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SUFFICIENT CONDITIONS FOR THE EXISTENCE OF CONVERGENT SUBSEQUENCES

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Let R be the real numbers, $S \subset R$ and E be an ordered topological vector space. Sufficient conditions are given that a sequence $\{y_k\}$, $y_k: S \to E$, will have a subsequence $\{h_k\}$ such that for each $t \in S$, $\{h_k(t)\}$ is either eventually monotone or else is convergent. In case E is a Banach space, sufficient conditions are given that $\{y_k\}$ have a subsequence $\{h_k\}$ so that $\{h_k(t)\}$ converges for each $t \in S$. Finally, if $E = R$, the concept of $\{y_k\}$ being equioscillatory is defined and it is shown that a necessary and sufficient condition for $\{y_k\}$ to have a subsequence that converges at every point of S is that $\{y_k\}$ have a subsequence which is pointwise bounded and equioscillatory. An application of these results to differential equations is treated briefly.

1. Introduction. The existence of solutions to initial and boundary value problems for both ordinary and partial differential equations is frequently shown by obtaining a convergent subsequence from a sequence of functions and showing that the limit function is the desired solution. For example, in the proof of the Picard-Lindelöf Theorem [1, Theorem 1.1, p. 8] and the Cauchy-Peano Existence Theorem [1, Theorem 2.1, p. 10] such techniques are used. The question arises then, for a given sequence of functions, what conditions suffice to allow extraction of a pointwise convergent subsequence. For a sequence $\{y_k\}$ with $y_k: I \to R$, where I is a real interval, there are many results which provide sufficient conditions for the existence of a convergent subsequence; for example, the Helly Selection Theorem and the Theorem of Ascoli.

Let $\{y_k\}$ be a sequence of functions from a nonempty subset S of the real numbers R into an ordered topological vector space E. Then we are interested in finding sufficient conditions that $\{y_k\}$ have a subsequence $\{h_k\}$ such that for each $s \in S$, $\{h_k(s)\}$ is a convergent sequence. Theorem 2.2 yields a subsequence $\{h_k\}$ such that for each $s \in S$, $\{h_k(s)\}$ is either eventually monotone or else is convergent. By adding conditions which will make these eventually monotone subsequences converge, the desired convergence result can be obtained. Such a result is given by Corollary 2.3. Furthermore, when $E = R$, we obtain a necessary and sufficient condition for a sequence $\{y_k\}$, $y_k: S \to R$, to have a subsequence which converges for each $s \in S$. This is stated in Corollary 2.5.

In §3 an application to differential equations is given. A more detailed description of the applications to boundary value problems for ordinary differential equations may be found in [4].

2. Primary results. We begin this section with the definition of a proper pair.

DEFINITION 2.1. Let S *be* a nonempty subset of real numbers and f be a function, $f: S \to E$, where E is an ordered vector space with positive cone K. Consider the set $\mathscr{P}$ of all finite nonempty partitions $P = \{x_1, x_2, \cdots, x_n\}$ of S where $n \geqq 1$, $x_i \in S$ for $i = 1, 2, \cdots, n$ and $x_1 < x_2 < \cdots < x_n$. If $f(s) \neq \theta$ for some $s \in S$, we say that (f, P) is a proper pair if $(-1)^i f(x_i) > \theta$ for $i = 1, 2, \cdots, n$ or else $(-1)^i f(x_i) < \theta$ for $i = 1, 2, \cdots, n$. If $f(s) = \theta$ for all $s \in S$ we say that (f, P) is a proper pair if P contains exactly one point.

