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REAL PARTS OF UNIFORM ALGEBRAS ON THE CIRCLE

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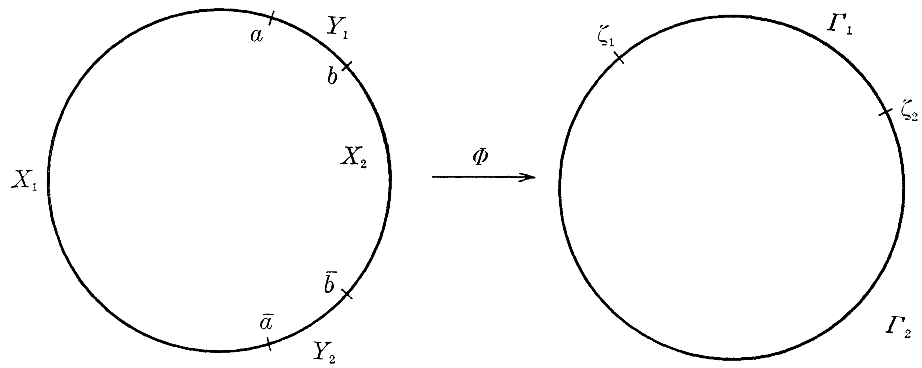
This paper is about uniform algebras on the unit circle Γ in the complex plane and specifically with the spaces of real parts of such algebras. The major portion of the paper is devoted to proving that if A is the disc algebra on Γ and B is any uniform algebra on Γ such that $\operatorname{Re} A \subset \operatorname{Re} B$, then either $B = C(\Gamma)$ or else $B = A \circ \Phi (= \{f \circ \Phi\colon f \in A\})$ for some homeomorphism Φ. We also show that any homeomorphism Φ for which $\operatorname{Re} A \subset \operatorname{Re} A \circ \Phi$ must be absolutely continuous.

By a uniform algebra on Γ we mean a subalgebra of $C(\Gamma)$ which is closed in the supremum norm, separates points, and contains the constant functions. The canonical example of a uniform algebra on Γ (other than $C(\Gamma)$, of course) is the disc algebra A: those members of $C(\Gamma)$ which have continuous extensions to the closure of the unit disc U so as to be analytic in U. In a recent paper [3], John M. F. O'Connell sets forth some of the consequences of assuming the relationship $\operatorname{Re} A = \operatorname{Re} B$, where $\operatorname{Re} A$ and $\operatorname{Re} B$ denote the spaces of real parts of the functions in A and B respectively. O'Connell shows that when $\operatorname{Re} A = \operatorname{Re} B$, there is a homeomorphism Φ of Γ onto Γ such that $B = A \circ \Phi$, and that any homeomorphism Φ for which $\operatorname{Re} A = \operatorname{Re} A \circ \Phi$ is necessarily absolutely continuous. Thus our main results, as stated in the opening paragraph, represent generalizations of O'Connell's work to settings in which only the inclusions $\operatorname{Re} A \subset \operatorname{Re} B$ and $\operatorname{Re} A \subset \operatorname{Re} A \circ \Phi$ are assumed.

THEOREM 1. *Let A be the disc algebra on Γ and B be any uniform algebra on Γ such that $\operatorname{Re} A \subset \operatorname{Re} B$. Then either $B = C(\Gamma)$ or there is a homeomorphism Φ of Γ onto Γ such that $B = A \circ \Phi$.*

As the proof is rather lengthy, we have chosen to proceed through a sequence of three lemmas. The last of these is perhaps interesting in its own right and is suggested by work that appears in O'Connell's paper. The first lemma is also due to O'Connell. Z will be used throughout to denote the identity function on Γ, $Z(z) = z$ for all $z \in \Gamma$.

