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DEVELOPABLE SURFACES IN HYPERBOLIC SPACE

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A developable surface in hyperbolic space is a ruled surface (the trace of a moving hyperbolic line) whose tangent hyperbolic plane is constant along elements. This paper discusses the following hyperbolic analogs of theorems on developable surfaces in euclidean space:

1. A ruled surface is developable if and only if the tangent to the directrix, the unit vector giving the direction of the element, and the covariant derivative of the latter along the directrix, are linearly dependent.

2. A developable surface consists of portions of cones, tangential surfaces, and geodesic cylinders (to be defined).

3. A developable surface is applicable on a hyperbolic plane.

4. A flat surface in hyperbolic space (a surface whose intrinsic curvature is the same as that of a hyperbolic plane) is necessarily developable.

This discussion will use Poincaré's model of hyperbolic space, the upper half-space $\{(x, y, z): z > 0\}$ with a metric defined by

$$ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2) .$$

Most of the results above can be obtained in somewhat greater generality from the general equations for submanifolds, particularly the Synge inequality (see Spivak [2]), without reference to a particular model. However, it seems worthwhile also to present a more concrete proof. Theorem 2 in particular has much more impact when one visualizes the surfaces in this model, and would suffer considerably by being cast into more abstract terms. The other theorems may be regarded mainly as exercises in developing intuition for hyperbolic space.

Poincaré's model is conformal; that is, the measure of an angle is the same as its euclidean measure. Hyperbolic lines (resp. planes) are semicircles (hemispheres) orthogonal to the $x-y$ plane, with vertical lines and planes as limiting cases. The vertical case sometimes presents notational problems; it can be avoided in any local discussion by a rotation of the space.

A ruled surface is defined by giving the directrix (x, y, z) as a function of one variable u, and another function (ξ, η, ζ) of the same variable which indicates the direction of the element. It seems helpful to set $\xi^2 + \eta^2 + \zeta^2 = 1$, so that $z(\xi, \eta, \zeta)$ is a unit vector in the hyperbolic norm. If v is arc length along the element, measured from the directrix, the surface is given by

$$X(u, v) = \left(x + \frac{z\xi}{1-\zeta^2}(\zeta - \cos\theta),\ y + \frac{z\eta}{1-\zeta^2}(\zeta - \cos\theta),\ \frac{z\sin\theta}{\sqrt{1-\zeta^2}}\right), \tag{1}$$

where

$$\tan\frac{\theta}{2} = e^v\sqrt{\frac{1-\zeta}{1+\zeta}}\,.$$

The auxiliary variable θ makes it easier to visualize the surface, but in fact (1) could be written without it. There is no real problem if $\zeta(u) = \pm 1$; for such values of u, $X(u, v) = (x, y, ze^{\pm v})$.

It is convenient to work with the following special parametrization: in a region in which no element is vertical, choose the ridgeline $\theta = \pi/2$ as a new directrix, and write

$$X(u, v) = (x - z\xi\cos\theta,\ y - z\eta\cos\theta,\ z\sin\theta) \tag{2}$$

with $\xi^2 + \eta^2 = 1$ and $\tan(\theta/2) = e^v$. Then

$$\begin{aligned}
X_1 &= (x' - z'\xi\cos\theta - z\xi'\cos\theta,\ y' - z'\eta\cos\theta - z\eta'\cos\theta,\ z'\sin\theta)\,,\\
X_2 &= z\sin\theta(\xi\sin\theta,\ \eta\sin\theta,\ \cos\theta)\,,\\
N &= (y'\cos\theta - z'\eta - z\eta'\cos^2\theta,\ -x'\cos\theta + z'\xi + z\xi'\cos^2\theta\,,\\
&\qquad (x'\eta - y'\xi)\sin\theta + z(\xi\eta' - \xi'\eta)\sin\theta\cos\theta)\,,
\end{aligned}$$

the last being a vector in the normal direction but not necessarily of unit length.

The tangent hyperbolic plane is the hemisphere with center

$$C(u, v) = X(u, v) - \lambda(u, v)N(u, v)$$

in the $x - y$ plane (except that if N is horizontal, the tangent plane is vertical); thus $\lambda = z[x'\eta - y'\xi + z(\xi\eta' - \xi'\eta)\cos\theta]^{-1}$. The surface is developable, by definition, if and only if $\partial C/\partial v = 0$.