THEOREM 2.2. *Let S be a nonempty subset of real numbers and $\{y_k\}$ be a sequence of functions, $y_k: S \to E$ where E is a sequentially complete ordered locally convex space with positive cone K. For each $t \in S$ assume that $\{y_k(t)\}$ is an eventually comparable sequence. Assume, for each $s \in S$, that E has a nested countable basis of circled sets at θ denoted by $\{U_s(n)\}$. For each $t \in S$ and each positive integer n assume that there are nonnegative integers $N(n, t)$, $H(n, t)$ and a number $\delta(n, t) > 0$ such that for all $k, j \geqq H(n, t)$ if $(y_k - y_j, P)$ is a proper pair then P contains at most $N(n, t)$ points x such that $y_k(x) - y_j(x) \notin U_x(n)$ and $t - \delta(n, t) < x < t + \delta(n, t)$. Then $\{y_k\}$ contains a subsequence $\{h_k\}$ such that for each $t \in S$, $\{h_k(t)\}$ is either eventually monotone or else is convergent.*

Proof. If $y_k(t)$ and $y_j(t)$ are comparable for all $k, j \geqq M(t)$ and $M(t)$ is the smallest positive integer having this property then let $A_i = \{t: t \in S, M(t) = i\}$ for $i = 1, 2, \cdots$. For any $t \in A_i$ we have $y_k(t)$ and $y_j(t)$ comparable for $k, j \geqq i$. We will prove the theorem assuming that $y_k(t)$ and $y_j(t)$ are comparable for all $t \in S$ and then a standard diagonalization argument where S is replaced by $A_1, A_2, \cdots$ yields the desired result.

We note that we may assume S is bounded because if the theorem is true for bounded sets a standard diagonalization argument yields the result for unbounded sets. Also, we may assume S is a closed ieterval because if the theorem is true for closed intervals, I, then we may choose I to be a closed interval containing the bounded set S and define a sequence of functions $\{z_k\}$, $z_k: S \to E$ by

$$\begin{aligned} z_k(t) &= y_k(t) \quad \text{for} \quad t \in S \\ &\qquad \theta \quad \text{for} \quad t \notin S \end{aligned}$$

then the sequence $\{z_k\}$ satisfies the hypotheses of the theorem on I and the result would follow for bounded sets S.

Furthermore, because of the compactness of S, we may assume that for each positive integer n there are nonnegative integers $N(n)$, $H(n)$ such that for all $k, j \geqq H(n)$ if $(y_k - y_j, P)$ is a proper pair then P contains at most $N(n)$ points x such that $y_k(x) - y_j(x) \notin U_x(n)$.

Let $\{J_i\}$ be an enumeration of all nonempty open subintervals of S with rational endpoints. Applying a slight modification of Corollary 2.2 of [3] to J_1, observe that either there is a subsequence of $\{y_k\}$, again denoted by $\{y_k\}$, such that $\{y_k\}$ is monotone on J_1 or else there is a subsequence of $\{y_k\}$, again denoted by $\{y_k\}$, such that for $k \neq j$, $y_k(t) > y_j(t)$ and $y_k(\tau) < y_j(\tau)$ hold for some $t, \tau \in J_1$. Now repeat the process described in the previous sentence consecutively on the intervals $J_2, J_3, \cdots$ and then take the diagonal subsequence, denoted by $\{y_k\}$ again. This sequence has the property that on J_i it is eventually monotone or else for every $k \neq j$ sufficiently large, depending on i, there is $t, \tau \in J_i$ such that $y_k(t) > y_j(t)$ and $y_k(\tau) < y_j(\tau)$.

Now using J_1 and $U_t(1)$ it follows from a slight modification of Corollary 2.3 of [3] that either there is a subsequence of $\{y_k\}$, again denoted by $\{y_k\}$, such that, for $k \neq j$, $y_k(t) - y_j(t) \in U_t(1)$ for all $t \in J_1$ or else there is a subsequence, again denoted by $\{y_k\}$, such that for $k \neq j$ there is a $t \in J_1$ with $y_k(t) - y_j(t) \notin U_t(1)$. Now repeat the process described in the preceding sentence for $U_t(2), U_t(3), \cdots$ and then take the diagonal subsequence, denote it by $\{y_k\}$. This sequence has the property that for J_1 and $U_t(n)$ either for all $k \neq j$ sufficiently large, depending on n, $y_k(t) - y_j(t) \in U_t(n)$ for all $t \in J_1$ or else for all $k \neq j$ sufficiently large, depending on n, there is some $t \in J_1$ such that $y_k(t) - y_j(t) \notin U_t(n)$.