LEMMA 1. *Suppose B is a uniform algebra on Γ such that $\operatorname{Re} A \subset \operatorname{Re} B$. Then there are two points $a, b \in \Gamma \cap \{z\colon \operatorname{Im} z \geqq 0\}$,* *and a function $\Phi \in B$ such that Φ maps Γ onto itself as in the diagram below. See, also, the notes following diagram.*



NOTES. X_1, X_2, Y_1, and Y_2 are the indicated *closed* arcs; $\Phi(X_j)$ is a singleton $\{\zeta_j\}$ and Φ takes Y_j homeomorphically onto Γ_j for $j = 1, 2$.

Proof. As $Z \in A$, there are functions ψ and ψ_1 in B such that $\operatorname{Re} Z = \operatorname{Re} \psi$ and $\operatorname{Im} Z = \operatorname{Im} \psi_1$. Now if ψ maps Γ onto a Jordan arc, then Mergelyan's theorem implies that the function $w \rightarrow \operatorname{Re} w$ is uniformly approximable on $\psi(\Gamma)$ by polynomials in ψ. Since B is a uniformly closed algebra, we have $\operatorname{Re} Z \in B$. Similarly, if also ψ_1 maps Γ onto a Jordan arc, then $\operatorname{Im} Z \in B$. Thus in the event that both $\psi(\Gamma)$ and $\psi_1(\Gamma)$ are Jordan arcs, we conclude by Fejér's theorem that $B = C(\Gamma)$. The conclusion of the lemma obviously holds in this case. So let us assume that at least one of $\psi(\Gamma)$ or $\psi_1(\Gamma)$, say the former, is *not* a Jordan arc. Remembering that $\operatorname{Re} \psi = \operatorname{Re} Z$, we see that it is necessarily the case that there exist $a, b \in \Gamma \cap \{z: \operatorname{Im} z \geqq 0\}$ with $\operatorname{Re} a < \operatorname{Re} b$ such that $\psi(a) = \psi(\bar{a})$, $\psi(b) = \psi(\bar{b})$, but $\psi(\zeta) \neq \psi(\bar{\zeta})$ for $\operatorname{Re} a < \operatorname{Re} \zeta < \operatorname{Re} b$. Let X_1, X_2, Y_1, Y_2 be the closed arcs indicated in the diagram. Then ψ maps each of Y_1 and Y_2 homeomorphically onto arcs and maps $Y_1 \cup Y_2$ onto a simple closed curve J. Let W be the bounded component of $\boldsymbol{C} - J$ and K be the union of $\psi(\Gamma)$ and the bounded components of $\boldsymbol{C} - \psi(\Gamma)$. By a famous theorem of Carathéodory, there is a homeomorphism ϕ of $W \cup J$ onto the closed unit disc such that ϕ is analytic on W; if we extend ϕ to K by defining $\phi(w) = \phi(\psi(a))$ for $\operatorname{Re} w < \operatorname{Re} \psi(a)$ and $\phi(w) = \phi(\psi(b))$ for $\operatorname{Re} w > \operatorname{Re} \psi(b)$, then ϕ is continuous on K and analytic on the interior of K. By Mergelyan's theorem, we see that ϕ is uniformly approximable on K by polynomials and so it follows that $\phi \circ \psi \in B$. The function $\Phi = \phi \circ \psi$ has the required properties.

For $X \subset \Gamma$, $B \mid X$ will denote the algebra of restrictions to X of the members of B.

LEMMA 2. *Let B be a uniform algebra on Γ such that* $\operatorname{Re} A \subset \operatorname{Re} B$ *and let $\Phi \in B$ map Γ onto Γ as in Lemma 1. Then $B \mid X_1 = C(X_1)$ and $B \mid X_2 = C(X_2)$.*