Simplification of $\partial C/\partial v$, using nothing more complicated than the identities $\xi^2 + \eta^2 = 1$, $\xi\xi' + \eta\eta' = 0$, leads to

$$zz'(\xi'\eta - \xi\eta')\eta = (x'\xi + y'\eta)(x'\eta - y'\xi)\eta$$

and

$$zz'(\xi'\eta - \xi\eta')\xi = (x'\xi + y'\eta)(x'\eta - y'\xi)\xi\,.$$

Since ξ and η cannot both vanish, it must be that

$$zz'(\xi'\eta - \xi\eta') = (x'\xi + y'\eta)(x'\eta - y'\xi) \ . \tag{3}$$

This is equivalent to the vanishing of the determinant

$$\begin{vmatrix} x' & \xi & z\xi' \\ y' & \eta & z\eta' \\ z' & 0 & x'\xi + y'\eta \end{vmatrix} ;$$

thus, a ruled surface is developable if and only if there is a linear dependence among the vectors

$T = (x', y', z')$, tangent to the directrix,
$Z = z(\xi, \eta, 0)$, a unit vector (in the hyperbolic norm) in the direction of the element, and
$DZ = (z\xi', z\eta', x'\xi + y'\eta)$, the covariant derivative of Z along the directrix.

The use of the special parametrization simplifies this computation, but it is not essential to the result. Indeed, for any regular curve on the surface, given by $v = f(u)$, the determinant (T, Z, DZ) using the given curve as a new directrix, is a nonzero multiple of the determinant above:

$$Z = X_2 = z \sin\theta(\xi \sin\theta, \eta \sin\theta, \cos\theta) ;$$

$$\begin{aligned} T = X_1 + f'X_2 = {} & \frac{zf' + x'\xi + y'\eta}{z} Z \\ & + (x' - z'\xi\cos\theta - z\xi'\cos\theta - x'\xi^2\sin^2\theta - y'\xi\eta\sin^2\theta \ , \\ & \quad y' - z'\eta\cos\theta - z\eta'\cos\theta - y'\eta^2\sin^2\theta - x'\xi\eta\sin^2\theta \ , \\ & \quad z'\sin\theta - (x'\xi + y'\eta)\sin\theta\cos\theta) \ , \end{aligned}$$

the last vector being orthogonal to Z; and

$$\begin{aligned} DZ = (& z\xi' + z'\xi\cos^2\theta - x'\cos\theta, z\eta' + z'\eta\cos^2\theta - y'\cos\theta \ , \\ & x'\xi\sin\theta + y'\eta\sin\theta - z'\sin\theta\cos\theta) \ . \end{aligned}$$

The determinant (T, Z, DZ) is easily reduced to

$$z\sin^3\theta \begin{vmatrix} x' & \xi & z\xi' \\ y' & \eta & z\eta' \\ z' + (x'\xi + y'\eta)\cos\theta & \cos\theta & x'\xi + y'\eta \end{vmatrix} ;$$

again using the identities $\xi^2 + \eta^2 = 1$, $\xi\xi' + \eta\eta' = 0$ this reduces to $z\sin^3\theta$ times the original determinant. This proves Theorem 1.

The condition of linear dependence suggests several classes of developable surfaces. If $Z = T$ along an appropriate directrix, the surface is tangential. $DZ = 0$ characterizes another class of surfaces, *geodesic cylinders*. The elements of a geodesic cylinder are geodesic parallels (in the sense of Levi-Civita) along a particular directrix, the *directrix of parallelism*, which is generally unique on the surface. In addition to the usual cone ($T = 0$ with the vertex as "directrix"), we define an *asymptotic cone* to be a ruled surface with all elements converging to a common point in the $x - y$ plane (an ideal point of the hyperbolic space).

Suppose, then, that $X(u, v)$ is a developable surface, represented as in (2). We have calculated T, Z, DZ for a curve $X(u) = X(u, f(u))$ on the surface. The surface is (i) a cone if there exists f such that $T = 0$; (ii) a tangential surface if there exists f such that T is a nonzero multiple of Z; (iii) a geodesic cylinder if there exists f such that $DZ = 0$. Asymptotic cones could be included in (i), with due care, but it seems easier to handle them separately. The claim of Theorem 2 is that any developable surface is made up of open regions, in each of which one of the above conditions applies, and boundaries of such regions.

Both (i) and (ii) require the vanishing of that part of T orthogonal to Z, which, since $\sin\theta > 0$, implies

$$z' - (x'\xi + y'\eta)\cos\theta = 0 \; . \tag{4}$$

This equation uniquely defines $\theta \in (0, \pi)$ provided $|z'| < |x'\xi + y'\eta|$. (iii) requires

$$x'\xi + y'\eta - z'\cos\theta = 0 \tag{5}$$

which uniquely defines θ when $|z'| > |x'\xi + y'\eta|$. For an asymptotic cone (iv) we set $\theta = 0$ or π in (2), and from $(x \mp z\xi, y \mp z\eta, 0) =$ const we obtain $z' = \pm(x'\xi + y'\eta)$.