We now repeat the entire process described in the preceding paragraph consecutively on the intervals $J_2, J_3, \cdots$ and then take the diagonal subsequence again denoted by $\{y_k\}$. This sequence has the property that for J_i and $U_t(n)$ either $y_k(t) - y_j(t) \in U_t(n)$ for all $t \in J_i$ and $k \neq j$ sufficiently large depending on i and n or else there is a $t \in J_i$ depending on k, j with $y_k(t) - y_j(t) \notin U_t(n)$, for $k \neq j$ sufficiently large depending on i and n.

We will now show by contradiction that for all but countably many values of $x \in S$ the sequence $\{y_k(x)\}$ is either convergent or eventually monotone. For $x \in S$ such that $\{y_k(x)\}$ is neither convergent nor eventually monotone let $\{F_{xi}\}$ be the subsequence of $\{J_i\}$ consisting of the intervals which contain x. There must be a smallest positive integer n_{xi}, such that $y_k(t) - y_j(t) \notin U_t(n_{xi})$ for all $k \neq j$ sufficiently large, depending on i, for some $t \in F_{xi}$ or else $\{y_k\}$ would be Cauchy on F_{xi} and hence would be convergent at each point in F_{xi}. In particular, $\{y_k(x)\}$ wold be convergent which contradicts the choice of x. If $\overline{\lim}_{i\to+\infty} n_{xi} = +\infty$ then there is a subsequence $\{n_{xi(\alpha)}\}$ of $\{n_{xi}\}$ such that $\lim_{\alpha\to+\infty} n_{xi(\alpha)} = +\infty$ and by the definition of $n_{xi(\alpha)}$ and the nestedness of $\{U_t(n)\}$ we have $y_k(t) - y_j(t) \in U_t(n_{xi(\alpha)} - 1)$ for all $k \neq j$ sufficiently large, depending on α, and all $t \in F_{xi(\alpha)}$. Thus $\{y_k(x)\}$ is Cauchy and hence convergent which is contrary to the choice of x so $\overline{\lim}_{i\to+\infty} n_{xi} = c_x < +\infty$. Let $d_x > c_x$ be an upper bound for the set $\{n_{xi}\}$.

If there are uncountably many values of $x \in S$ at which $\{y_k(x)\}$ is neither convergent nor eventually monotone then there is some fixed positive integer d so that $d_x \leqq d$ holds for uncountably many $x \in S$ at which $\{y_k(x)\}$ is neither convergent nor eventually monotone. Denote this uncountable set of x's by A. We now have $x \in A$ and $k \neq j$ sufficiently large, depending on i, implies $y_k(t) - y_j(t) \notin U_t(d)$ for some $t \in F_{xi}$.

Choose $N > N(d)$ and $u(1) \in A \cap S^\circ$ and $F_{u(1)i(1)} \in \{F_{u(1)i}\}$ such that $(S - F_{u(1)i(1)}) \cap A$ is uncountable. Choose $u(2) \in (S - F_{u(1)i(1)}) \cap (A \cap S^\circ)$ and $F_{n(2)i(2)} \in \{F_{u(2)i}\}$ with $F_{u(1)i(1)} \cap F_{u(2)i(2)} = \varnothing$ and