Proof. We have $\Phi(X_1) = \{\zeta_1\}$ while $\Phi(\zeta) \in \Gamma - \{\zeta_1\}$ for $\zeta \in \Gamma - X_1$. Therefore X_1 is a peak set for B and consequently $B \mid X_1$ is sup-norm closed in the space $C(X_1)$. [5, Lemma 12.3.] In fact, an application of the generalized Stone-Weierstrass theorem will show that $B \mid X_1 = C(X_1)$. For to begin with, it follows from the theory of conformal mapping of Jordan regions (already used once in the proof of Lemma 1) that there is a function f in A which takes Γ homeomorphically onto the (boundary of) the closed unit square and is such that $f(X_1) = \{x\colon 0 \leqq x \leqq 1\}$. We have $\operatorname{Re} A \subset \operatorname{Re} B$, so choose $g \in B$ such that $\operatorname{Re} f = \operatorname{Re} g$. As $\operatorname{Re} f$ is one-to-one on X_1, so too is $\operatorname{Re} g$; thus g maps X_1 onto a Jordan arc. By Mergelyan's theorem once more, there is a sequence $\{P_n\}$ of polynomials such that $P_n(w) \to \operatorname{Re} w$ uniformly on $g(X_1)$; hence $P_n \circ g(\zeta) \to \operatorname{Re} g(\zeta)$ uniformly on X_1. As $B \mid X_1$ is closed we have $\operatorname{Re} g \mid X_1 \in B \mid X_1$. But $\operatorname{Re} g = \operatorname{Re} f$ is one-to-one on X_1. Thus $B \mid X_1$ contains a real-valued function (viz. $\operatorname{Re} g \mid X_1$) which is one-to-one. In particular, the real-valued functions in $B \mid X_1$ separate the points of X_1. It follows from the generalized Stone-Weierstrass theorem [5, Theorem 12.1] that $B \mid X_1 = C(X_1)$ as required. Exactly the same argument applies to show that $B \mid X_2 = C(X_2)$.

Before turning to the third and final lemma needed for the proof of Theorem 1, let us stop and note how O'Connell's result follows readily from what we have already done. That is, assume we actually have the equality $\operatorname{Re} A = \operatorname{Re} B$. Then $\operatorname{Re} A \mid X_1 = \operatorname{Re} B \mid X_1 = C_R(X_1)$, the last equation being a consequence of Lemmas 1 and 2. By a result of Sidney and Stout [5, Theorem 20.9] we conclude that $A \mid X_1 = C(X_1)$. But as is well-known, $A \mid X_1 = C(X_1)$ only if the "arc" X_1 is a single point. Of course the same argument yields that X_2 is a single point. Consequently, (see diagram again) $a = \bar{a} = -1$ and $b = \bar{b} = 1$, and hence Φ is one-to-one. Now $A \circ \Phi \subset B$ implies $A \subset B \circ \Phi^{-1}$, so $B \circ \Phi^{-1}$ is a uniform algebra on Γ which lies between A and $C(\Gamma)$. By Wermer's maximality theorem, $B \circ \Phi^{-1} = A$ or $B \circ \Phi^{-1} = C(\Gamma)$. The latter equality is impossible ($\operatorname{Re} A \neq C_R(\Gamma)$), so $B = A \circ \Phi$ and we have O'Connell's result.

As preparation for Lemma 3, we need to introduce some additional terminology and recall a few facts. Let m denote normalized Lebesgue one-dimensional measure on Γ and let $L^p(m)$ have its usual meaning. $H^p(m)$ is the subspace of $L^p(m)$ consisting of those $f \in L^p(m)$ whose Fourier coefficients with negative indices are zero: thus $f \in H^p(m)$ iff $f \in L^p(m)$ and $\int_\Gamma Z^n f dm = 0$ for $n = 1, 2, \cdots$. For $f \in L^1(m)$, let $P[f]$ denote the Poisson integral of f. It is a basic fact that $P[f]$ is a (complex-valued) harmonic function on U and $P[f]$ is analytic if and only if $f \in H^1(m)$. Also, $P[f]$ has radial limits equal to f a.e., and when $f \in H^1(m)$, then $f(\zeta) = 0$ on a set of positive measure if and only if $P[f]$ is identically zero.

