d) Assume first that K is a $3'$-group. Then $\mathscr{A}_3 = \{1\}$ and $S = 1$. Since $Z(J(H)) \subseteq A$ for every $A \in \mathscr{A}(H)$, $O_3(Z(J(H))) = 1 = Z(J(S))$, as desired.

Now assume that K is not a $3'$-group. Then $S \neq 1$. Let $T = O_3(Z(J(H)))$ and $U = Z(J(S))$. By Lemma 6, $UO_{3'}(K) \lhd K$. Since $O_{3'}(K) \subseteq O_{3'}(G)$ and $K = C_G(O_{3'}(G))$,

$$UO_{3'}(K) = U \times O_{3'}(K).$$

Hence

$$1 \subset U = O_3(UO_{3'}(K)) \lhd K. \tag{12}$$

As $O_{\pi'}(G) = 1$ and $1 \subset U \subseteq O_3(K) \subseteq O_3(G)$, $3 \in \pi$.

Suppose $A \in \mathscr{A}(H)$. By (b), $A_3 \subseteq H \cap K$. Let $A^* = O_{3'}(A)$ and let S^* be a Sylow 3-subgroup of $H \cap K$ that contains A_3. Since $K \lhd G$ and $3 \in \pi$, $H \cap K$ is a Hall π-subgroup of K and S^* is a Sylow 3-subgroup of K. As S^* and S are conjugate in K, (12) yields that

$$U = ZJ(S^*). \tag{13}$$

By (a), $A^* \subseteq O_{3'}(G)$. Therefore, S^* centralizes A^*. Since $A = A_3 \times A^*$, $A_3 \in \mathscr{A}(S^*)$ and $d_3 = |A_3| = d(S^*) = d(S)$. By (13), $U \subseteq A_3 \subseteq A$. As A is an arbitrary element of $\mathscr{A}(H)$, $U \subseteq Z(J(H))$. So,

$U \subseteq T$. On the other hand, $T \subseteq A_3$ for every $A \in \mathscr{A}(H)$. Consequently, $T \subseteq B$ for every $B \in \mathscr{A}(S)$, by (b), and hence $T \subseteq U$. Thus $T = U$.

By (12), $U = Z(J(R))$ for every Sylow 3-subgroup R of K. Therefore, U is a characteristic subgroup of K and hence a normal subgroup of G. This completes the proof of (d) and thus of Proposition 4.

4. Proof of Theorems.

We first prove Theorem 2. Parts (a) and (b) follow directly from Proposition 2 and 3. Since

$$Z(J(H)) = \langle O_p(Z(J(H))) \mid p \in \pi \rangle,$$

(c) follows from Proposition 4. To prove (d), assume $2 \notin \pi$ and let π_1, π_2, and π_3 be the sets of prime divisors of $|Z(J(H))|$, $d(H)$, and $|F(G)|$ respectively. Since $Z(J(H)) \subseteq A$ for every $A \in \mathscr{A}(H)$,

$$\pi_1 \subseteq \pi_2. \tag{14}$$

Take S as in Proposition 4. Note that $O_3(G) \subseteq K$, so $O_3(G) \subseteq S$. Therefore,

$$3 \in \pi_1 \text{ if and only if } 3 \in \pi_3, \tag{15}$$

by Proposition 4(d). By parts (b) and (d) of Proposition 4,

$$\text{if } 3 \in \pi_2, \text{ then } \mathscr{A}_3 \neq \{1\},\ S \neq 1, \text{ and } 3 \in \pi_3. \tag{16}$$

Now (14), (15), and (16) yield that 3 belongs to all of π_1, π_2, and π_3 or none of them. Parts (a) and (c) of Proposition 4 yield an analogous statement for each prime greater than 3. This completes the proof of Theorem 2.

Finally, we prove Theorem 1. For each prime p, define $d(p)$ to be the highest power of p that divides $d(H)$. Let σ be the set of all odd primes. We may and will assume that $2 \notin \pi$. Define d_3 as in Proposition 4.

Parts (a) and (b) of Theorem 1 are special cases of Theorem 2. By Proposition 4,

$$d(3) = d_3 \text{ and } d(p) = d(O_p(G)) \text{ for every prime } p > 3.$$

Hence $d(H) = d_3 \Pi_{p>3}\, d(O_p(G))$. Thus, $d(H)$ does not depend on the choice of π, provided that $\pi \subseteq \sigma$ and $O_{\pi'}(G) = 1$. As G is a Hall σ-subgroup of G, $d(G) = d(H)$. A similar argument from Proposition 4 shows that $Z(J(G)) = Z(J(H))$.