With these remarks in mind, we partition the u-domain as follows:

E_1: the open set in which $|z'| < |x'\xi + y'\eta|$;

E_2: the open set in which $|z'| > |x'\xi + y'\eta|$;

E_3: the interior of the set in which $z' = \pm(x'\xi + y'\eta) \neq 0$;

E_4: the interior of the set in which $z' = x'\xi + y'\eta = 0$.

Any of these sets may be empty. Their union omits only a set containing no open interval.

In E_1, note that $x'\xi + y'\eta \neq 0$. Using the value of θ uniquely defined by (4), we find

$$
\begin{aligned}
x' &- z'\xi\cos\theta - z\xi'\cos\theta - x'\xi^2\sin^2\theta - y'\xi\eta\sin^2\theta \\
&= x' - z\xi'\cos\theta - x'\xi^2 - y'\xi\eta - \xi\cos\theta[z' - (x'\xi + y'\eta)\cos\theta] \\
&= -z\xi'\cos\theta + x'\eta^2 - y'\xi\eta \\
&= -\frac{zz'\xi'}{x'\xi + y'\eta} + (x'\eta - y'\xi)\eta \qquad \text{(using equation (4))} \\
&= -\frac{zz'\xi'}{x'\xi + y'\eta} + \frac{zz'(\xi'\eta - \xi\eta')\eta}{x'\xi + y'\eta} \qquad \text{(using equation (3))} \\
&= 0\,.
\end{aligned}
$$

Similarly, $y' - z'\eta\cos\theta - z\eta'\cos\theta - y'\eta^2\sin^2\theta - x'\xi\eta\sin^2\theta = 0$; thus $T = (zf' + x'\xi + y'\eta)Z/z$. The surface is therefore tangential in open regions in which $T \neq 0$, conic in any open u-interval in which $T = 0$. (Note that $f(u) = v = \log\tan\theta/2$ is determined by (4).)

In E_2, using the value of θ defined in (5),

$$
\begin{aligned}
z\xi' + z'\xi\cos^2\theta - x'\cos\theta &= z\xi' + \xi\cos\theta(x'\xi + y'\eta) - x'\cos\theta \\
&= z\xi' + (y'\xi - x'\eta)\eta\cos\theta \\
&= z\xi' + \frac{(y'\xi - x'\eta)(x'\xi + y'\eta)\eta}{z'} \qquad \text{(using equation (5))} \\
&= z\xi' - z(\xi'\eta - \xi\eta')\eta \qquad \text{(using equation (3))} \\
&= 0
\end{aligned}
$$

and similarly $z\eta' + z'\eta\cos^2\theta - y'\cos\theta = 0$, so that the surface is a geodesic cylinder.

In E_3, we note that equation (3), which holds for all developable surfaces, reduces to $x'\eta - y'\xi = \pm z(\xi'\eta - \xi\eta')$. Setting $\theta = 0$ or π in (2), we obtain $X(u, \pm\infty) = (x \mp z\xi, y \mp z\eta, 0)$ which is constant in each component of E_3; that is, each component gives a portion of asymptotic cone.

E_4 can be further partitioned into cases analogous to the ones above. The vanishing of z' and $x'\xi + y'\eta$ simplify formulas, so that the surface is tangential or conic if there is $\theta(u)$ such that

$$
\begin{aligned}
x' - z\xi'\cos\theta &= 0 \\
y' - z\eta'\cos\theta &= 0\,,
\end{aligned}
$$

cylindrical if there is $\theta(u)$ such that

$$
\begin{aligned}
z\xi' - x'\cos\theta &= 0 \\
z\eta' - y'\cos\theta &= 0\,.
\end{aligned}
$$

There is a unique solution to one of these systems, and no solution to the other, except in the case $|z\xi'| = |x'|$, $|z\eta'| = |y'|$. In an open interval in which at least one of these quantities is nonzero, the surface is an asymptotic cone; if all vanish, the surface degenerates to a single line.

Therefore, any developable surface consists of open portions (the union of elements passing through an open subset of the directrix) in which the surface is tangential, open portions of cones, open portions of geodesic cylinders, and a remaining set containing no open subset of the directrix. This is Theorem 2.

It is interesting to note that in the projective model of hyperbolic space (see Spivak [2], Chapter 7) the developable surfaces are precisely those which are also developable in the euclidean geometry — that is, portions of (euclidean) cones, cylinders and tangentials contained inside the unit ball. However, the classification above does not correspond to the euclidean classification. For example, a (euclidean) cone or tangential whose vertex or edge lies outside the ball, is a hyperbolic cylinder, and not a cone or tangential. A hyperbolic tangential surface as defined above has an edge consisting of real hyperbolic points, and a hyperbolic cone has a vertex which is a real or ideal hyperbolic point.