LEMMA 3. *Suppose S is a subalgebra of $L^\infty(m)$, $h \in H^1(m)$ with $h \not\equiv 0$, and $hf \in H^1(m)$ for each $f \in S$. Then $S \subset H^\infty(m)$.*

Proof. We can assume that S is closed in $L^\infty(m)$ as the closure of S will satisfy the hypothesis if S does. Now for any $f_1, f_2 \in S$, $P[h]\cdot P[f_1 f_2 h]$ and $P[f_1 h]\cdot P[f_2 h]$ are each of bounded characteristic (belong to the Nevanlinna class N, cf. [4], Chapter 17) and, further, have the same radial limits a.e. Thus $(P[h]\cdot P[f_1 f_2 h])(z) = (P[f_1 h]\cdot P[f_2 h])(z)$ for all z in U. This means that for any fixed z where $P[h](z) \neq 0$, the functional

$$f \longrightarrow \frac{P[fh]}{P[h]}(z) \qquad (f \in S)$$

is a multiplicative linear functional on the Banach algebra S; thus it is continuous and has norm $\leqq 1$. Consequently, for each $f \in S$, the meromorphic function $P[fh]/P[h]$ is actually bounded on U and since $P[fh]/P[h]$ has radial limits equal to f a.e., we conclude that $f \in H^\infty(m)$. So $S \subset H^\infty(m)$ as required.

Proof of Theorem 1. Let $\Phi \in B$ map Γ onto Γ as in Lemma 1. We have $B \mid X_1 = C(X_1)$ and $B \mid X_2 = C(X_2)$ by Lemma 2. Now let us consider the remaining restriction algebras $B \mid Y_1$ and $B \mid Y_2$, and the two possibilities: (1) $B \mid Y_1 = C(Y_1)$ and $B \mid Y_2 = C(Y_2)$, (2) $B \mid Y_1 \neq C(Y_1)$ or $B \mid Y_2 \neq C(Y_2)$. If (1) holds then $B = C(\Gamma)$ because of the following theorem of R. E. Mullins [2, Theorem 3, p. 272]: "Let A be a function algebra on a compact metric space X. Let $F_1, \cdots, F_n$ be n closed sets such that $X = \bigcup_{i=1}^n F_i$ and $A \mid F_i = C(F_i)$, $i = 1, 2, \cdots, n$. Then $A = C(X)$." (A generalization of Mullins' result can be found in [5], Theorem 13.11.)

Suppose then that (2) holds with say $B \mid Y_1 \neq C(Y_1)$. By a theorem of Glicksberg [5, Theorem 20.16] there is a complex regular Borel measure μ on Γ such that $\int_\Gamma g d\mu = 0$ for all $g \in B$, but the restriction $\mu \mid Y_1$ is *not* the zero measure. The same theorem implies, moreover, that each of $\mu \mid X_1$ and $\mu \mid X_2$ *is* the zero measure. Denote by μ^* the measure on Γ induced by Φ; that is, $\mu^*(E) = \mu(\Phi^{-1}(E))$ for each Borel set E. Since Φ maps Y_1 homeomorphically onto Γ_1 and $\mu \mid Y_1$ is not the zero measure, it follows that μ^* is not the zero measure either. For each $g \in B$, let g^* be defined on $\Gamma - \{\zeta_1, \zeta_2\}$ by $g^*(\zeta) = g(\Phi^{-1}(\zeta))$. We claim that $d\mu^* = hdm$ where $h \in H^1(m)$ and that $g^*h \in H^1(m)$. To see this let $g \in B$ and n be a positive integer. Then

$$0 = \int_\Gamma \Phi^n g d\mu = \int_{\Gamma - (X_1 \cup X_2)} \Phi^n g d\mu = \int_{\Gamma - \{\zeta_1, \zeta_2\}} Z^n g^* d\mu^* = \int_\Gamma Z^n g^* d\mu^* .$$