5. Some examples.

EXAMPLE 1. Let q be a power of a prime p. Let $E = GF(q)$ and $F = GF(q^2)$. Take a fixed element μ of F-E and define B to be the set of all ordered pairs of the form (α, β) for $\alpha \in F$ and $\beta \in E$. Define multiplication on B by the rule

$$(\alpha, \beta)\,(\gamma, \delta) = (\alpha + \gamma, \beta + \delta + \alpha\mu\gamma^q + \alpha^q\mu^q\gamma).$$

By calculation one may show that B is a group of order q^3. Moreover, for $(\alpha, \beta) \in B$,

$$C_B((\alpha, \beta)) = \{(\gamma, \delta) | \gamma \in \alpha E, \delta \in E\} \quad \text{if} \quad \alpha \neq 0.$$

By further calculations,

$$d(B) = q^2 \text{ and } B' = \Phi(B) = Z(B) = \{(0, \beta) | \beta \in E\}. \tag{17}$$

Take a nonzero element γ of F that has multiplicative order $q + 1$. The mapping ϕ: $B \to B$ given by

$$\phi((\alpha, \beta)) = (\alpha\gamma, \beta)$$

is an automorphism of B that has order $q + 1$. Let G be the semidirect product of B by $\langle\phi\rangle$. Embed $\langle\phi\rangle$ and B in G in the natural manner. Let $A = \langle\phi, B'\rangle$. Then A is Abelian and $|A| = (q + 1)q > d(B)$, by (17). A short argument shows that $C_G(b) \subseteq B$ for every $b \in B$-B' and that $d(G) = (q + 1)q$ and $A \in \mathscr{A}(G)$.

The group of automorphisms $\langle\phi\rangle$ yields an example of the 'extreme' cases of Lemmas 1 and 2, that is, $|\langle\phi\rangle| = 1 + p^k$ for $p^k = q$. Since B is nilpotent and AB is not nilpotent, G violates the conclusion of Proposition 1; here, B is not Abelian, B is a 2-group if $p = 2$, and $|A|$ is even if $p \neq 2$.

Let π be the set of all prime divisors of $|G|$ and let $H = G$. Then G violates various conclusions of Theorems 1 and 2. For every $r \in \pi$-$\{p\}$, $O_r(A) \neq 1$ and $O_r(G) = 1$, although it is possible that $r \geqq 5$. Furthermore, every element of π divides $d(G)$, but p is the only prime divisor of $|Z(J(G))|$ and is the only prime divisor of $|F(G)|$. Note, however, that obviously $Z(J(H)) \lhd G$.

EXAMPLE 2. Let $F = GF(3)$ and let V be a 3-dimensional vector space over F. Then there exists a group K of linear transformations of V over F such that K has order 39 and is not cyclic. Define T and M as in Lemma 7, and define K to be an operator group on T by the rule $t^g = t^{M(g)}$ for $t \in T$, $g \in K$.

Let G be the semi-direct product of T by K and embed T and K in G in the natural manner. Let π be $\{3\}$ and H be a Sylow 3-subgroup of G. Then T is an extra-special group of order 3^7, $T = F(G)$, and $d(H) = d(T) = 3^4$. There exists $A \in \mathscr{A}(H)$ such that $A \not\subseteq T$. Then $A = O_3(A) \not\subseteq O_3(G) = T$. Thus, part (a) of Theorem 1, part (b) of Theorem 2, and the corollary of Theorem 2 cannot be extended to include the case in which $p = 3$.

EXAMPLE 3. Here G is defined as in Example 2 except that K is taken to be isomorphic to the alternating group of degree 4.

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Received December 26, 1973. The first author wishes to express his gratitude to his dissertation advisor, Professor M. Herzog, of the Department of Mathematical Sciences, Tel-Aviv University, for his guidance and concern. During the preparation of this paper, the second author was supported by a National Science Foundation Senior Postdoctoral Fellowship and a grant from the Science Research Council and enjoyed the hospitality of the University of Oxford and the Weizmann Institute of Science. He gratefully acknowledges the support of these institutions.

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