In euclidean space, a complete developable surface with no singularities is a cylinder; an elementary proof of this fact was given by Stoker [3]. In hyperbolic space this is not the case; a complete, smooth developable surface can be patched together from geodesic cylinders and asymptotic cones. To preserve the similarities between the two spaces, one might call asymptotic cones, "Lobachevskian cylinders," their elements being parallel in the classical hyperbolic sense. In either case, it is worth noting that hyperbolic space has an "extra" class of developable surfaces.

To prove Theorem 3, let $X(u, v)$ be a ruled surface, represented as in (2), and assume that u measures arc length on the directrix.

$$g_{22} = 1;$$

$$g_{12} = \frac{x'\xi + y'\eta}{z} = \cos\phi$$

where ϕ is the angle between T and Z;

$$\begin{aligned} g_{11} = \frac{1}{z^2\sin^2\theta}[&x'^2 + y'^2 + z'^2 + z^2(\xi'^2 + \eta'^2)\cos^2\theta \\ &- 2z'(x'\xi + y'\eta)\cos\theta - 2z(x'\xi' + y'\eta')\cos\theta] \,. \end{aligned}$$

Since $x'^2 + y'^2 + z'^2 = z^2 = z^2\sin^2\theta + z^2\cos^2\theta$,

$$\begin{aligned} g_{11} = 1 + &\frac{\cos^2\theta}{\sin^2\theta}(1 + \xi'^2 + \eta'^2) \\ &- \frac{2\cos\theta}{\sin^2\theta}\left[\frac{z'(x'\xi + y'\eta)}{z^2} + \frac{x'\xi' + y'\eta'}{z}\right]. \end{aligned}$$

From $\tan(\theta/2) = e^v$ follow $\sin\theta = \operatorname{sech} v$, $\cos\theta = -\tanh v$. Moreover, $DZ = (z\xi', z\eta', x'\xi + y'\eta)$, whence

$$||DZ||^2 = \xi'^2 + \eta'^2 + \cos^2\phi \quad \text{(hyperbolic norm)}$$

and

$$\frac{z'(x'\xi + y'\eta)}{z^2} + \frac{x'\xi' + y'\eta'}{z} = \langle DZ, T\rangle$$

(hyperbolic inner product) .

Thus $g_{11} = 1 + \sinh^2 v(||DZ||^2 + \sin^2\phi) + \sinh 2v\langle DZ, T\rangle$.

If the surface is developable, the determinant $(T, Z, DZ) = 0$; moreover, $\langle Z, DZ\rangle = 0$, so that, unless $DZ = 0$,

$$T = aZ + bDZ = \cos\phi\, Z + \frac{\sin\phi}{||DZ||}DZ;\ \langle DZ, T\rangle = \sin\phi\, ||DZ|| \,.$$

Obviously the last equation holds also when $DZ = 0$. Thus, for a developable surface,

$$g_{11} = 1 + \sinh^2 v\,(||DZ||^2 + \sin^2\phi) + \sinh 2v \sin\phi\, ||DZ||;$$

that is, the metric on a developable surface is determined by the angle between directrix and element, and the norm $||DZ||$.

It suffices now to exhibit a parametrization of a hyperbolic plane as a ruled surface with any specified ϕ and $||DZ||$. Such a parametrization is

$$Y(u, v) = \left(x + \frac{z}{\xi}(\zeta - \cos\theta), 0, \frac{z}{\xi}\sin\theta\right),\ \tan\frac{\theta}{2} = e^v\sqrt{\frac{1-\zeta}{1+\zeta}}$$

where $x(u)$, $z(u)$, $\xi(u)$ and $\zeta(u)$ are chosen as follows:

Let $\alpha(u)$ be a solution of $\alpha' + \cos\alpha = \phi'(u) - ||DZ||$. Choose x, z so that $x' = z\cos\alpha$, $z' = z\sin\alpha$, and set $\xi = \cos(\alpha - \phi)$, $\zeta = \sin(\alpha - \phi)$. Then along the directrix ($v = 0$), $(x'\xi + z'\zeta)/z = \cos\phi$ while $\xi'\zeta - \xi\zeta' - x'/z = ||DZ||$.

Theorem 4 can also be derived in the Poincaré model, but not without introducing additional material, particularly the second fundamental form and hyperbolic analogs of the equations of Gauss, Mainardi and Codazzi. This material is of interest in a general study of hyperbolic space and much of it can be found in Rozenfel'd [1], Chapter 6.

References

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