If we take $g \equiv 1$, then the preceding calculation, together with the classical F. and M. Riesz theorem, implies that $d\mu^* = hdm$ where $h \in H^1(m)$. This establishes the claim. Note also that since μ^* is not the zero measure, h cannot vanish on a set of positive m-measure. Now let S be the subalgebra of $L^\infty(m)$ consisting of $\{g^*: g \in B\}$. Lemma 3 applies and we conclude that $S \subset H^\infty(m)$. So $g^* \in H^\infty(m)$ for each $g \in B$. It is clear that the bounded analytic extension to U of g^* is continuous on $U \cup \Gamma - \{\zeta_1, \zeta_2\}$ and that, furthermore, $\lim_{\substack{\zeta \to \zeta_1 \\ \zeta \in \Gamma_1 - \{\zeta_1\}}} g^*(\zeta) = g(a)$ while $\lim_{\substack{\zeta \to \zeta_1 \\ \zeta \in \Gamma_2 - \{\zeta_1\}}} g^*(\zeta) = g(\bar{a})$. By an old theorem due to E. Lindelöf [1, p. 43], we deduce that $g(a) = g(\bar{a}) = \lim_{\substack{\zeta \to \zeta_1 \\ \zeta \in \bar{U} - \{\zeta_1\}}} g^*(\zeta)$. In the same way, $g(b) = g(\bar{b}) = \lim_{\substack{\zeta \to \zeta_2 \\ \zeta \in \bar{U} - \{\zeta_2\}}} g^*(\zeta)$. As B separates points, we conclude that $a = \bar{a}$ and $b = \bar{b}$ so that Φ is one-to-one. The above argument also shows that then $g \circ \Phi^{-1} \in A$ for each $g \in B$. So $B \subset A \circ \Phi$. But we already have $A \circ \Phi \subset B$, hence $A \circ \Phi = B$ and the proof of the theorem is now complete.

REMARK. Although Wermer's maximality theorem was used to give a short proof of O'Connell's result, the above proof of the generalization does not require the maximality theorem. What the proof does use is some of the techniques that appear in Wermer's original argument.

A compact subset $K \subset \Gamma$ is an interpolation set for A if $A \mid K = C(K)$. Say that a continuous map $\Phi: \Gamma \to \Gamma$ *preserves interpolation sets* if $A \mid K = C(K)$ implies $A \mid \Phi(K) = C(\Phi(K))$.

THEOREM 2. *Suppose Φ is a continuous map of Γ into Γ such that* $\operatorname{Re} A \subset \operatorname{Re} A \circ \Phi$. *Then Φ preserves interpolation sets.*

Proof. Let K be an interpolation set for A so that $C(K) = A \mid K$. Then $C_R(K) = \operatorname{Re}(A \mid K) \subset \operatorname{Re}(A \circ \Phi \mid K)$. This implies that $C_R(\Phi(K)) \subset \operatorname{Re} A \mid \Phi(K)$. Hence by the Sidney-Stout result once more, we conclude that $C(\Phi(K)) = A \mid \Phi(K)$ as reguired.

Note that the preceding simple result is true in the setting of general uniform algebras. It would be interesting to know whether; in general, a map Φ satisfying the hypothesis of Theorem 2 necessarily preserves peak sets. In certain special cases interpolation sets are automatically peak sets.

COROLLARY. *Let Φ be a homeomorphism of Γ such that* $\mathrm{Re}\, A \subset \mathrm{Re}\, A \circ \Phi$. *Then Φ is absolutely continuous.*

Proof. By Theorem 2, Φ maps interpolation sets to interpolation sets. But by the Rudin-Carleson theorem these are the closed sets of measure zero. So the homeomorphism Φ maps closed sets of measure zero to closed sets of measure zero. It follows that Φ is absolutely continuous.

Although Theorem 1 generalizes in a trivial way to the case where Γ is the boundary of an arbitrary Jordan region in the plane, this is the only extension of Theorem 1 that we know of. It would be of interest to determine its validity in some other cases.

Added in proof. Results of E. L. Arenson can be used to show that in the setting of general uniform algebras, a homeomorphism Φ satisfying the hypothesis of Theorem 2 necessarily preserves peak sets.

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