$$(S - (F_{u(1)i(1)} \cup F_{u(2)i(2)})) \cap A$$

is uncountable. Continuing in this manner we get $\{u(1), u(2), \cdots, u(2N+1)\}$ in $A \cap S^\circ$ and $\{F_{u(1)i(1)}, F_{u(2)i(2)}, \cdots, F_{u(2N+1)i(2N+1)}\}$ which are mutually disjoint. By renaming the points $u(i)$ we may assume $u(1) < u(2) < \cdots < u(2N+1)$. So choose $k \neq j$, $k, j > H(d)$, sufficiently large that for each odd positive integer α, $1 \leqq \alpha \leqq 2N+1$, $y_k(x(\alpha)) - y_j(x(\alpha)) \notin U_{x(\alpha)}(d)$ for some $x(\alpha) \in F_{u(\alpha)i(\alpha)}$, and for each positive even integer α, $2 \leqq \alpha \leqq 2N$, $y_k(t_\alpha) - y_j(t_\alpha) < \theta$ holds for some $t_\alpha \in F_{u(\alpha)i(\alpha)}$ and $y_k(\tau_\alpha) - y_j(\tau_\alpha) > \theta$ holds for some $\tau_\alpha \in F_{u(\alpha)i(\alpha)}$. Now consider the partition $P_0 = \{\beta_1, \beta_2, \cdots, \beta_n\}$ where $\beta_\alpha = x(\alpha)$ if α is odd; β_α is omitted from P_0 if α is even and $y_k(x(\alpha-1)) - y_j(x(\alpha-1)) < \theta$ and $y_k(x(\alpha+1)) - y_j(x(\alpha+1)) > \theta$ or the opposite inequalities hold; β_α is taken to be t_α if $y_k(x(\alpha-1)) - y_j(x(\alpha-1)) > \theta$ and $y_k(x(\alpha+1)) - y_j(x(\alpha+1)) > \theta$ and β_α is taken to be τ_α if $y_k(x(\alpha-1)) - y_j(x(\alpha-1)) < \theta$ and $y_k(x(\alpha+1)) - y_j(x(\alpha+1)) < \theta$. Then the partition P_0 is such that $(y_k - y_j, P_0)$ is a proper pair and $y_k(x(\alpha)) - y_j(x(\alpha)) \notin U_{x(\alpha)}(d)$ for α odd, $x(\alpha) \in P_0$, and there are $N+1$ such $x(\alpha)$. This is contrary to the hypothesis of the theorem.

We conclude that the conclusion of theorem holds for all but countably many values of x. By choosing a monotone subsequence of $\{y_k(x)\}$ for each such x and diagonalizing, the subsequence, again denoted by $\{y_k\}$, is either eventually monotone or convergent for each x in S.

NOTE. If one wishes to consider sequences $\{y_k\}$, $y_k \in \prod_{s\in S} E_s$ where each E_s is an ordered topological vector space then the definition of a proper pair given in Definition 2.1 may be modified by replacing $f: S \to E$ by $f \in \prod_{s \in S} E_s$, E by E_s, K by K_s, and θ by θ_s. With the corresponding changes in the statement and proof of Theorem 2.2 this remains a valid resnlt.

COROLLARY 2.3. *Let B be a reflexive ordered Banach space with normal positive cone K and S be a nonempty subset of R. Let $\{y_k\}$, $y_k: S \to B$ be such that for each $s \in S$, $\{y_k(s)\}$ is an eventually comparable norm bounded sequence. If there are nonnegative integers $N(n)$ and $H(n\}$ such that for all $k, j \geqq H(n)$ $(y_k - y_j, P)$ is a proper pair then P contains at most $N(n)$ points x such that $y_k(x) - y_j(x) \notin U_x(n)$ then $\{y_k\}$ contains a subsequence $\{h_k\}$ which converges at each point of S.*

Proof. It follows from Theorem 2.2 that there is a subsequence which at each point s of S is either eventually monotone or else is convergent. By [2, Proposition 3.7, p. 93] it follows that this subsequence conveges at every point of S.

DEFINITION 2.4. Let S be a nonempty set of real numbers and $\{y_k\}$ be a sequence of functions, $y_k: S \to R$. We say that the sequence $\{y_k\}$ is equioscillatory if for each $s \in S$ there exists a neighborhood basis of 0 of radii $\varepsilon(n, s)$ and for each positive integer n there exist positive integers $N(n)$ and $H(n)$ such that if $k, j \geqq H(n)$ and $(y_k - y_j, P)$ is a proper pair then P contains no more than $N(n)$ points x for which $|y_k(x) - y_j(x)| > \varepsilon(n, x)$.

COROLLARY 2.5. *Let S be a nonempty subset of real numbers and $\{y_k\}$ be a sequence of functions, $y_k: S \to R$. The sequence $\{y_k\}$ has a subsequence which is pointwise convergent if and only if it has a subsequence which is pointwise bounded and equioscillatory.*

Proof. The sufficiency follows from Theorem 2.2. The necessity is trivial since if $N(n) = 0$ in Definition 2.4 we see that this is equivalent to saying that $\{y_k\}$ is pointwise Caucny.

3. Applications. In this section we examine some examples which serve to illustrate the results obtained in §2.

EXAMPLE 1. Let H be a complex Hilbert space and E be the ordered locally convex space, over the reals, of continuous linear Hermitian operators on H with the strong operator topology. Let the order for E be determined by $A \geqq \theta \Leftrightarrow (Ax, x) \geqq 0$ for all $x \in H$ and $A \geqq D \Leftrightarrow A - D \geqq \theta$ for $A, D \in E$. Let $A_k(t)$ be a sequence of functions from the real interval I into E which satisfies the hypotheses of Theorem 2.2. It is known that monotone sequences in E which are topologically bounded are convergent in the strong operator topology. Thus it follows from Theorem 2.2 that if $\{A_k(t)\}$ is topologically bounded for each $t \in I$ then there is a subsequence $\{D_j\}$ of $\{A_k\}$ such that $\{D_j(t)\}$ is convergent in the strong operator topology on E for every $t \in I$.

EXAMPLE 2. If in Example 1 we take H to be the d-dimensional complex Hilbert space C^d and B to be the $d \times d$ Hermitian matrices with the usual operator norm then B is a reflexive Banach with a normal positive cone. Thus a sequence $\{A_k\}$ of functions from a real interval I into B which satisfies the hypotheses of Corollary 2.3 must contain a subsequence which converges in norm for every $t \in I$.

Consider the sequence of linear differential equations

$$y' = A_k(t)y + f_k(t),\ y(t_k) = y_k \tag{3.1$_k$}$$

where $y \in R^d$, $A_k(t)$ is a $d \times d$ matrix and $f_k(t)$ a $d \times 1$ matrix each with continuous real entries for $t \in I$. Assume that $A_k(t)$ can be partitioned independent of k and t into square submatrices, possibly 1×1 such that each of the sequences of square submatrices satisfies the hypotheses of Corollary 2.3. Assume also that $f_k(t)$ can be partitioned independent of k and t into square submatrices, necessarily 1×1, such that each sequence of square submatrices satisfies the hypotheses of Corollary 2.3. Then the sequence $\{A_k\}$, $A_k: I \to B$ must contain a subsequence, $\{A_{k_j}\}$, which converges in the operator norm on the $d \times d$ matrices for each $t \in I$ and hence converges in R in each entry for each $t \in I$. Let us denote this limit by $A_0(t)$. Also, $\{f_{k_j}\}$ must contain a subsequence which converges in R in each entry for each $t \in I$ to a function we will denote by $f_0(t)$. If $t_k \to t_0$ and $y_k \to y_0$ as $k \to +\infty$ where $t_k \in I$ for $k = 0, 1, \cdots$ then it follows that the sequence of solutions of $(3.1)_k$ contains a subsequence which converges at every point of I to a function y which is a solution of $(3.1)_0$ almost everywhere on I.

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